Model of vortex boundary of plane laminar hydraulic jet

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Abstract. Reason for interest in jet streams is their great importance for various branches of engineering. Liquid and gas jet streams occur in rockets, airplanes, heat engines, turbines, boilers, combustion chambers, furnaces, ovens, hydraulic facilities, chemical and technological devices, ventilation devices and fluidics. In particular, during the hydrosystem operation, environmental problems arise associated with the erosion pool formation in the tailrace at considerable velocities of the flow. Riverbed erosion in the tailrace leads to emergency situations by the loss of hydraulic structures stability. Development of the hydrosystem stability assessment is complicated by the lack of an integrated approach to the analysis of riverbed processes, agreed with the plane hydraulic jet theory. To solve the problem of the plane laminar hydraulic jet propagation, the vortex boundary model, based on the scheme of rolling vortices, is applied in this paper. It is found that the geometric locus of equilibrium centres of vortices rolling without sliding along the jet boundary is a catenary line. To assess the adequacy of the proposed vortex model, it is compared with known solutions of the problem of the infinitely thin jet-source motion based on the boundary layer theory. The proof of the hypothesis about the identity of dependencies for the kinematic characteristics of the plane laminar hydraulic jet, which are derived from the vortex theory and the boundary layer theory, is associated with the phased development of new hydromechanics propositions and the analysis of existing ones. First of all, an equation of the plane laminar hydraulic jet boundary, based on the scheme of rolling vortices, is formulated. Then we investigate the statement of the problem of the infinitely thin jet-source motion in terms of the boundary layer theory. The self-similarity (affine similarity) of the longitudinal jet velocity profiles, based on the law of conservation of momentum carried through the jet cross-section, is proved. There is provided information on the application of the similarity and dimension theory to the formulation of partial differential boundary layer equations and a ordinary differential third-order equation for the boundary layer flow function in a dimensionless form. As a result of substitution, a dimensionless ordinary differential equation is formulated, based on the vortex jet boundary model. Conversion of the partial differential third-order equation into the ordinary differential equation for the flow function is generalized by calculating invariants**.** The partial differential boundary layer equation and the ordinary differential equation in regard to the flow function under the given vortex jet boundary nature are integrated in MathCad. Conditions of conversion of solutions of the dimensionless plane hydraulic jet equations, obtained by different authors, into a dimensional form are estimated.

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Introduction

Investigation of jet streams is sometimes either purely theoretical or experimental, but the most common works have semi-empirical character.

In some works, the authors study only turbulent jets based on semi-empirical theories, in other works - mostly laminar jets based on strict hydrodynamic equations [1].

Turbulent jets are the most important for practice. However, mechanisms of laminar jet propagation can serve as the theoretical model of turbulent jets. It is assumed that a turbulent jet can be analysed mathematically similarly to a laminar one.

Number of problems on the incompressible liquid motion, solution of which is possible on the basis of the accurate Navier-Stokes equations, is extremely limited. One of the most universal problems is L. D. Landau's problem [2] of ***a submerged laminar*** axisymmetric jet, flowing from a thin tube in an unlimited space, filled with the same liquid. However, the problem of the plane laminar jet-source propagation (analogue of L. D. Landau's problem) based on the Navier-Stokes equations is still not solved.

In this regard, along with numerical solutions of specific problems in the laminar jet theory, general solution methods based on the boundary layer theory are important.

Calculation methods currently used in the turbulent jet theory based on semi-empirical theories are not universal.

Empirical constants, borrowed from the experiment, reflecting the influence of different, not explicitly considered, factors, are also not universal.

Many analytical solutions of the problems of the jet propagation are related to the motions created by the jet-source. It is so, for example, for a significant part of the so-called "self-similar" solutions.

For a medium with the same physical properties as the liquid moving within the jet boundaries, the jet propagation is reduced substantially to the gradual equalization of initial velocity profile. Due to the viscosity, the jet involves the surrounding liquid in motion, passing part of the initial momentum to it. The velocity on axis and in cross-sections of the jet decreases.

The difficulty of obtaining the strict analytical solutions of the Navier-Stokes equations necessitated the use of several rather general assumptions about the nature of liquid motion.

The first assumption is considered to be the hypothesis about the existence of elementary geometric structures formed by the plane jet propagation. These hypotheses include the idea of the existence of the jet stream source pole, the linear nature of the core, the boundaries of the mixing layer and of the submerged hydraulic jet propagation, quite clearly reflected in the monograph by G. N. Abramovich [3].

Concerning kinematic characteristics, self-similarity of the longitudinal velocity profile propagation is postulated, at least within the transition and the main areas of the jet.

The assumption that there is no pressure gradient and therefore the independence of the longitudinal component of the momentum from the longitudinal coordinates of the jet is a very strong limitation of a dynamic nature.

Methodologies for calculating the geometric, kinematic and dynamic parameters of the hydraulic jets have a finished analytical nature only for a limited class of problems. First of all, they include the problem of the infinitely thin plane laminar jet-source propagation in the motionless liquid.

In the Cartesian coordinate system, coincident with the centre of the slit source and the axis x_1, situated in a symmetry plane, under the action of viscosity, the jet expands axially y_1 downstream, while the average flow velocity decreases. Due to the viscosity, the jet captures part of the motionless liquid, and as the result the stream with threads appears.

In this formulation, the problem of the plane laminar ***submerged jet*** was first solved by H. Schlichting [4].

According to L.G. Loitsyansky [5], despite the extremely thin cross-section of the source, the jet has a finite value of the total momentum J due to the high outflow velocity.

The practical value of the idea about the jet-forming source is determined by self-similar nature of the induced jet stream. It allows greatly to simplify the mathematical formulation of the problem and to go from the partial differential second-order boundary-layer equations to the ordinary differential third-order equation for the flow function.

The problem is solved using methods of the boundary-layer theory and usually includes several main stages.

In particular, in the monograph by N. A. Slezkin [6] there are the following methodological steps.

1. Statement of problem of the infinitely thin jet-source motion in terms of the boundary layer theory.

2. Application of the law of conservation of total momentum, carried through the jet cross-section, for justification of self-similarity (affine similarity) of longitudinal jet velocity profiles.

3. Application of the similarity and dimension theory for the transition to a dimensionless form of the boundary layer equation.

4. Transition from the boundary layer equations to the partial differential equation for the flow function.

5. Integration of the partial differential equation by the method of characteristics.

6. Derivation of the ordinary differential third-order equation for the flow function from the partial differential equation.

7. Integration of the ordinary differential equation for the flow function.

8. Expression of the solution obtained in a dimensional form.

In general, methods of solving the problem of the thin laminar jet propagation by H. Schlichting [4], L. G. Loitsiansky [5] and N. A. Slezkin [6] are the same in number and composition of stages. An exception isintegration of the partial differential equation by the method of characteristics by N. A. Slezkin.

For comparative analysis of these methodologies, including the shortened methodology by L. A. Vulis [1], uniform designations of the jet stream parameters are introduced. To reduce the comparative analysis of the mentioned works for each stage of the methodology, the summary variants of calculations, supplemented with commentaries by significant differences in the stage interpretation by various authors, are given below.

The proposed method of calculating the parameters of the submerged plane laminar hydraulic jet propagation is based on the equation of the vortex equilibrium on the curvilinear plane jet boundary [7]. In this regard, correspondence of the vortex jet boundary model to the solution of the jet propagation problem by methods of the boundary layer theory is analysed step-by-step. In the comparative analysis of methods, symbolic and numerical solutions in MathCad are used, which, in particular, refers to theintegration of the ordinary differential equations.

Derivation of the ordinary differential third-order equation for the flow function from the partial differential equation is based on an invariant transformation, similar to that in [8], for the equations of the boundary layer momentum and continuity.

Main part

1. We consider the planar motion of a material point n (Fig. 1), which position is determined by the radius vector $\bar{\rho}$ and the polar angle θ, with velocity $\bar{u}_\tau$. Vector of tangent to the motion path $y = f(x)$ of the velocity $\bar{u}_\tau$ at the point n in the Cartesian coordinate system xoy has vectors of projections on the axis x and y equal to $\bar{u} = \bar{u}_\tau \cos\varphi$ and $\bar{v} = \bar{u}_\tau \sin\varphi$ respectively. The tangent $\xi\xi$ to the curve $y = f(x)$ has the inclination angle φ relative to the abscissa axis Ox [9]. Acceleration $\bar{a}$ of the point n is the desired vector.

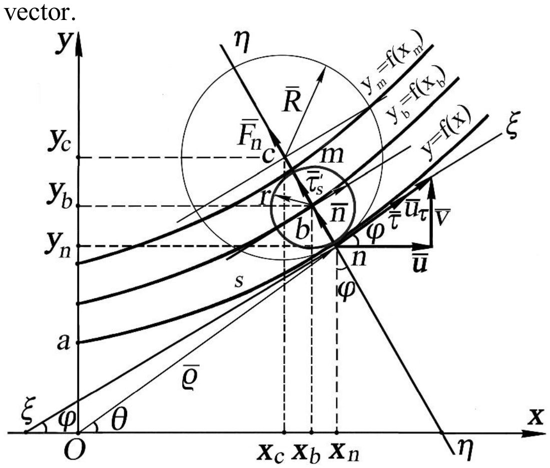


Fig.1. Motion of the point along the plane curve

$y = f(x)$ - equation of the curve s; R - radius of the curvature;

$\bar{\tau}$ - tangent to the curve; $\bar{n}$ - normal to the curve;

$\bar{\tau}'_s = \dfrac{d\bar{\tau}}{ds} = k\bar{n}$ - derivative of the tangent vector $\bar{\tau}$ along the arc length s

For the parameter dependence of the radius vector of the point $\bar{\rho}$ on the arc length s of the curve $y = f(x)$, we write $\bar{\rho} = \bar{\rho}(s)$. Derivative $\dfrac{d\bar{\rho}}{ds}$ is denoted by $\bar{\tau}$ and is the unit vector of the tangent to the path, so

$$\bar{u}_\tau = \frac{d\bar{\rho}}{dt} = \frac{d\bar{\rho}}{ds}\frac{ds}{dt} = u_\tau \bar{\tau}, \quad (1)$$

where $u_\tau = |\bar{u}_\tau|$ – modulus of velocity.

