Consideration of Buffon's Needle Problem in Geometric Probability

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Abstract: The research on geometric probability began as a turning point of the Buffon's needle problem. In this paper, based on the idea of the geometric probability, we first describe two methods in the Buffon needle problem to determine the probability that the needle intersects the parallel lines in the Buffon needle problem. Next, in the case of a convex curve arc [a curve of a needle] instead of a needle, we consider a problem of determining the expectation value that a convex curve arc intersects the parallel lines, and later we prove Barbier's theorem applying our results.

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1. Introduction

The concept of geometric probability is that when all events are represented by a set Ω on a plane and the event of interest is represented by a fractional set A of Ω, the probability of the occurrence of the event of interest is determined by the relation「area of A ÷ area of Ω」. The idea of the geometric probability began as a turning point of the Buffon needle problem. "You drop a needle of length l on a flat plane on which are drawn countless number of parallel lines at intervals of d, and find the probability of the needle crossing the parallel lines." It is a so-called Buffon's needle problem.

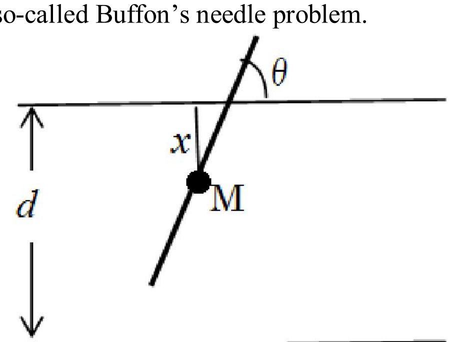

Figure-1 Buffon's needle problem

Assuming that X is the number of lines the convex curve of length l crosses the parallel lines, using constant α that does not depend on the convex curve shows that the expected value E (X) of X becomes E (X) = $\alpha\, l$.

Lastly, as an application of this theorem, we prove the theorem of Barbier which states that the length of the circumference of the convex set of constant width d in the plane becomes πd.

2. Material and Methods

Buffon's Needle Problem: Using the concept of geometric probability of Buffon needle described previously, it is usually interpreted as when X is the distance from M parallel lines closest to the center A, and θ is the angle of the needle making with parallel lines, then

$0 \le \theta \le \pi, 0 \le x \le \frac{d}{2}$ because θ and X are independent, the all events Ω is (0, π) x (0, $\frac{d}{2}$). If $l < d$, the needle crosses the parallel lines at most at one point, and the condition for the intersection is $\frac{1}{2}\sin\theta \geqq x$. That is, when the needle intersects the parallel lines the event C, shown by the shaded portion in Figure-2, is expressed by the relation given below.

$$C = \left\{(\theta, x) \in \Omega \,\middle|\, \frac{1}{2}\sin\theta \geqq x\right\}$$

The concept of geometric probability is that the probability of occurrence of event C of interest out of all events Ω is obtained by the simple relation [area of C ÷ area of Ω]. In other words, if P is the probability of occurrence of event C and l < d, then P is obtained by

$$p = \frac{2}{\pi d}\int_0^{\pi} \frac{l}{2}\sin\theta d\theta = \frac{2l}{\pi d}$$

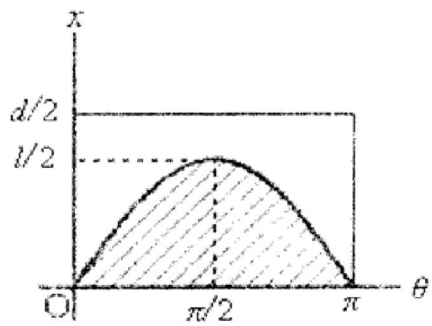

Figure-2 Event *C* when $l < d$

Next, consider the case of l ≥ d. Here, $\frac{1}{2}\sin\theta \geq x$ is the condition for the needle to cross the parallel lines, which is the same as when l < d. The needle, however, may intersect a plural number of parallel lines. In Figure-3, the shaded portion shows the event C when the needle crosses the parallel lines.

