# On the stability of circular and spiral orbits in nonstationary gravitational field

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**Abstract.** In nonstationary gravitating systems with axial symmetry, there is a certain class of the circular and spiral orbits, which play a special role in the dynamics of such systems. The stability of the circular and spiral orbits in the motion of a material point in this nonstationary axially symmetric gravitational field is investigated. The conditions of existence and stability of a wide class of circular and spiral orbits are obtained. The variety of the considered circular and spiral orbits determined by the law changes and the rate of change in the function of time, which characterizes the nonstationarity of the system, the size of the sector velocity of a material point, a set of initial parameters of the orbits.

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## I
## ntroduction

The nonstationary gravitating systems with axial symmetry have a certain class of circular and spiral orbits, which play a special role in the dynamics of such systems [1-7]. We clarify the stability conditions of circular and spiral orbits in nonstationary gravitational field. In this case we use the investigating method of nonautonomous dynamical systems stability [8]. Since the real gravitating systems are essentially nonstationary, such nonstationary circular and spiral orbits are of interest to some astronomical problems.

The existence and stability of spiral and circular orbits are studied in [9, 10] in considering the motion of a material point in the nonstationary axisymmetric gravitational fields. In these studies the stability of a wide class of spiral and circular motions for the gravitational fields of different structures is established. The variety of considered spiral and circular orbits is determined by the rate of the change of time function, characterizing nonstationarity of the system, by the value of sector velocity of a material point, the set of initial parameters. Generalization [9] of the stability conditions of S. Chandrasekhar [11], V.G. Demin [12] circular motions are obtained. Application of sustainability criteria of the circular orbits to the stability of the ring structure of peculiar galaxies is given in [13].

In the most important case of axially symmetric gravitating systems we can distinguish several types of force fields of such systems. According to Chandrasekhar [11], the most general form of the gravitational potential $U(r, z, t)$ of the nonstationary system (galaxy) is given by the expression

$$U(r, z, t) = \frac{\ddot{\varphi}}{2\varphi}\left(r^2 + z^2\right) + \frac{1}{\varphi^2}\widetilde{U}\left(\frac{r}{\varphi}, \frac{z}{\varphi}\right), \qquad (1)$$

where $r$, $z$ are cylindrical coordinates; $\varphi$ is a rather arbitrary function of time; $\widetilde{U}$ - here and in all subsequent cases rather arbitrary function of its arguments.

The force function of nonstationary gravitating system can belong, moreover, to one of the following types:

$$U(r, z, t) = \gamma^2 \widetilde{U}(r\gamma, z\gamma); \qquad (2)$$

$$U(r, z, t) = \gamma \widetilde{U}(r, z), \qquad (3)$$

where $\gamma$ is a function of time.

An example of the force function of the type (2) is a force function of a bounded rectilinear three-body problem with variable mass. An example of the force function of the type (3) is the force function of the problem of two fixed centers with variable mass or a variable gravitational constant [9, 10, 13].

Hereinafter we consider gravitating systems with axial symmetry, force functions of which belong to one of these types (1)-(3). The investigation of the motion stability in such force fields consider on the basis of analysis of the nonautonomous dynamical systems stability, led to an autonomous form [8]. Below we find out the conditions of existence and stability of a wide class of spiral and circular motions in nonstationary axisymmetric gravitational fields of these types.

Let the force function in cylindrical coordinates $\rho, \lambda, z$ with axis $z$, coinciding with the axis of symmetry, has the form

$$U(\rho, z, t) = f(t)\widetilde{U}(\rho, z), \qquad (4)$$

where $f$ is time function of the expression

$$f(t) = \frac{1}{\alpha t + \beta} \quad (\alpha > 0, \beta > 0) \qquad (5)$$

Let's consider the motion in the field (1) at the presence of the friction force

$$\vec{F}_{fr} = \nu(t)\dot{\vec{r}} \qquad \left(\nu = \frac{\dot{f}}{2f}\right), \qquad (6)$$

where $\vec{r}$ is the radius vector of a material point.