Differential of the arc in Cartesian coordinates can be written for the point n in the form [10]

$$ds = |d\bar{\rho}| = |dx\bar{i} + dy\bar{j}| = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx, \quad (2)$$

where y' denotes the time differentiation.

Since $|\bar{\tau}(s)| = 1$, the vector $\dfrac{d\bar{\tau}}{ds}$ is perpendicular to the vector $\bar{\tau}$, so the straight line $\xi\xi$ (Fig. 1) is tangent to the path $y = f(x)$ and the straight line $\eta\eta$ parallel to the direction of the derivative vector $\dfrac{d\bar{\tau}}{ds} = \bar{\tau}'_s$ is normal to the curve at the point n. The length of the vector $\bar{\tau}'_s = \dfrac{d\bar{\tau}}{ds}$ is the curvature of the curve $\bar{k}$, defined in the form [10]

$$\left|\frac{d\bar{\tau}}{ds}\right| = k, \ \frac{d\bar{\tau}}{ds} = k\bar{n}, \quad (3)$$

where $\bar{n}$ – unit normal vector.

Having differentiated the expression $\bar{u}_\tau = u_\tau \bar{\tau}$ with respect to time, we obtain

$$\bar{a} = \frac{d^2\bar{\rho}}{dt^2} = \frac{du_\tau}{dt}\bar{\tau} + u_\tau \frac{d\bar{\tau}}{ds}\frac{ds}{dt} = \frac{du_\tau}{dt}\bar{\tau} + u_\tau^2 k\bar{n}. \quad (4)$$

Having substituted the acceleration vector value $\bar{a}$ in the formula of Newton's second law, we write

$$\bar{F} = m\frac{du_\tau}{dt}\bar{\tau} + mu_\tau^2 k\bar{n}. \quad (5)$$

This implies that the active force has the tangential component

$$\bar{F}_\tau = F_\tau \bar{\tau} = m\frac{du_\tau}{dt}\bar{\tau} \quad (6)$$

and the normal $\overline{F}_n = F_n \overline{n} = m u_\tau^2 k \overline{n}$ component. (7)

From the formula for the tangential component, we obtain

$$u_\tau F_\tau = u_\tau m \frac{du_\tau}{dt} = \frac{d}{dt}\left(\frac{m u_\tau^2}{2}\right) = \frac{dT}{dt}, \quad (8)$$

so the loss of kinetic energy by the material point for overcoming the motion resistance is

$$T_1 - T_2 = -\int_{t_1}^{t_2} u_\tau F_\tau dt = -\int_{t_1}^{t_2} F_\tau ds. \quad (9)$$

Consequently, the work is produced only by the tangential component.

Preservation of the material point momentum along the curve $dJ = m u_\tau^2 = const$ is equivalent to the absence of the kinetic energy loss, which allows to rewrite (8) and (9) in the form of $\frac{d}{dt}\left(\frac{m u_\tau^2}{2}\right) = u_\tau F_\tau = 0$ and

$$T_1 - T_2 = -\int_{t_1}^{t_2} u_\tau F_\tau dt = -\int_{t_1}^{t_2} F_\tau ds = 0$$

respectively.

The normal component of the force produces trajectory bending with the curvature

$$k = \frac{F_n}{m u_\tau^2}. \quad (10)$$

Derivation of the last formula is produced in accordance with known guides to higher mathematics [10] and applied mechanics [11].

Based on the definition of the curve arc differential (2), dependences are derived to calculate the curvature

$$k = \frac{|y''|}{\left(1 + y'^2\right)^{3/2}}, \quad (11)$$

the radius of curvature $R = \frac{1}{k}$, and inclination angle cosine φ of the tangent to the curve at the point n

$$\cos\varphi = \frac{1}{\sqrt{1 + y'^2}}. \quad (12)$$

We place a layer of circular vortices of constant radius r over the plane curve $y = f(x)$ so that each of the vortices will roll without sliding along the curve $y_m = f(x_m)$ of the equidistant initial curve. Centres of the vortex rotation in a composite motion form the curve $y_b = f(x_b)$, equally spaced in the normals direction to the curves $\eta\eta$ from the curves $y = f(x)$ and $y_m = f(x_m)$ (Fig. 1). Thus, the curve $y_m = f(x_m)$ is the boundary of the motionless liquid, separated from the boundary thread $y = f(x)$ of the jet propagation area by the vortex layer. Scheme of the rolling vortices introduced in hydromechanics by M. D. Millionshchikov [12] showed sufficient efficiency during the simulation of flow around solid curved surfaces in the incompressible liquid [13].

Circulation along the boundary circle of the vortex is $\Gamma = 2\pi r u_\tau = 2\pi C$, where $C = \frac{\Gamma}{2\pi}$ – vortex strength (Fig. 2).

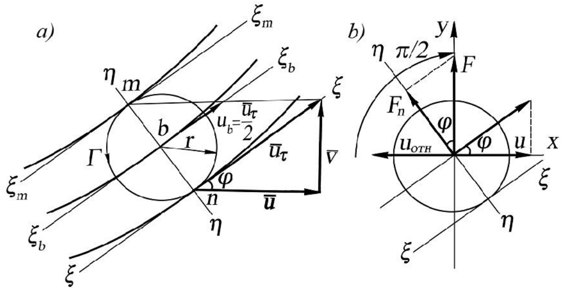


Fig. 2. Scheme of the vortices rolling along the plane curve $y_m = f(x_m)$

Kinematic characteristics of the vortex have the following dimensions: Γ, [m^2/s]; $C = \frac{\Gamma}{2\pi}$, [m^2/s]; u_τ, [m/s]; r, [m]. Dimension of volume values is determined by the conditions of the plane problem and is L^2, [m^2], so the vortex mass is $m = \rho\pi R^2$, [kg/m].

The relative velocity of the vortex centre $u_{отн}$ is modulo equal to projection of tangent to the velocity motion path $\overline{u}_\tau$ at the point n, but of opposite sign

$$\left|\overline{u}\right| = u = \frac{u_\tau}{2} cos\,\varphi. \qquad (13)$$

The magnitude of the force acting on a rolling vortex, in view of (13) is determined according to Zhukovsky's [14] formula

$$F = \rho u\Gamma = 2\pi r\rho u u_\tau = \pi r\rho u_\tau^2 \cos\varphi. \quad (14)$$

In this case, the vortex moves relative to the liquid with the velocity $u_{омн}$, so, according to Zhukovsky's formula, the force $\overline{F}$ is directed along the ordinate axis (Fig. 2).

Projection of the force $\overline{F}$ on the direction of the unit normal $\overline{n}$ is

$$F_n = F\cos\varphi = \pi r\rho u_\tau^2 \cos^2\varphi. \qquad (15)$$

The trajectory curvature (10) after the substitution of the force F_n from the equation (15) takes the form

$$k = \frac{F_n}{m u_\tau^2} = \frac{\pi r\rho u_\tau^2 \cos^2\varphi}{\pi r^2 \rho u_\tau^2} = \frac{\cos^2\varphi}{r}. \quad (16)$$

Using the definition of the curvature (11) and the expression for the inclination angle cosine φ of the tangent to the curve (12), we obtain from the equality (16) the differential second-order equation

$$\frac{y''}{\sqrt{1+y'^2}} = \frac{1}{r}. \qquad (17)$$

Having replaced the derivative $y' = p$ and separated the variables, we find [15]

$$\frac{\frac{dp}{dx}}{\sqrt{1+p^2}} = \frac{1}{r}. \qquad (18)$$

Separation of the variables and integration $\int \frac{dp}{\sqrt{1+p^2}} = \int \frac{dx}{r}$ gives

$$ln\left(p+\sqrt{1+p^2}\right) = \frac{x}{r} + C_1. \qquad (19)$$

With regard to the expression $ln\left(p+\sqrt{1+p^2}\right) = Arsh(p)$ [15] the solution of the equation (19) has the form

$$p = sh\left(\frac{x}{r} + C_1\right). \qquad (20)$$

Since $p = y'$, the last equation is the ordinary differential equation relative to the desired function $y(x)$

$$y' = sh\left(\frac{x}{r} + C_1\right). \qquad (21)$$

After its integration, we obtain the equation of catenary

$$y = r\cdot ch\left(\frac{x}{a} + C_1\right) + C_2. \qquad (22)$$

Under $x = 0$, the derivative $y' = 0$ and $sh(C_1) = 0$, so $C_1 = 0$. Under the condition $y(0) = r$, we obtain $r\cdot ch\left(\frac{0}{r}\right) + C_2 = r$ or $C_2 = 0$.

Thus, the desired boundary equation has the form

$$y = r\cdot ch\left(\frac{x}{r}\right). \qquad (23, a)$$

If under $x = 0$, the derivative $y' = 0$ and simultaneously $y(0) = 0$, then under $C_1 = 0$ we obtain $r\cdot ch\left(\frac{0}{r}\right) + C_2 = 0$ or $C_2 = -r$.