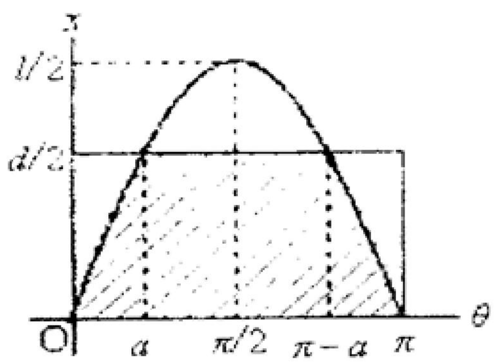

Figure-3 C event for l ≥ d

Let's compute the probability p when l ≥ d and the needle and parallel lines intersect. If a, in 0 < θ ≦ π/2, is the value of θ satisfying $l\sin\theta = d$, then p becomes

$$p = \frac{2l}{\pi d}\left\{2\int_0^a \frac{1}{2}\sin\theta d\theta + \frac{d}{2}(\pi - 2a)\right\}$$
$$= 1 + \frac{2l\cos a}{\pi d} - \frac{2a}{\pi a} - \frac{2l\cos a}{\pi d}$$

Let N be the number of parallel lines intersecting with a needle. When l < d, either N = 1 or N = 0. As calculated above, since $\frac{2l}{\pi d}$ was the probability of N to be 1, and if we write E (N) the expected value of N, then E (N) = $\frac{2l}{\pi d}$ is established.

When l ≥ d, we consider the length of the needle divided into several segments that are shorter in length than d. Using the linearity of the expectation value, it can be seen that E (N) = $\frac{2l}{\pi d}$ still holds.

Consider d ≦ l < 2d, and take any value 0, 1, 2 of N. Let p_k (k = 0, 1, 2) be the probability of N = k. Then the probability p the needle crosses the parallel lines is

$p = p_1 + p_2$; $E(N) = p_1 + 2p_2 = p + p_2$.

It follows that

$$p_2 = \frac{2l}{\pi d} - p = \frac{2a}{\pi} + \frac{2l\cos a}{\pi d} - 1$$

From here we obtain,

$$p_1 = p - p_2 = 2 + \frac{2l}{\pi d} - \frac{4a}{\pi} - \frac{4l\cos a}{\pi d}$$

Buffon's Needle Problem (Another solution): Let Y be the distance of parallel lines just below the center M of the needle from M. Here, θ has the same meaning as in the previous section. Considering (θ, y) uniformly distributed over (0, π) x (0, d) let us investigate the probability of the needle intersecting the parallel lines.

Let E_1 be the event the needle crosses the straight line just below M (Figure-4a). The condition for event E_1 to occur is $\frac{1}{2}\sin\theta \geq y$. Again, let E_2 be the event the needle crosses the straight line just above M (Figure-4b). The condition for event E_2 to occur is $\frac{1}{2}\sin\theta \geq d - y$.