In this case the circular motion is permitted

$$\rho = \rho_0; \quad \dot{\rho} = 0; \quad \Omega = \frac{\sigma_0}{\sqrt{\alpha t + \beta}}; \quad z = z_0; \quad \dot{z} = 0, \qquad (7)$$

with a continuously decreasing areal velocity $\Omega/2 = \rho^2\dot{\lambda}/2$.

The motion (7) is examined for stability in the sense of Lyapunov with respect to the values $\rho, \dot{\rho}, \Omega, z, \dot{z}$. For this purpose the Lyapunov function in the form of the first integrals bunch of the perturbed motion equations is constructed using the method N.G. Chetaev [9]. As a result, the stability conditions of circular orbits (7) have the form

$$\begin{cases} \left(\ddot{U}_{\rho\rho} + \frac{3}{\rho}\dot{U}_{\rho}\right)_0 < 0; \\ \left[\left(\ddot{U}_{\rho\rho} + \frac{3}{\rho}\dot{U}_{\rho}\right)\ddot{U}_{zz} - \ddot{U}_{\rho z}^2\right]_0 > 0. \end{cases} \qquad (8)$$

where the subscript 0 means that the value of the function $\rho_0$, $z_0$ is taken at the point.

If the field force is symmetrical relative to a plane $z = z_0$, that is $(\ddot{U}_{\rho z})_0 = 0$, then the stability condition (8) are simplified and have the form

$$\left(\ddot{U}_{\rho\rho} + \frac{3}{\rho}\dot{U}_{\rho}\right)_0 < 0; \quad (\ddot{U}_{zz})_0 < 0. \qquad (9)$$

These stability conditions can lead to a more compact form. In fact, if we consider the motion within the gravitating medium of continuous density, then the Poisson equation for the force function has the following form:

$$\ddot{U}_{\rho\rho} + \ddot{U}_{zz} + \frac{1}{\rho}\dot{U}_{\rho} = -4\pi G\mu(P) \qquad (10)$$

where $G$ is gravitational constant; $\mu(P)$ is a density of the inner point of the medium $P$. Using the equation (10), we represent the stability conditions (9) in the form

$$\left(\ddot{U}_{\rho\rho} + \frac{3}{\rho}\dot{U}_{\rho}\right)_0 < 0; \quad \left(\ddot{U}_{\rho\rho} + \frac{3}{\rho}\dot{U}_{\rho}\right)_0 > -4\pi G\mu + \left(\frac{2}{\rho}\dot{U}_{\rho}\right)_0 \qquad (11)$$

and after simple transformations we obtain the stability condition

$$\left[2\rho^2\dot{U}_{\rho} - 4\pi G\rho^3\mu\right]_0 < \left[\frac{\partial}{\partial\rho}\left(\rho^3\dot{U}_{\rho}\right)\right]_0 < 0, \qquad (12)$$

which is a generalization of the stability conditions of stationary circular motions given in [11, 12].

The stability of circular motions can be shown similar to the above mentioned researches

$$\rho = \rho_0; \quad \dot{\rho} = 0; \quad \Omega = \sigma_0 f^{1/2}; \quad z = z_0; \quad \dot{z} = 0 \qquad (13)$$

at the force function of (1), in which the function $f$ varies according to the Eddington-Jeans:

$$\dot{f} = \alpha_1 f^n, \quad (\alpha_1 < 0, \quad n > 3/2) \qquad (14)$$

and functions $\nu$

$$\nu = \frac{\alpha_1}{2} f^{n-1}, \qquad (15)$$

The variety of circular orbits (7) and (14) are defined by a set of values $\rho_0$, $z_0$, by the value of areal velocity $\Omega/2$ and function $f$. Note that the circular orbits (7) at the large values of the time $t$ are close to the corresponding stationary circular orbits with a small areal velocity in a weak force field.