In this case, the boundary will start from the origin, its equation becomes

$$y = r\cdot ch\left(\frac{x}{r}\right) - r. \qquad (23, б)$$

In addition, the derivative (21) takes the form

$$y' = sh\left(\frac{x}{a}\right). \qquad (24)$$

Having replaced the derivative in the expression (12) in accordance with (24), we obtain

$$\cos\varphi = \frac{dx}{ds} = \frac{1}{\sqrt{1+y'^2}} = \frac{1}{\sqrt{1+sh^2\left(\frac{x}{a}\right)}} = \frac{1}{ch\left(\frac{x}{a}\right)} = sec\,h\left(\frac{x}{a}\right) \qquad (25)$$

Since from the definition of the derivative for the tangent to the curve $y = f(x)$ it follows that $y' = tg\varphi$, then

$$sin\,\varphi = y'\,cos\,\varphi = \frac{y'}{\sqrt{1+y'^2}} = \frac{sh\left(\frac{x}{a}\right)}{ch\left(\frac{x}{a}\right)} = th\left(\frac{x}{a}\right). \quad (26)$$

The velocity components u_τ in the projection on the axis of the Cartesian coordinate system x and y are, respectively:

$$u = u_\tau\,cos\,\varphi = u_\tau \frac{1}{ch\left(\frac{x}{a}\right)} = u_\tau\,sec\,h\left(\frac{x}{a}\right); \quad (27)$$

$$v = u_\tau\,sin\,\varphi = u_\tau th\left(\frac{x}{a}\right). \quad (28)$$

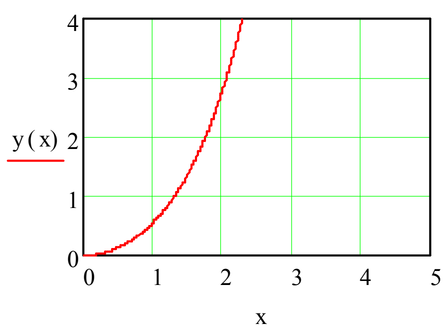
Curve of the vortex boundary equation can be constructed by numerical solution of the differential second-order equation in MathCad (Fig. 3).

The velocity components, derived from the vortex propagation boundary model, u_τ in the projection on the coordinate axes (27) and (28), satisfy the equations of the boundary layer momentum and continuity, as illustrated by the symbolic solution in MathCad (Fig. 4). Direct substitution of the jet boundary equations leads to the satisfaction of the boundary layer equations for both forms of the vortex boundary equations (23, a) and (23, б).

Analytic dependence of the vortex boundary with r = 1

$r := 1$

$y(x) := r \cdot \cosh\left(\frac{x}{r}\right) - r$



Solution of the differential second-order equation

Given

$$\frac{\frac{d^2}{dx^2}y(x)}{\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2}} = \frac{1}{r}$$

boundary conditions

$y(0) = 0.01 \qquad y'(0) = 0$

$Y := \text{Odesolve}(x, 10) \qquad x := 0, 0.01 .. 5$

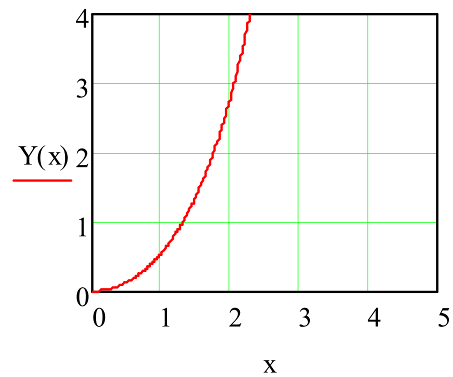


Fig. 3. Comparison of the analytic dependence of the vortex boundary (23, b) with the results of numerical solution of the differential equation (17)

Equationof the vortex boundaryaxial and transverse velocities of the jet

$r \equiv 1 \qquad y \equiv r \cdot \cosh\left(\frac{x}{r}\right) - r \qquad u \equiv \frac{1}{\cosh\left(\frac{x}{r}\right)} \qquad v \equiv \frac{\sinh\left(\frac{x}{r}\right)}{\cosh\left(\frac{x}{r}\right)}$

Continuity and momentum equation

$$\frac{d}{dx}u + \frac{d}{dx}\left(-v \cdot \frac{1}{\frac{d}{dx}y}\right) \to 0$$

$$u \cdot \frac{d}{dx}u + v \cdot \frac{d}{dx}u \cdot \frac{1}{\frac{d}{dx}y} - \frac{d}{dx}\left(\frac{d}{dx}u \cdot \frac{1}{\frac{d}{dx}y}\right) \to 0$$

Fig. 4. Substitution of the jet velocity components in the boundary layer equations

From the physical formulation of the problem it follows that in the initial section under $x = 0$ $\varphi = 0$ and $cos\,\varphi = 1$. Consequently, the vortex strength can be defined as

$$C_o = \frac{\Gamma_0}{2\pi} = \frac{2\pi r u_0}{2\pi} = r u_0, \quad (29)$$

A. Ya. Milovich reports [16] that application of the plain vortex layer theory to the flow around a circular cylinder based on the assumption of a constant vortex strength C led to insolubility of the problem in a finite form. In this regard, the problem was solved by introducing the alternating voltage proportional to the sine of the polar angle of the incoming flow.

In this case, the vortex strength along the curve $y = f(x)$ can be accepted inversely proportional to the cosine of the angle φ of the incoming flow, directed along the jet axis.

In this formulation, the equation of the jet boundary curve becomes

$$y = \frac{\Gamma_0}{2\pi u_0} \cdot ch\left(\frac{2\pi u_0 x}{\Gamma_0}\right) = \frac{C_0}{u_0} \cdot ch\left(\frac{u_0 x}{C_0}\right) \quad (30)$$

and vanishes under $x = 0$.

In the following analysis, the dimensional geometric and kinematic parameters of the jet are marked by subscript "1", the dimensionless functions of one independent variable ζ – by lower-case subscript, and the dimensionless functions of two variables x and y – by upper-case subscript.

2. The jet, flowing from the nozzle of the final diameter with limited initial velocity for the motion area remote from the jet source, in all mentioned calculation methods [1, 2, 4, 5, 6] is modelled by the source extremely thin in cross-section with a finite momentum J [2].

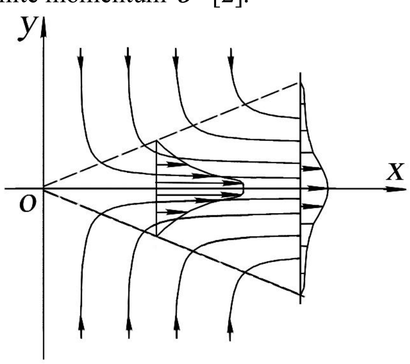


Fig. 5. Scheme of the plane jet

The thin plane jet has the source at the beginning of the Cartesian coordinate system (Fig. 5). The axis x_1 lies in the plane of symmetry, and the axis y_1 – perpendicular to this plane. The passing jet mass $\rho u_1 dy_1$ creates the elementary momentum $\rho u_1 dy_1 u_1$ through the segment dy_1, ***so*** the total jet momentum through the straight line parallel to the axis y_1, is represented as

$$J = \int_{-\infty}^{\infty} \rho u_1^2 dy_1 , \quad (31)$$

where ρ – density of the liquid; u_1 – longitudinal velocity of the jet.

Flow of the momentum caused by the jet is assumed to be given with a known velocity distribution in the initial section.

Dimensional form of the boundary layer momentum equation has the form

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} = U_1 \frac{\partial U_1}{\partial x_1} + \nu \frac{\partial^2 u_1}{\partial y_1^2} . \quad (32)$$

In case of the plane submerged jet flowing into the motionless liquid, the flow velocity at infinity is zero $U_1 = 0$, and the pressure in the space surrounding the jet $p = const$, so the equation (32) takes the form

$$u_1 \frac{\partial u_1}{\partial x_1} + v \frac{\partial u_1}{\partial y_1} = \nu \frac{\partial^2 u_1}{\partial y_1^2} . \quad (33)$$

The momentum equation (33) is supplemented by the continuity equation

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0 . \quad (34)$$

On the line of symmetry, the longitudinal component of the velocity vector must be the greatest, and the transverse component must vanish. Thus, for the line $y_1 = 0$ we have the following boundary conditions:

$$\frac{\partial u_1}{\partial y_1} = 0 \text{ and } v_1 = 0 \text{ under } y_1 = 0 . \quad (35)$$

Assuming that the jet propagates into infinity, we write the additional condition

$$u_1 \to 0 \text{ under } y_1 \to \infty . \quad (36)$$

3. ***Zero*** boundary conditions (35) and (36) in y_1 direction and the absence of the boundary condition in x_1 direction lead to ***the physically unjustified*** solution $u_1 = 0$, $v_1 = 0$ in the entire flow area. To eliminate contradictions, L. G. Loitsyansky [5] writes the boundary layer momentum equation, using the continuity equation, as

$$\frac{\partial \left(u_1^2\right)}{\partial x_1} + \frac{\partial \left(u_1 v_1\right)}{\partial y_1} = \nu \frac{\partial^2 u_1}{\partial y_1^2} , \quad (37)$$

and then integrates both its parts in y_1 direction within $-\infty$ to ∞

$$\frac{d}{dx_1} \int_{-\infty}^{\infty} u_1^2 dy_1 + u_1 v_1 \Big|_{-\infty}^{\infty} = \nu \frac{\partial u}{\partial y}\Big|_{-\infty}^{\infty} . \quad (38)$$

It is assumed that the integral exists in the infinite limits of the first summand and that change of the order of differentiation and integration operations is possible.

Substituting in the left side of the equality (38) vanishes as a result of passage to the limit. The right side of the equality (38) also vanishes on the assumption of smoothness of the transition ($\partial u_1 / \partial y_1 \to 0$ under $y \to \pm\infty$), so $\frac{d}{dx_1}\int_{-\infty}^{\infty} \rho u_1^2 dy_1 = 0$ is followed by

$$J = \int_{-\infty}^{\infty} \rho u_1^2 dy_1 = \rho u_0^2 = const. \quad (39)$$

The equation (39) reflects the theorem of change of the liquid momentum in the area between two lines parallel to the axis y_1, provided the constant pressure, and means that the total ***momentum***, carried through the jet cross-section, ***is the same for all sections.*** Assigning of the characteristic constant J makes the problem of the jet propagation concrete.

Thus, the study of the liquid motion in the plane jet is reduced to solving the equations (33) and (34) under the boundary conditions (35) and (36) and the integral invariant (39).

However, by the solution of the differential boundary layer equations, the integral condition (39) is not used directly by the authors studying the plane jet [1, 4, 5, 6], but a similarity transformation is applied instead of the condition (39).

4. In our opinion, some of the assumptions made by L. G. Loitsiansky can be improved on the basis of the generalized function theory, in particular, as the result of the useof the delta function.