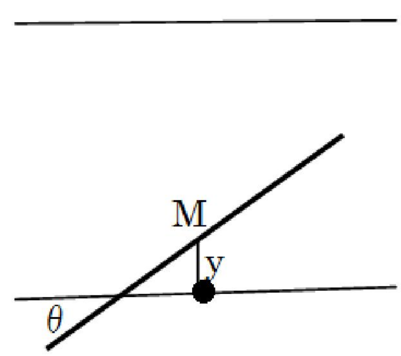

Figure-4a The straight line is just below M

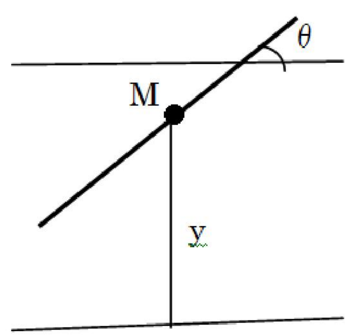

Figure-4b The straight line is Just above M

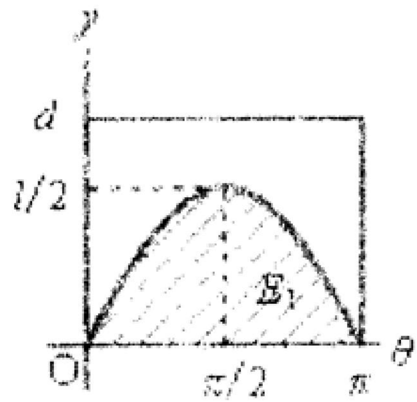

Figure-5a Event E_1

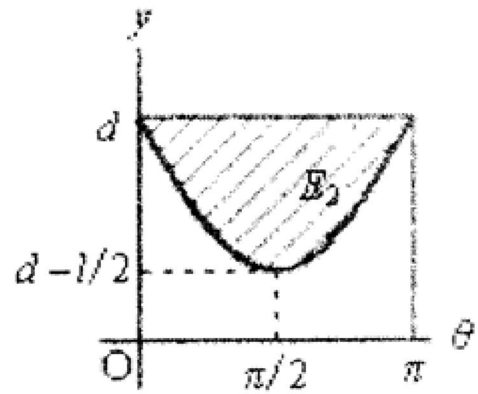

Figure-5b Event E_2

In Figure-5a and Figure-5b, events E_1 and E_2 are shown by their respective regions in plane (θ, y). The probability of occurrence of E_1 and E_2 is given by, respectively

$$P(E_1) = \frac{1}{\pi d}\int_0^{\pi}\frac{1}{2}\sin\theta d\theta = \frac{1}{\pi d}$$

$$P(E_2) = \frac{1}{\pi d}\int_0^{\pi}\left\{d - \left(d - \frac{1}{2}\sin\theta\right)\right\}d\theta = \frac{1}{\pi d}$$

An event of the needle intersecting the parallel lines is expressed by E. In case $l < d$, because the needle crosses the straight lines at most once, $E_1 \cap E_2 = \varphi$. Therefore, the probability P (E) of the occurrence of event E is obtained by the equation below.

$$P(E) = P(E_1) + P(E_2) = \frac{2l}{\pi d}$$

Also, in the case of $d \leq l < 2d$, since event $E_1 \cap E_2$ in which the needle intersects two parallel lines is represented by the shaded area in Figure-6, its probability P ($E_1 \cap E_2$) is calculated as follows:

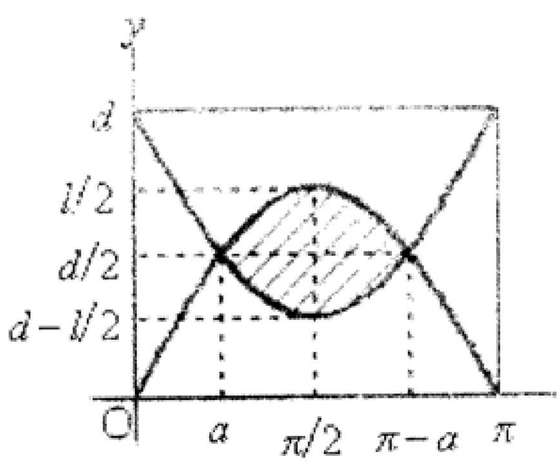

Figure-6 Event ($E_1 \cap E_2$) when d ≦ l <2d

$$P(E_1 \cap E_2) = \frac{1}{\pi d}\int_0^{\pi - a}\left\{\frac{1}{2}\sin\theta - \left(d - \frac{1}{2}\sin\theta\right)\right\}d\theta$$

$$= -\frac{1}{\pi}(\pi - 2a) + \frac{1}{\pi d}(\cos a - \cos(\pi - a))$$

$$= \frac{2a}{\pi} - 1 + \frac{2l\cos a}{\pi d}$$

This is consistent with the results of the previous one.