Let the force function has the form

$$U(r, z, t) = \gamma^2\widetilde{U}(r\gamma, z\gamma), \qquad (16)$$

where

$$\gamma(t) = \frac{1}{\alpha t + \beta}, \quad (\alpha, \beta - const) \qquad (17)$$

In this case, in the field (16) the spiral motions may take place

$$r = r_0(\alpha t + \beta); \quad \dot{r} = \alpha r_0; \quad \Omega = \Omega_0; \quad z = z_0(\alpha t + \beta); \quad \dot{z} = \alpha z_0, \qquad (18)$$

Spiral motion (18) is examined for stability in the sense of Lyapunov with respect to the values $r, \dot{r}, \Omega, z, \dot{z}$. Similar to [10] the Lyapunov function is found by the method of integral connection by N.G. Chetaev. As a result, we have the conditions of motion stability (18)

$$\begin{cases} \left(U''_{rr} + \frac{3}{r}U'_r\right)_0 < 0; \\ \left[(U''_{zz})\left(U''_{rr} + \frac{3}{r}U'_r\right) - (U''_{rz})^2\right]_0 > 0, \end{cases} \quad (19)$$

where the subscript 0 means that the value of function is taken at the point $r_0(\alpha t + \beta)$, $z_0(\alpha t + \beta)$.

Particular solutions of (18) represent a class of spiral orbits, the variety of which is determined by the function $\gamma$, its rate of change and a set of values $r_0$, $z_0$, $\Omega_0$. The particular case of these orbits are the flat spiral orbits $(z_0 = 0)$. In the case of very slow change of function $\gamma$ with time $(\alpha \approx 0)$ the considered spiral orbits are nearly circular:

$$r = \beta r_0; \quad \dot{r} = 0; \quad \Omega = \Omega_0; \quad z = \beta z_0; \quad \dot{z} = 0. \quad (20)$$

Thus, there are stable spiral orbits (18) in nonstationary axisymmetric gravitational field of (16) under conditions (19). At the termination of nonstationarity of the force field the spiral orbits turn into the corresponding stable circular orbits.

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**References**

1. King, I.R., 2002. An introduction to classical stellar dynamics. M: URSS, pp: 288.
2. Canon, R.D., C. Lloyd and V.M. Penston, 1970. Ring galaxies. Observatory., N 977(Vol. 90): pp 153-154.
3. Thomson, L.A., 1977. Possible ring galaxies near rich clusters. Astroph. J, N 3(Vol. 211 part I): pp 684-692.
4. Theys, J.S. and A.E. Spiegel, 1976. Ring galaxies. I.. Astroph. J., N 3(Vol. 208 part I): pp 650-661.
5. Freeman, K.C. and G. de Vaucouleurs, 1974. An interpretation of ring galaxies and the properties of intergalactic gas clouds. Astroph. J., N 3(Vol. 194 part I.): pp 569-585.
6. Sersic, J.L., 1970. Transient annular structures in exploding galaxies. "Periodic orbits, Stability and Resonances", (ed. G.E.O.Giacaglia)., Dordrecht-Holland, pp: 314-321.
7. Sersic, J.L., 1973. On the structure of the peculiar galaxies. Bulletin of the astron., inst. of Czechoslovakia (issue Vol. 24, N 3.), pp: 150-157.
8. Bekov, A.A., 1998. On the stability of non-autonomous dynamical systems. Proc. of Kazakhstan Ac. Sci., Series Phys.-Math.,, № 4: pp 57-60.
9. Bekov, A.A., 1981. On the stability of non-stationary circular orbits in resisting medium. Proc. Astrophys.Inst. Sci.Ac. KazUSSR, № 2(V. 35): pp 50-54.
10. Bekov, A.A., 1982. On the stability of spiral orbits in non-stationary axisymmetric gravitation field. Proc. Astrophys.Inst. Sci.Ac. KazUSSR, № 6(V. 39): pp 42-46.
11. Chandrasekhar, S.-., 1942. Principles of stellar dynamics. Chicago, Illinois, USA.: The University of Chicago press, pp: 251.
12. Demin, V.G., 1968. Motion of artificial satellite in non-stationary gravitation field. M: Nauka, pp: 990.
13. Bekov, A.A., 2013. Dynamics of double stationary gravitating systems. Almaty: Sci, pp: 170.

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