Delta function (Dirac delta function) is defined as a generalized singular function, i.e. as a linear continuous functional on the functional space of the main functions [11, 17].

Formally, delta function is defined by the relation

$$\int_{\Re^n} \delta(x-a) f(x) dx = f(x) \quad (40)$$

for any continuous function $f(x)$.

For one-variable delta-function, the following equalities are true:

$$\delta(x) = 0,\ \forall \neq 0;\ \int_{-\infty}^{\infty} \delta(x) dx = 1. \quad (41)$$

Dirac delta function is non-zero only at the point $x = 0$, where it becomes infinite, so that its integral in each neighborhood $x = 0$ is equal to one.

Continuity means that if $f_n \to \varphi$, then $\langle \delta; f_n \rangle \to \langle \delta; \varphi \rangle$. Where $\langle \delta; f \rangle$ – the functional value of the function f. For convenience, it is written as a formal equality

$$\langle \delta; f \rangle = \int_{-\infty}^{\infty} \delta(x) f(x) dx. \quad (42)$$

Let the condition of the delta function normalization be satisfied $\int_{-\infty}^{\infty} \delta(x) dx = 1$, then the sequence $f_n(x) = nf(nx)$ converges weakly to delta function.

Using delta function, the spatial density of the physical quantity concentrated at the point can be determined. For example, the density of the point unit mass located at the point a of Euclidean space $\Re^n$ is recorded using delta function in the form $\delta(x-a)$.

Satisfaction of condition of the total jet momentum constancy (39) is achieved not only with the infinite value of the initial jet velocity u_0 and the final mass density ρ in the initial section, but with a finite value of the initial velocity and infinite mass density.

Due to the fact that the longitudinal component of the total momentum of the jet flowing from the point source at the origin J should not depend on the coordinate of the longitudinal axis of the jet x with constant pressure p =const outside the jet boundaries in the liquid resting at infinity, the average density ρ_ε is a function only of the coordinate y.

We define the density created by the material mass point m, located at the origin, following V. S. Vladimirov [17].

Having distributed the density inside the two-dimensional sphere U_ε, we find the value of the average density

$$\rho_\varepsilon(y) = \begin{cases} \frac{m}{\pi \varepsilon^2}, & |y| \le \varepsilon; \\ 0, & |y| > \varepsilon. \end{cases} \quad (43)$$

Pointwise limit of the sequence of average densities $\rho_\varepsilon(y)$ under $\varepsilon \to +0$ is equal to

$$\rho(y) = \lim_{\varepsilon \to 0} \rho_\varepsilon(y) = \begin{cases} +\infty, & y = 0; \\ 0, & y \neq 0. \end{cases} \quad (44)$$

On the other hand, the area integral S of a plane figure is equal to mass contained within its boundaries, or

$$\int_S \rho(y)dy = \begin{cases} m, & 0 \in S; \\ 0, & 0 \overline{\in} S. \end{cases} \quad (45)$$

It follows from (44) that the integral (45) must be zero. Consequently, the pointwise limit of the sequence $\rho_\varepsilon(y)$ under $\varepsilon \to 0$ does not determine the density $\rho(y)$.

The generalized function theory is applied to specify the representation of the density propagation created by the jet flowing from the point source at the origin, and, on this basis, to determine the physical meaning of the condition of the total jet momentum constancy (39).

We introduce the dimensionless density δ from the relation $\delta(y) = \frac{\rho(y)}{m}$, where m – dimensional mass of the liquid, kg.

We calculate the weak limit of the functions sequence $\rho_\varepsilon(y)$ under $\varepsilon \to 0$. Moreover, for every continuous function $f(x)$ we find the limit of the numerical sequence $\int \rho_\varepsilon(y)\varphi(y)dy$ under $\varepsilon \to 0$.

We show that

$$\lim_{\varepsilon \to 0} \int \rho_\varepsilon(y) f(y) dy = f(0). \quad (46)$$

From the continuity of the function $f(y)$ for any number ζ >0 there is ε_0 >0 such that $|f(y) - f(0)| < \zeta$ under $|y| < \varepsilon_0$.

Therefore for all $\varepsilon \leq \varepsilon_0$, we find

$$\left|\int \rho_\varepsilon(y) f(y) dy - f(0)\right| =$$

$$= \frac{m}{\pi\varepsilon^2}\left|\int_{|y|<\varepsilon} [f(y) - f(0)]dy\right| \leq \frac{m}{\pi\varepsilon^2}\int_{|y|<\varepsilon} |f(y) - f(0)|dy < \zeta \frac{m}{\pi\varepsilon^2}\int_{|y|<\varepsilon} dy = \zeta,$$

which proves the statement (46).

Consequently, the weak limit of sequence of the functions $\rho_\varepsilon(y)$ under $\varepsilon \to 0$ is the functional $f(0)$, defined by the function value at the point $y = 0$. The specified functional $f(0)$ is taken for the density determination $\rho(y) = m\delta(y)$, where $\delta(y)$ – Dirac delta function.

For any continuous function $f(y)$, the limit relation

$$\int \rho_\varepsilon(y) f(y) dy \to (\delta, f) \text{ is right under } \varepsilon \to 0, \quad (47)$$

where (δ, f) corresponds to the number $f(0)$ – the value of the functional action $\delta(y)$ on the function $f(y)$.

In case of mass concentration m at the point $x = 0$, the density will be equal to $m\delta(y)$. If the mass m is concentrated at the point y_0, then the corresponding density is equal to $m\delta[y - y_0]$, therefore $(m\delta(x - x_0), f) = mf(x_0)$.

In the figure 6 there is MathCad document, in which delta function properties are software implemented. To denote δ-function, the identifier Δ is used in the listing.

Determination of the momentum under the given vortex boundary equation $y = r \cdot ch\left(\frac{x}{r}\right)$ is shown in the figure 7.

5. If, based on the similarity and dimension theory, we express the longitudinal coordinate x_1 in the length scale L, and the transverse coordinate y_1 – in scale $Y = \frac{L}{\sqrt{Re}} = \sqrt{\frac{\nu L}{U}}$, where $Re = \frac{UL}{\nu}$ Reynolds number, U – scale of the longitudinal velocity u_1, ν – kinematic viscosity coefficient, then for the jet-source the boundary layer equations (33, 34) can be reduced to the ordinary differential equation. In this case, the only specified dimensional quantity is the jet momentum and consequently the length L and velocity U scales are related by

$$J = \alpha \frac{\rho U^2 L}{\sqrt{Re}}, \quad (48)$$

where α – unspecified dimensionless number. Using this relation, the length scale can be expressed through the velocity scale

$$L = \frac{J^2}{\alpha^2 \rho^2 \nu U^3}, \qquad (49)$$

and the velocity scale through the length scale, respectively, as

$$U = \sqrt[3]{\frac{J^2}{\alpha^2 \rho^2 \nu L}}. \qquad (50)$$

With this choice of the length scale, the formulae of transition from dimensionless coordinates and velocities to dimensional ones are

$$\left.\begin{aligned} x_1 &= Lx = \frac{J^2}{\alpha^2 \rho^2 \nu U^3} x, \\ y_1 &= Yy = \frac{L}{\sqrt{Re}} y = \frac{J}{\alpha \rho U^2} y, \\ u_1 &= Uu, \\ v_1 &= \frac{U}{\sqrt{Re}} v = \frac{\alpha \rho \nu U^2}{J} v. \end{aligned}\right\} \qquad (51)$$

The boundary layer equations (33, 34) during the transition to dimensionless coordinates and velocities take the form

$$\left.\begin{aligned} u_1 \frac{\partial u_1}{\partial x_1} + v \frac{\partial u_1}{\partial y_1} &= \nu \frac{\partial^2 u_1}{\partial y_1^2}, \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} &= 0. \end{aligned}\right\} \qquad (52)$$

From the definition of the longitudinal velocity as the derivative of the flow function in the transverse coordinate $u_1 = \frac{\partial \psi_1}{\partial y_1}$ and the given scale of Y the transverse coordinate y_1, we can conclude that the dimension of the flow function is defined by the dependence

$$\Psi_m = UY = U\sqrt{\frac{\nu L}{U}} = \sqrt{\nu U L}. \qquad (53)$$

Functional of delta function ?(x) equates a function f(y) to the value of the function when the argument y=0:

$$\int_{-\infty}^{\infty} \Delta(y) \cdot f(y)\, dy \to f(0)$$

The function $f\left(\frac{y}{x^{2/3}}\right)$ gets a similar importance

$$\int_{-\infty}^{\infty} \Delta(y) \cdot f\left(\frac{y}{x^{2/3}}\right) dy \to f(0)$$

Filtering property of delta function for the function $f(x)$ is defined by the equality

$$\int_{-\infty}^{\infty} \Delta(x - x_0) \cdot f(x)\, dx \to f(x_0)$$

Taking into account the property of delta function $\delta(x-y)=0$ with $x \neq y$ the value of this integral does not change, if the function $f(x)$ is replaced by the function $\tilde{f}(x)$, which is equal to $f(x)$ at the point $x = y$, and at the remaining points has arbitrary values. For example, we select $\tilde{f}(x) = f(y)$=const, then factor $f(y)$ outside the integral sign, and using the condition in the definition of the delta function $\int_{-\infty}^{\infty} \delta(x) dx = 1$, obtain the equality

$$\int_{-\infty}^{\infty} \Delta(x - ó) \cdot f(x)\, dx \to f(ó) \quad .$$

Similarly, replacing the variables, we find

$$\int_{-\infty}^{\infty} \Delta(y - x) \cdot f(y)\, dy \to f(x) \quad .$$

As a result of the density determination, we obtain

$$\int_{-\infty}^{\infty} \rho \cdot \Delta(y) \cdot f(x)\, dy \to \rho \cdot f(x) \qquad \int_{-\infty}^{\infty} \rho \cdot \Delta(y) \cdot f\left(\frac{y}{x^{2/3}}\right)^2 dy \to \rho \cdot f(0)^2$$

$$\int_{-\infty}^{\infty} \rho \cdot \Delta(y) \cdot \left(\frac{u_0}{\cosh(y)}\right)^2 dy \to \rho \cdot u_0^2 \qquad \int_{-\infty}^{\infty} \rho \cdot \Delta\left(\zeta - \frac{y}{x^{2/3}}\right) \cdot f(\zeta)\, d\zeta \to \rho \cdot f\left(\frac{y}{x^{2/3}}\right)$$