Curved Needle: Here, we discuss a case when the needle is a convex curve arc. In preparation, first think of it as if needle L is a polygonal line. Let L_1, L_2….L_n be short line segments that make up the line. And let X_k be the number of parallel lines intersecting with L_k ($1 \leqq k \leqq n$).

Let X be the number of parallel lines that intersect L, then $X = \sum_{k=1}^{n} X_k$. From the discussion in Section 2, if length l_k of the line segment L_k is smaller than d, the expected value E (X_k) of X_k was E (X_k) = αl_k. Here, a is a constant which is independent of k (specifically, $\alpha = \frac{2}{\pi d}$). The following relationship can be established from the linearity of the expected value[4].

$$E(X) = \sum_{k=1}^{n} E\left(X_k\right) = \sum_{k=1}^{n} \alpha\left(l_k\right) = \alpha l$$

Here, l is the length of the line L.

Even if L is a general convex curve arc, it shows that similar results for the expectation value for X can be obtained.

3. Results and Discussion

Theorem 1: Let L be a convex curve arc. The expectation value E(X) of the number X of parallel lines intersecting L is equal to the product of constant a, which does not depend on L, and length l of L. That is, it establishes the relation E (X) = αl.

Proof: Let A, and B, respectively, be the start and end points on L. From the property of the length of the curve, there exist L ': $P_0 P_1 \ldots P_m$ (however, P_0 = A, P_m = B) that inscribes L, and L″: $Q_0, Q_1, \ldots Q_n$ (however, Q_0 = A、Q_n = B) that circumscribe L for a given positive number ε, so as to establish the following equations [8]:

$$1 - \varepsilon \leqq \sum_{i=1}^{m} \overline{P_{i-1}P_i} \leqq 1$$

$$1 \leqq \sum_{i=1}^{n} \overline{Q_{i-1}Q_i} \leqq 1 + \varepsilon$$

Let Y_ε be the number of parallel lines that intersect L ', and Z_ε be the number of parallel lines that intersect L". From the convexity of the L curve, $Y_\varepsilon \leqq X \leqq Z_\varepsilon$ is established. On the other hand, considering the case of a polygonal line, the following equation is established:

$$E(Y_\varepsilon) = \alpha \sum_{i=1}^{m} \overline{P_{i-1}P_i} \qquad E(Z_\varepsilon) = \alpha \sum_{i=1}^{n} \overline{Q_{i-1}Q_i}$$

Therefore, $\alpha(1-\varepsilon) \leqq E(Y_\varepsilon) \leqq E(X) \leqq E(Z_\varepsilon) \leqq \alpha(1+\varepsilon)$.

Since ε was larger than zero that is, $\varepsilon > 0$, we get E (X) = αl. (Proven)

If we especially take circumference of diameter d as L, and since X = 2 always, then from Theorem 1 we get 2 = E (X) =α. πd. From this, a must be equal to $\frac{2}{\pi d}$ that is, $\alpha = \frac{2}{\pi d}$.

Using Theorem 1, we can show Barbier's theorem on the length of the curve of constant width such as the Reuleaux triangle (Rouleau).

Theorem 2: In the plane, πd is the length of the circumference of a convex set whose width is of constant value d.

Proof: Let L be a convex closed curve formed of the boundary of a convex set, and l its length. Since L is a curve of constant width, X = 2 always. Hence, 2 = E(X) = αl = $\frac{2l}{\pi d}$ and, consequently, it is established that $l = \pi d$. (Proven).

Conclusion

The results obtained from Buffon's needle problem is consistent with the results obtained by another way i.e., Buffon's needle (Another solution). It is also shown with the help of theorem 1 the Barbier's theorem on the length of the curve of constant width such as the Reuleaux Triangle (Rouleau).

The expectation value E(X) [Mean of X] of the number X of parallel lines intersecting, will be equal to αl, here α is defined as any constant value and l, which is independent of α, is defined as the length of L. It is also proved by theorem 1 and Theorem 2.

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