Fig. 6. Properties of delta function

The momentum with the unit impulse function, which determines the density of mass concentrated at the point $y=0$

$$\int_{-\infty}^{\infty} \rho \cdot \Delta(y) \cdot u_0^2 \cdot \frac{1}{\cosh(y)^2}\, dy \to \rho \cdot u_0^2$$

The momentum for the given function of the vortex boundary

$$\int_{0}^{\infty} \frac{\rho \cdot u_0^2}{r \cdot \cosh\left(\frac{x}{r}\right)^2}\, dx \to \rho \cdot u_0^2 \cdot \lim_{x \to \infty} - \tanh\left(\frac{x}{r}\right) \qquad \lim_{x \to \infty} \tanh\left(\frac{x}{r}\right) = 1$$

$$\int_{0}^{\infty} x^{\frac{1}{3}} \cdot \frac{\rho \cdot u_0^2}{\cosh\left(\frac{y}{x^{2/3}}\right)^2}\, dy \to \lim_{y \to \infty} - \left(\rho \cdot u_0^2 \cdot x \cdot \tanh\left(\frac{y}{x^{2/3}}\right)\right)$$

Fig. 7. The momentum when using the vortex boundary equation $y = r \cdot ch\left(\frac{x}{r}\right)$

Therefore, we rewrite the integral condition (39) in the form

$$\int_{-\infty}^{\infty} \rho \left(\frac{\partial \psi_1}{\partial y_1}\right)^2 dy_1 = J \qquad (54)$$

or, selecting the dimensionless components,

$$\frac{\rho \Psi_m^2}{Y} \int_{-\infty}^{\infty} \left(\frac{\partial \Psi}{\partial y} \right)^2 dy = J \,. \qquad (55)$$

Having assumed that $\Psi_m^2 = \frac{J}{\rho}\sqrt{\frac{\nu L}{V}}$, we construct the condition (39, 54) in a dimensionless form

$$\int_{-\infty}^{\infty} \left(\frac{\partial \Psi}{\partial y} \right)^2 dy = 1 \,, \qquad (56)$$

where Ψ – dimensionless flow function depending on two variables x and y, $\Psi(x, y)$.

Having compared the expressions for the length scales L, velocity scales U and the momentum J, we find another expression for the scale of the flow function

$$\Psi_m = \frac{J}{\rho U} \,. \qquad (57)$$

6. Transition from the boundary layer equations to the partial differential equation for the flow function is proved by N. A. Slezkin [6] on the basis of the fact that by the very nature of the problem, the velocity scale U should not be included in the solution result.

It follows directly from (51) that the transition from dimensional coordinates to dimensionless ones is performed according to the formulae

$$x = \frac{1}{L} x_1 = \frac{\alpha^2 \rho^2 \nu U^3}{J^2} x_1 \; \textit{and}$$

$$y = \frac{\sqrt{Re}}{L} y_1 = \frac{\alpha \rho U^2}{J} y_1 . \qquad \textit{(58)}$$

Derivative of the function $\Psi = \Psi[x_1(U), y_1(U)]$ using the parameter U has the form

$$\frac{d\Psi}{dU} = 3 \frac{\partial \Psi}{\partial x} \frac{\alpha^2 \rho^2 \nu U^2}{J^2} x_1 + 2 \frac{\partial \Psi}{\partial y} \frac{\alpha \rho U}{J} y_1 \,. \quad (59)$$

If we assume

$$\Psi(x, y) = \Psi\left(\frac{\alpha^2 \rho^2 \nu}{J^2} U^3 x_1, \frac{\alpha \rho U^2}{J} y_1 \right), \quad (60)$$

the requirement of independence of the dimensional velocity on the scale U gives

$$\frac{d}{dU}\left[U \Psi\left(\frac{\alpha^2 \rho^2 \nu U^3}{J^2} x_1, \frac{\alpha \rho U^2}{J} y_1 \right) \right] = 0 \,. \quad (61)$$

Having differentiated, we find

$$\frac{d(\Psi \cdot U)}{dU} = \Psi + U \frac{d\Psi}{dU} = \Psi + 3 \frac{\partial \Psi}{\partial x} \frac{\alpha^2 \rho^2 \nu U^3}{J^2} x_1 + 2 \frac{\partial \Psi}{\partial y} \frac{\alpha \rho U^2}{J} y_1 = 0 \,. \quad (62)$$

Using the selected length scales (51), we obtain the equation

$$\Psi + 3 \frac{\partial \Psi}{\partial x} x + 2 \frac{\partial \Psi}{\partial y} y = 0 \,. \qquad (63)$$

7. Having introduced the flow function dependence, in consideration of broadening the jet half-width, resulting from the vortex jet boundary theory, along the direction of its motion in the form $y = r \cdot ch\left(\frac{x}{r}\right)$ and in consideration of the expression of the transition from dimensionless coordinates to dimensional ones (51), we write

$$x_1 = Lx = \frac{J^2}{\alpha^2 \rho^2 \nu U^3} x,$$

$$y_1 = \frac{L}{\sqrt{Re}} y = \frac{J}{\alpha \rho U^2} y = \frac{J}{\alpha \rho U^2} r \cdot ch\left(\frac{x}{r}\right). \qquad (64)$$

In consideration of the jet boundary function differential

$$dy = sh\left(\frac{x}{r}\right) dx \qquad (65)$$

and the expression of the transition from the dimensional velocities to the dimensionless ones

$$\Psi(x, y) = \Psi\left(\frac{\alpha^2 \rho^2 \nu U^3}{J^2} x_1, \frac{\alpha \rho U^2}{J} y_1 \right),$$

we perform differentiation of the function $u = f(x, y)$ on the velocity scale U

$$\frac{d\Psi}{dU} = \Psi + 3 \frac{\partial \Psi}{\partial x} \frac{\alpha^2 \rho^2 \nu U^3}{J^2} x_1 + 2 \frac{\partial \Psi}{\partial y} \frac{\alpha \rho U^2}{J} y_1 =$$

$$= \Psi + 3 \frac{\partial \Psi}{\partial x} \frac{\alpha^2 \rho^2 \nu U^3}{J^2} x_1 + 2r \frac{\partial \Psi}{\partial x} \frac{\alpha \rho U^2}{J} \frac{ch\left(\frac{x}{r}\right)}{sh\left(\frac{x}{r}\right)} .$$

The requirement of independence of the dimensional velocity on the scale U in this case gives

$$\frac{d}{dU}[U \Psi(x, y)] = \Psi + 3x \frac{\partial \Psi}{\partial x} + 2r \cdot cth\left(\frac{x}{r}\right) \frac{\partial \Psi}{\partial x} = 0 \,. \quad (66)$$

Hence, the ordinary differential equation follows

$$\Psi + \left(3x + 2r \cdot cth\left(\frac{x}{r}\right)\right)\frac{d\Psi}{dx} = 0. \quad (67)$$

Further solution is made in MathCad (Fig. 8).

$r := 1$

Given

$\psi_1(x_1) + \left(3 \cdot x_1 + 2 \cdot \coth\left(\frac{x_1}{r}\right)\right) \cdot \frac{d}{dx_1}\psi_1(x_1) = 0 \qquad \psi_1(0.001) = r$

$\psi_1 :=$ Odesolve$(x_1, 10)$

$x_1 := 0, 0.01 .. 10 \qquad u_1(x_1) := \psi_1(x_1)$

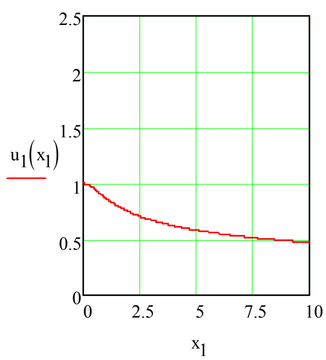


Fig. 8. Calculation of the longitudinal velocity of the jet $u(\zeta)$ as a function of the dimensionless independent variable ζ under the jet boundary variation law

8. N. A. Slezkin [6] solves the partial differential equation (63), using the method of characteristics,

$$\frac{d\Psi}{-\Psi'} = \frac{dx}{3x} = \frac{dy}{2y}.$$

Integrals of these characteristics equations are the functions:

$$\Psi x^{1/3} = C_1, \; yx^{-2/3} = C_2,$$

and the solution of the equation (63) is presented in the form

$$u_1 = \Psi(x, y) = x^{-1/3}\psi\left(yx^{-2/3}\right). \quad (68)$$

Thus, a new independent dimensionless variable is

$$\zeta = yx^{-2/3}, \quad (69)$$

and for this variable we have:

$$\frac{d\zeta}{dy} = x^{-2/3},$$

$$\frac{d\zeta}{dx} = -\frac{2}{3}yx^{-5/3} = -\frac{2}{3}x^{-1}\zeta. \quad (70)$$

9. Introduction of the dimensional flow function, m²/s, $\psi_1(x_1, y_1)$ from the conditions $u_1 = \frac{\partial \psi_1}{\partial y_1}$ and $v_1 = -\frac{\partial \psi_1}{\partial x_1}$ allows to combine the equations of the plane jet momentum and continuity into a single third-order equation

$$\frac{\partial \psi_1}{\partial y_1}\frac{\partial^2 \psi_1}{\partial x_1 \partial y_1} - \frac{\partial \psi_1}{\partial x_1}\frac{\partial^2 \psi_1}{\partial y_1^2} = \nu \frac{\partial^3 \psi_1}{\partial y_1^3}. \quad (71)$$

with the boundary conditions

$$\frac{\partial^2 \psi_1}{\partial y_1^2} = 0, \; \psi_1 = 0 \text{ under } y_1 = 0,$$

$$\frac{\partial \psi_1}{\partial y_1} \to 0 \text{ under } y \to \pm\infty. \quad (72)$$

We write the general solution in the dimensionless coordinates $\Psi = \Psi(x, y)$ and the dimensional coordinates

$$\psi_1 = \Psi_m \Psi\left(\frac{x_1}{L}, y_1\sqrt{\frac{U}{\nu L}}\right) = \frac{J}{\rho U}\Psi\left(\frac{\rho^2 \nu U^3}{J^2}x_1, \frac{\rho U^2}{JK}y_1\right)$$

In order that the right side of this expression may not depend on U, the general form of the solution should be as follows:

$$\Psi = x^{1/3}\psi\left(y / x^{2/3}\right). \quad (73)$$

Indeed, this is

$$\psi_1 = \frac{J}{\rho U}\frac{\sqrt[3]{\rho^2 \nu U^3 x_1}}{J^{2/3}}\psi\left(\frac{\rho U^2}{J}\frac{J^{4/3}}{\rho^{4/3}\nu^{2/3}U^2}\frac{y_1}{x_1^{2/3}}\right) = \sqrt{\frac{\nu J x_1}{\rho}}\psi\left(\sqrt[3]{\frac{J}{\rho \nu^2}}\frac{y_1}{x_1^{2/3}}\right)$$

Having assumed

$$\frac{y}{x^{2/3}} = \sqrt[3]{\frac{J}{\rho \nu^2}}\frac{y_1}{x_1^{2/3}} = \zeta,$$

we substitute the expression Ψ from (73), rewritten in the form

$$\Psi = x^{1/3}\psi(\zeta), \quad (74)$$

in (71), converted for the dimensionless coordinates

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial^3 \Psi}{\partial y^3}, \quad (75)$$

where ψ – dimensionless flow function depending on the variable ζ, $\psi(\zeta)$.

Preliminarily we calculate (primes denote derivatives of ζ)

$$u = \frac{\partial \Psi}{\partial y} = x^{-1/3}\psi',$$

$$v = -\frac{\partial \Psi}{\partial x} = \frac{1}{3}x^{-1/3}\left[2\zeta\psi' - \psi\right],$$

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \Psi}{\partial x \partial y} = -\frac{1}{3}x^{-4/3}\left[\psi' + 2\zeta\psi''\right], \quad \textbf{(76)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 \Psi}{\partial y^2} = x^{-1}\psi'',$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 \Psi}{\partial y^3} = x^{-5/3}\psi'''.$$

When substituting the equalities (76) in the first equation (52), we arrive to the integration of ordinary differential third-order equation

$$\psi''' + \frac{1}{3}\left(\psi'^2 + \psi\psi''\right) = 0 \qquad (77)$$

under the boundary conditions

$$\psi = 0,\ \psi'' = 0, \text{ under } \zeta = 0, \quad (78)$$

$$\psi' \to 0 \text{ under } \eta \to \pm\infty$$

and the integral condition (5), which can be written as

$$\int_{-\infty}^{\infty} \psi'^2(\zeta) d\zeta = 1. \qquad (79)$$

10. Derivation of the ordinary differential third-order equation for the flow function from the partial differential equations can be performed on the basis of invariant transformation. Moreover, in contrast to the use of the momentum and continuity equations for invariant transformations, as it is produced by L. A. Vulis [1], or is given in the guides for the solution of differential equations [8, 18], the following direct integral transformation of the partial third-order equation for the flow function in the ordinary differential equation is given below.

On the conditions $u_1 = \frac{\partial \psi_1}{\partial y_1}$ and $v_1 = -\frac{\partial \psi_1}{\partial x_1}$, we write the equations of momentum and continuity of the plane laminar jet (33, 34) in the form of the third-order equation for the flow function $\psi(x_1, y_1)$

$$\frac{\partial \psi_1}{\partial y_1}\frac{\partial^2 \psi_1}{\partial x_1 \partial y_1} - \frac{\partial \psi_1}{\partial x_1}\frac{\partial^2 \psi_1}{\partial {y_1}^2} = \nu \frac{\partial^3 \psi_1}{\partial {y_1}^3}.$$

Having made the similarity transformation [8]

$$x_1 = c \cdot \bar{x}_1,\ y_1 = c^k \cdot \bar{y}_1,\ \psi_1 = c^l \cdot \bar{\psi}_1, \quad (80)$$

we obtain

$$\frac{c^l}{c^k}\frac{c^l}{c \cdot c^k} \cdot \frac{\partial \bar{\psi}_1}{\partial \bar{y}_1}\frac{\partial^2 \bar{\psi}_1}{\partial \bar{x}_1 \partial \bar{y}_1} - \frac{c^l}{c}\frac{c^l}{c^{2k}} \cdot \frac{\partial \bar{\psi}_1}{\partial \bar{x}_1}\frac{\partial^2 \bar{\psi}_1}{\partial {\bar{y}_1}^2} = \nu \frac{c^l}{c^{3k}} \cdot \frac{\partial^3 \bar{\psi}_1}{\partial {\bar{y}_1}^3}$$

or

$$c^{2l-2k-1} \cdot \frac{\partial \bar{\psi}_1}{\partial \bar{y}_1}\frac{\partial^2 \bar{\psi}_1}{\partial \bar{x}_1 \partial \bar{y}_1} - c^{2l-2k-1} \cdot \frac{\partial \bar{\psi}_1}{\partial \bar{x}_1}\frac{\partial^2 \bar{\psi}_1}{\partial \bar{y}_1^2} = \nu c^{l-3k} \cdot \frac{\partial^3 \bar{\psi}_1}{\partial \bar{y}_1^3} \quad (81)$$

Having divided all members of the equation $c^{2l-2k-1}$ we obtain $l - 3k - 2l + 2k + 1 = 0$. Consequently, for the invariance of the initial equation and the equation resulting from the similarity transformation, the equality $l = 1 - k$ must be performed.

Substitution of the similarity transformations gives

$$x_1 = c \cdot \bar{x}_1,\ y_1 = c^k \cdot \bar{y}_1,\ \psi_1 = c^{1-k} \cdot \bar{\psi}_1. \quad (82)$$

Having eliminated the parameter c ($c = \frac{x_1}{\bar{x}_1}$), we find:

$$y_1 = \left(\frac{x_1}{\bar{x}_1}\right)^k \bar{y}_1 \Rightarrow$$

$$y_1 x_1^{-k} = \bar{y}_1 \bar{x}_1^{-k} = I_1 = \zeta; \qquad (83)$$

$$\psi_1 = \left(\frac{x_1}{\bar{x}_1}\right)^{1-k} \bar{\psi}_1 \Rightarrow$$

$$\psi_1 x_1^{k-1} = \bar{\psi}_1 \bar{x}_1^{k-1} = I_2. \qquad (84)$$

The solution is sought in the form $I_2 = \psi(I_1)$, i.e. $\psi(\zeta) = x_1^{k-1}$.

Consequently,

$\psi_1(x_1, y_1) = x_1^{1-k}\psi_1(\zeta)$ and $\zeta = y_1 x_1^{-k}$, where k – arbitrary constant.

Having calculated the derivatives of the flow function $\psi(x_1, y_1)$ by the flow function variables x_1 and y_1, taking into account the expressions

$\frac{d\zeta}{dx_1} = -kx_1^{-1}\zeta$ and $\frac{d\zeta}{dy_1} = x_1^{-k}$, we find sequentially:

$$\frac{\partial\psi_1}{\partial x_1} = (1-k)x_1^{-k}\psi - kx_1^{-k}\zeta\psi',$$

$$\frac{\partial\psi_1}{\partial y_1} = x_1^{1-2k}\psi', \quad \frac{\partial^2\psi_1}{\partial y_1^2} = x_1^{1-3k}\psi'',$$

$$\frac{\partial^3\psi_1}{\partial y_1^3} = x_1^{1-4k}\psi''',$$

$$\frac{\partial^2\psi_1}{\partial x_1\partial y_1} = (1-2k)x_1^{-2k}\psi' - kx_1^{-2k}\zeta\psi''. \quad (85)$$

By substituting the found derivative values in the differential third-order equation for the flow function, we obtain

$$(2k-1)\psi'^2 + (1-k)\psi\psi'' + \psi''' = 0. \quad (86)$$

When the condition $2k-1=\frac{1}{3}$ is fulfilled, we obtain for the arbitrary constant $k=\frac{2}{3}$.

After substituting this value in the differential equation for the flow function, we find

$$\psi''' + \frac{1}{3}\left(\psi'^2 + \psi\psi''\right) = 0. \quad (87)$$

This special case of the equation coincides with the equations obtained by L. G. Loitsiansky [5] and N. A. Slezkin [6] as a result of transformation of the plane laminar boundary layer equations into the dimensionless form.

The boundary conditions for determination of the flow function take the form:

$$\psi = 0,\ \psi'' = 0 \text{ under } \zeta = 0, \quad (88)$$

$$\psi' \to 0 \text{ under } \eta \to \pm\infty$$

and the integral condition (39), which can be written as

$$\int_{-\infty}^{\infty} \psi'^2(\zeta)d\zeta = 1. \quad (89)$$

Symbolic substitution of the flow function $\psi(\zeta) = a \cdot tg\left(\frac{\zeta}{a}\right)$ in the differential boundary layer equation in MathCad shows (Fig. 9) that the equation is satisfied identically.

Differential third-order equation for the flow function

$$\frac{d}{dy}\psi(x,y)\cdot\frac{d}{dx}\frac{d}{dy}\psi(x,y) - \frac{d}{dx}\psi(x,y)\frac{d^2}{dy^2}\psi(x,y) = \frac{d^3}{dy^3}\psi(x,y)$$

Substitution of the flow function for the given numerical values of the parameters a and k

$$r := \sqrt{6} \qquad \psi(\zeta) := r\cdot\tanh\left(\frac{\zeta}{r}\right) \qquad k := \frac{2}{3}$$

Substitution of the flow function in the differential boundary layer equation in an invariant form

$$(2\cdot k - 1)\cdot\left(\frac{d}{d\zeta}\psi(\zeta)\right)^2 + (1-k)\cdot\psi(\zeta)\frac{d^2}{d\zeta^2}\psi(\zeta) + \frac{d^3}{d\zeta^3}\psi(\zeta) \to 0$$

Fig. 9. Symbolic substitution of the flow function $\psi(\zeta) = a \cdot tg\left(\frac{\zeta}{a}\right)$ in the differential third-order equation for the flow function

Using the invariant transformation shows that under the arbitrary constant values k, other numerical factors can be obtained in the momentum equation.

11. Directly we find the first integral of the equation (77, 87) in the form

$$\psi'' + \frac{1}{3}\psi\psi' = C. \quad (90)$$

On the basis of the boundary conditions (78, 88), we assume the constant C equal to zero $C=0$.

Having performed a further integration of the equation (90), we find:

$$\psi' + \frac{1}{6}\psi^2 = D. \quad (91)$$

On the basis of the equations (76, 85) and (51), the dimensional longitudinal velocity component is

$$u_1 = Uu = (\nu x_1)^{-1/3}\left(\frac{J}{\rho\alpha}\right)^{2/3}\psi'(\zeta). \quad (92)$$

We choose the constant α so that

$$\psi'(0) = 1. \quad (93)$$

Under this and the first condition (78, 88), the integration constant D must be equal to one. Thus, we obtain for the function $\psi(\zeta)$ the following first-order equation:

$$\psi' + \frac{1}{6}\psi^2 = 1. \quad (94)$$

Having solved this equation by separation of variables, and using the first condition (78, 88), we obtain the final expression for the required function in the form

$$\psi(\xi)=\sqrt{6}\cdot th\left(\frac{\zeta}{\sqrt{6}}\right). \qquad \textbf{(95)}$$

On the basis of (95) and the first equalities (76), we obtain the following expressions for the dimensionless velocities:

$$\left.\begin{aligned} u &= x^{-1/3}\frac{1}{ch^2\frac{\zeta}{\sqrt{6}}}, \\ v &= \frac{1}{3}x^{-2/3}\left[\frac{2\eta}{ch^2\frac{\zeta}{\sqrt{6}}}-\sqrt{6}\cdot th\frac{\zeta}{\sqrt{6}}\right]. \end{aligned}\right\} \qquad \textbf{(96)}$$

For the maximum velocity on the symmetry line, we have

$$u_{1m}=x^{-1/3}. \qquad (97)$$

Relation between the longitudinal velocity and the maximum one is expressed as

$$u=u_m\frac{1}{ch^2\frac{\zeta}{\sqrt{6}}}. \qquad (98)$$

Calculation of the longitudinal velocity of the jet $u(\zeta)=\psi'(\zeta)$ as a function of the independent dimensionless variable ζ on the basis of the analytical solution of the partial differential boundary layer equation made in MathCad is shown in the figure 10.

$$\psi(\zeta) := \sqrt{6}\cdot\tanh\left(\frac{\zeta}{\sqrt{6}}\right)$$

$$\zeta := 0, 0.01 .. 10 \qquad \psi'(\zeta) := \frac{d}{d\zeta}\psi(\zeta)$$

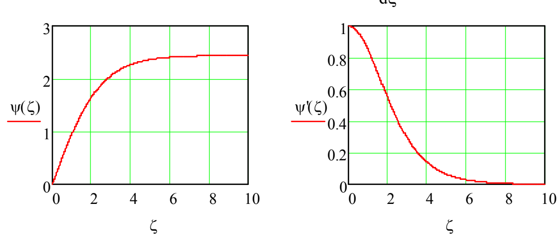


Fig. 10. Dimensionless flow function and its derivative according to the analytical solution by N. A. Slezkin

Due to the fact that the dimensionless parameter according to (69) is the function $\zeta=yx^{-2/3}$, the figure 11 shows curves of the axial velocity $u_1=f(\zeta)$ under different x_1 and the function $u_1=f(x_1)$ under the change of ζ.

Then we give the curves (96) for the transverse jet velocity (inflow velocity) $v=f(\zeta)$ under different x_1, as well as the functions $u_1=f(x_1)$ under different ζ.

$$u_1(x_1,\zeta) := x_1^{-\frac{1}{3}}\cdot\frac{1}{\left(\cosh\left(\frac{\zeta}{\sqrt{6}}\right)\right)^2}$$

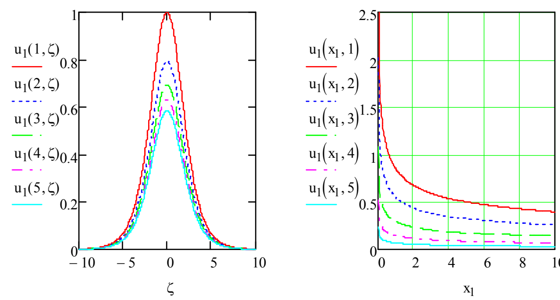


Fig. 11. Axial jet velocity function of the dimensionless variable

Calculation of the longitudinal jet velocity $u(\zeta)$ as the function of the dimensionless independent variable ζ, based on the solution of the partial differential boundary layer equation by the characteristics method, is reduced to solving of the two-point boundary-value problem for the ordinary differential third-order equation with initial and boundary conditions. Solution in MathCad is shown in the figure 13.

$$v_1(x_1,\zeta) := \frac{1}{3}x_1^{-\frac{2}{3}}\left[\frac{2\cdot\zeta}{\left(\cosh\left(\frac{\zeta}{\sqrt{6}}\right)\right)^2}-\sqrt{6}\cdot\tanh\left(\frac{\zeta}{\sqrt{6}}\right)\right]$$

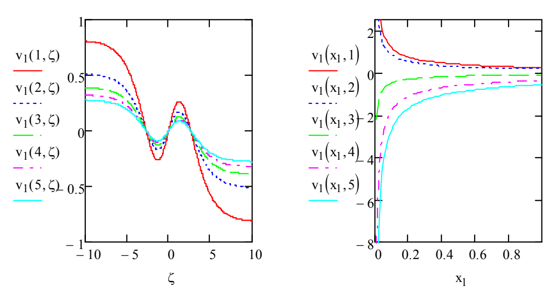


Fig. 12. Transverse jet velocity function of the dimensionless variable

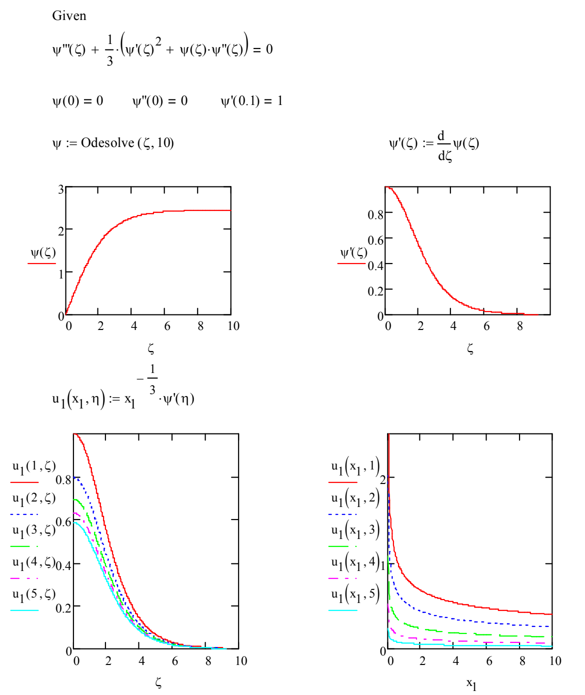


Fig. 13. Calculation of the longitudinal velocity of the jet $u(\zeta)$ based on the solution of the differential boundary layer equation

Comparison of the results of the analytical and numerical solutions of the differential boundary layer equation shows that they are identical (Fig. 10, 11 and 13).

In turn, comparison of the functions $u(\zeta)$, given in the figures 8, 11 and 13, shows the coincidence of the solutions obtained on the basis of the analytical and numerical solutions of the differential boundary layer equations with the results of applying the vortex jet boundary model.

12. Having passed in the equality (33) to dimensionless quantities on the basis of (51), (70) and (96), we obtain the following expression for the number α

$$\alpha = \int_{-\infty}^{\infty} u^2 dy = \int_{-\infty}^{\infty} \psi'^2(\zeta) d\eta = \int_{-\infty}^{\infty} \frac{d\zeta}{ch^4 \frac{\zeta}{\sqrt{6}}} = \frac{4}{3}\sqrt{6} = 3{,}27 . \quad (99)$$

In conclusion, we calculate the flow rate through the infinite straight line parallel to the axis y_1

$$Q = \int_{-\infty}^{\infty} u_1 dy_1 = \frac{J}{\rho\alpha} U^{-1} \int_{-\infty}^{\infty} u dy = \frac{J}{\rho\alpha} x^{-1/3} U^{-1} \int_{-\infty}^{\infty} \psi'(\zeta) d\zeta = \quad (100)$$

$$= \left(\frac{\nu J x_1}{\rho\alpha}\right)^{1/3} \int_{\infty}^{\infty} \frac{d\zeta}{ch^2 \frac{\zeta}{\sqrt{6}}} = 2\sqrt{6}\left(\frac{\nu J x_1}{\rho\alpha}\right)^{1/3} .$$

Since the numerical value of the coefficient is equal to $2\sqrt{6}\left(\frac{1}{3{,}27}\right)^{\frac{1}{3}} = 3{,}301$, then under $\nu = 1$ the flow formula coincides with the dependencies by H. Schlichting and L. G. Loitsiansky.

Thus, the flow through the initial section of the jet ($x = 0$) is equal to zero, and then the flow increases due to the inflow from the sides of the jet. Nature of the threads is defined by the equation

$$x_1^{1/3}\psi(\zeta) = const . \quad (101)$$

Discussion of results

For the analytical solution of the jet-source problem, L. A. Vulis and V. P. Kashkarov [1] considerthat it is sufficient to replace assignment of the initial conditionswith the determination of the total momentum in the projection on the jet symmetry axis $J = \int_{(s)} \rho u_{x_0}^2 ds$, where ρ – density of the liquid, u_{x_0} – axial velocity component in the initial section s of the jet, which is replaced with the effective source.

Combined equations of the liquid motion problem caused by the plane hydraulic jet (3, 4), on the basis of self-similarity transformation $\frac{u}{u_m} = \psi'(\zeta)$, $u_m = Ax^\alpha$, $\zeta = Byx^\beta$ under the equality $\frac{A}{B^2} = 6\nu$ and the self-similarity constants α and β allow to reduce the problem to the ordinary differential equation

$$\psi''' + 2\left(\psi\psi'' + \psi'^2\right) = 0 \quad (102)$$

with the boundary conditions:

$$\psi = 0,\ \psi'' = 1 \text{ under } \zeta = 0,$$

$$\psi' = 0 \text{ under } \zeta = \pm\infty . \quad (103)$$

The solution of this equation has the form

$$\psi = th\zeta,\ \psi' = \frac{1}{ch^2\zeta} = 1 - th^2\zeta . \quad (104)$$

Peculiarity of the solution by L. A. Vulis and V. P. Kashkarov [1] is the conclusion on the basis of invariant transformation of self-similarity of the ordinary differential equation $\psi''' + 2\left(\psi\psi'' + \psi'^2\right) = 0$, which differs from the similar equations by L. G.

Loitsiansky [5] and N. A. Slezkin [6] in a multiplier before the sum of terms up to the second order. H. Schlichting [4], as a result of applying the similarity and dimension theory, gets another form of the equation $\psi''' + \psi\psi'' + \psi'^2 = 0$, in which all the coefficients of the derivatives are equal to one. However then, H. Schlichting after a single integration of the ordinary differential third-order equation for the flow function uses the second similarity transformation $\xi = \alpha\zeta$, $\Psi(\zeta) = 2\alpha\psi(\zeta)$, and after replacing the variable gets the second-order equation $2\psi\psi' + \psi'' = 0$.

While the jet momentum $J = \int_{-\infty}^{+\infty} \rho u^2 dy = const$, values of the constants A, B and the self-similarity constants α and β are equal to

$$A = \frac{1}{2}\sqrt[3]{\frac{3J^2}{4\rho^2\nu}},\ B = \frac{1}{2}\sqrt[3]{\frac{J}{6\rho\nu^2}},$$

$$\alpha = -\frac{1}{3},\ \beta = -\frac{2}{3}. \qquad (105)$$

The figure 14 shows the results of numerical and analytical calculation of the flow function and the universal velocity profile in the plane jet cross-section in MathCad.

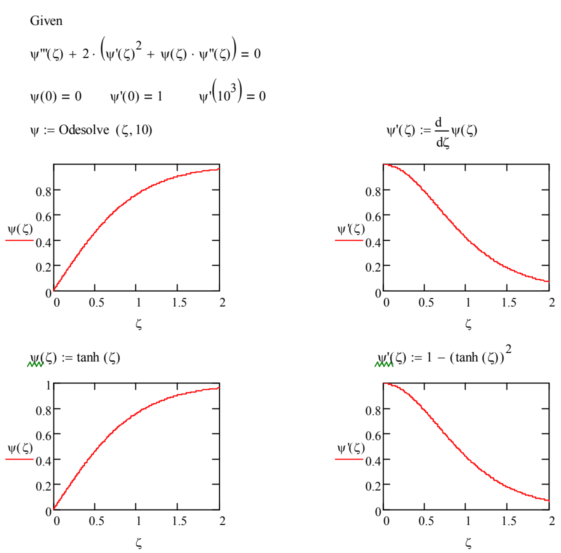


Fig. 14. The flow function and the universal velocity profile in the plane jet cross-sections, obtained on the basis of the analytical and numerical solutions of the Cauchy problem for the ordinary differential second-order equation

Laws of change of the mass flow G and the kinetic energy flux E in the plane jet are defined, respectively:

$$G = \int_{-\infty}^{+\infty} \rho u dy = \rho \frac{A}{B} x^{1/3}[\psi(+\infty) - \psi(-\infty)] = \sqrt[3]{36\rho^2 \nu J x^{1/3}}; \qquad (106)$$

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} \rho u^3 dy = \frac{J}{30\mu}\sqrt[3]{6\rho\nu^2 J^2} x^{-1/3}. \qquad (107)$$

Summary

1. Analysis of the hydraulic jet theories shows that the most common concept of their formation is the idea that the jet surface is the vorticity concentration area in the form of a potential of the ordinary or double vortex layer. The jet boundary as a result of continuous expansion by the addition of transit flows takes a curved shape. A classic example of jet streams is the axisymmetric jet outflow from the jet-forming nozzle of the circular section in unlimited water space.

2. Analysis of plane motion of a material point in the natural coordinates found that the normal component of the force produces motion trajectory bending. The force acting on the curved jet boundary can be defined as Zhukovsky force applied to the free vortex in a plane homogeneous flow of the ideal liquid. As a result, we obtained the vortex equilibrium centre locus, which has the form of a catenary equation. It is established that the vortex strength is inversely proportional to the cosine of the angle of rotation of the boundary of the flow flowing from the initial cross-section. Comparison of the analytic dependence of the vortex boundary with the results of the numerical solution of the ordinary differential second-order equation for the condition of the dynamic equilibrium of the vortex, rolling along a curved path, in MathCad showed their coincidence. The velocity components, derived from the vortex propagation boundary model, of the tangents to the thread in the projection on the coordinate axes satisfy the equations of the boundary layer momentum and continuity, as illustrated by the symbolic solution in MathCad.

3. As the study of existing formulations of the problems showed, the law of conservation of momentum to the area between two straight lines parallel to the normal in the jet direction, with a constant pressure, leads to self-similarity (affine similarity) of the longitudinal jet velocity profiles. The full***per-second momentum***, carried through the jet cross-section remains***the same for all sections, which***makes the jet propagation problem concrete.

4. Dependencies for the boundary and kinematic characteristics of the plane jet, found on the basis of the vortex theory, are presented in the form of

hyperbolic functions of the longitudinal jet coordinate x. Kinematic characteristics of the jet, resulting from the application of the boundary layer theory, appear to be hyperbolic functions of independent variable $\zeta = yx^k$, expressed as the transverse coordinate relation y to the power function of the longitudinal coordinate x with fractional negative index k. Therefore, to assess the impact of the total jet momentum condition on the uniqueness of the obtained solutions, it is offered to use elements of the generalized function theory and, in particular, delta function.

5. It is found that, due to the filtering properties of delta function, the functional effect of the generalized liquid density distribution within the boundaries of the plane jet, flowing from the point source at the origin, on the total jet momentum leads to the satisfaction of the conditions of momentum conservation regardless of the type of argument determining the kinematic characteristics of the jet. Filtering property of delta function and calculation of the momentum, when using the generalized distribution of the liquid density and the vortex boundary equation, confirming the main hypothesis about the identity of dependencies of the kinematic characteristics of the plane jet, derived from the vortex theory and the boundary layer theory, are illustrated in MathCad. This hypothesis is confirmed by consideration of interchangeability of expressions for the flow function differential $\partial\psi_1 = u_1\partial y_1 = -v_1\partial x$ assuming the existence of the integral equation $\psi_1 = \int u_1 dy_1 = -\int v_1 dx$, that allows replacement of the functions $\psi_1(y_1)$ and $\psi_1(x_1)$ subject to the existence of explicit longitudinal velocity expressions in the form $u_1(y_1)$, and transverse jet velocity expressions - in the form $v_1(x_1)$.

6. Pursuant to the known provisions of the similarity theory resulting from the expression of coordinates and kinematic characteristics of the jet in the scale associated with the Reynolds number for the jet-source, the boundary layer equations are reduced to a dimensionless form.

7. Transition from the boundary layer equations to the partial third-order equation for the flow function, in well-known works on the plane laminar jet theory is performed by substituting the longitudinal and transverse velocities, expressed in terms of derivatives of the flow function.

8. From the classic works on the subject of the study it is found that the conversion of the dimensionless partial differential third-order equation for the flow function into the ordinary differential equation is possible either because the jet momentum is assumed as a given dimensional quantity, or it is assumed that the longitudinal velocity scale is excluded from the solution result.

9. Application of eliminating the longitudinal velocity scale from the solution results allowed, based on the vortex jet boundary model, to derive the dimensionless ordinary differential equation from the boundary layer equations for the flow function and thereby to establish a deep inner connection between the proposed vortex model and the classical approach to the plane laminar jet problem.

10. It is found that derivation of the ordinary differential third-order equation for the flow function from the partial differential equations can be performed on the basis of invariant transformation. Symbolic substitution of the flow function in the differential third-order equation for the flow function in MathCad confirmed the made calculations.

11. There are given data on the integration of the partial differential boundary layer equation by the characteristics method, using which the longitudinal jet velocity is calculated as a function of the dimensionless independent variable under the given vortex jet boundary law. In MathCad, a curve of the longitudinal jet velocity is created.

12. Integration of the ordinary differential third-order equation with respect to the flow function is performed directly throughthe symbolic substitution of the analytical expression for the flow function in the boundary layer equations and in a numerical form in MathCad. Comparison of the results of the analytical and numerical solutions of the differential boundary layer equation showed that they are identical. In turn, comparison of the results of solving the boundary layer equations with the results of applying the vortex boundary model indicates the coincidence of the solutions for the kinematic characteristics of the plane jet.

13. When expressing the obtained solution results in a dimensional form, the numerical value of the coefficient is found, with which the jet flow formulae obtained by various authors coincide.

14. Validity of the obtained results is confirmed by obtaining identical dependencies as a result of applying the vortex model and the boundary layer theory to calculate the kinematic characteristics of the submerged plane laminar hydraulic jet-source.

Thus, the hypothesis about the identity of dependencies for the kinematic characteristics of the plane laminar hydraulic jet-source derived from the vortex theory and the boundary layer theory is considered to be proven.

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