On the bounds of Euler's constant γ

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Abstract: In this paper, we deduced the following double inequality

$$\frac{1}{2n}-\sum_{k=1}^{2m+1}\frac{B_{2k}}{2k\,n^{2k}}<\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma<\frac{1}{2n}-\sum_{k=1}^{2m}\frac{B_{2k}}{2k\,n^{2k}};\qquad m=0,1,2,\ldots$$

with sharp bounds, where γ is the Euler's constant and B_j are the Bernoulli numbers.

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1. Introduction

The n^{th} partial sum of the harmonic (divergent) series

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}+\ldots \qquad (1)$$

is called the harmonic number and is denoted by H_n , $n=1,2,3,\ldots$. In 1734, the Swiss mathematician L. Euler defined one of the most useful constants in mathematics by the limit of the sequence

$$\gamma=\lim_{n\to\infty}(H_n-\ln n), \qquad (2)$$

which is called Euler's constant and it is also known as the Euler-Mascheroni constant, in recognition of the work of the Italian mathematician L. Mascheroni (1750-1800), who the first to use the symbol γ to denote this constant as he extended several results of Euler [8].

The constant γ has many applications in analysis, special functions, number theory, probability and physics. For an interesting historical discussion about this constant and its different formulas, see J. Havil [9].

Because of the importance of the constant γ there exists a very rich literature on its inequalities. Here some examples [21], [22], [12], [2], [5], [7], [4], [6]:

$$\frac{1}{2n+2}<H_n-\ln n-\gamma<\frac{1}{2n-2}, \qquad (3)$$

$$\frac{1}{2n+2}<H_n-\ln n-\gamma<\frac{1}{2n}, \qquad (4)$$

$$\frac{1}{2n+2/5}<H_n-\ln n-\gamma<\frac{1}{2n+1/3}, \qquad (5)$$

$$\frac{1}{2n+\dfrac{2\gamma-1}{1-\gamma}}<H_n-\ln n-\gamma<\frac{1}{2n+1/3}, \qquad (6)$$

$$\frac{1}{24(n+1)^2}<H_n-\ln(n+1/2)-\gamma<\frac{1}{24n^2}, \qquad (7)$$

$$\frac{1}{2n}-\frac{1}{12n^2+\dfrac{2(7-12\gamma)}{2\gamma-1}}<H_n-\ln n-\gamma<\frac{1}{2n}-\frac{1}{12n^2+6/5}, \qquad (8)$$

$$\frac{-1}{180n^4}<H_n-\frac{1}{2}\ln(n^2+n+1/3)-\gamma<\frac{-1}{180(n+1)^4}, \qquad (9)$$

$$\frac{8}{2835(n+1)^6}<H_n-\frac{1}{4}\ln[(n^2+n+1/3)^2-1/45]-\gamma$$

$$< \frac{8}{2835 n^6}. \qquad (10)$$

The sequence $\gamma_n = H_n - \ln n$ converges toward its limit γ very slowly like $\frac{1}{n}$, so there are many quicker approximations of the constant γ were established. C. Mortici open a new direction to accelerate the convergence of the sequence $H_n - \ln n - \gamma$ and other sequences see [13]-[20].

By utilizing the Euler-Maclaurin summation formula, the function H_n is asymptotically equal to the (divergent) series

$$\ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, n^{2k}}, \qquad (11)$$

where the B_k ($k = 0,1,2,\ldots$) are the Bernoulli numbers defined by [1]

$$\sum_{j=0}^{\infty} \frac{B_j}{j!} t^j = \frac{t}{e^t - 1}.$$

The first question which arises whether it is possible to determine the sign of the following function for $n \in N$

$$H_n - \ln n - \gamma - \frac{1}{2n} + \sum_{k=1}^{m} \frac{B_{2k}}{2k\, n^{2k}}; \quad m = 0,1,2,\ldots .$$

The second question whether the choice of the constants is the best choice. In this paper we will answer about these two questions, hence we can refine some of the above mentioned inequalities of the sequence $H_n - \ln n - \gamma$.

2. Main Results.

The diagamma function $\psi(x)$ is the logarithmic derivative of the gamma function $\Gamma(x)$.

Using the relations [1]

$$\psi(x+n) = \frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n-1} + \psi(x), \quad n = 1,2,3,\ldots$$

and

$$\psi(1) = -\gamma$$

we have

$$H_n - \gamma = \psi(n+1).$$

Lemma 2.1.

Let $m \geq 0$ be an integer. The function

$$M_m(x) = \ln x + \frac{1}{2x} - \psi(x+1) - \sum_{k=1}^{2m} \frac{B_{2k}}{2k\, n^{2k}}, \qquad (12)$$

is strictly completely monotonic on $(0,\infty)$.

Proof

Firstly consider the function

$$F_m(x) = \ln\Gamma(x) - (x - 1/2)\ln x + x - \ln\sqrt{2\pi} - \sum_{k=1}^{2m} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}, \quad m = 0,1,2,\ldots .$$

Then

$$F'_m(x) = -M_m(x).$$

In [3], Alzer proved that the function $F_m(x)$ is strictly completely monotonic on $(0,\infty)$, that is,

$$(-1)^{r+1} F_m^{(r+1)}(x) > 0, \quad r \geq 1.$$

Then

$$(-1)^r M_m^{(r)}(x) > 0, \quad r \geq 0. \qquad (13)$$

Similarly, using that the function [3]

$$G_m(x) = -F_m(x) + \frac{B_{4m+2}}{(4m+2)(4m+1)x^{4m+1}}, \quad m = 0,1,2,\ldots$$

is strictly completely monotonic on $(0,\infty)$, that is,

$$(-1)^{r+1} G_m^{(r+1)}(x) > 0, \quad r \geq 1.$$

we can conclude the following result:

Lemma 2.2.

Let $m \geq 0$ be an integer. The function

$$K_m(x) = \psi(x+1) - \ln x - \frac{1}{2x} + \sum_{k=1}^{2m+1} \frac{B_{2k}}{2k\, n^{2k}}, \qquad (14)$$

is strictly completely monotonic on $(0,\infty)$, that is,

$$(-1)^r K_m^{(r)}(x) > 0, \quad r \geq 0. \qquad (15)$$

Theorem 1.

For any natural number $n \in N$,

$$\frac{1}{2n} - \sum_{k=1}^{2m+1} \frac{B_{2k}}{2k\, n^{2k}} < H_n - \ln n - \gamma < \frac{1}{2n} - \sum_{k=1}^{2m} \frac{B_{2k}}{2k\, n^{2k}}; \quad m = 0,1,2,...$$

with sharp bounds.

Proof.

Using the relations (13) and (15) at $r = 0$ and $x = n;\ \ n \in N$, we obtain the inequality (16). Now we will prove that the constants in it can not be improved. By the definition of the asymptotic expansion [10], the expansion of a function $F(x)$ obtained from Euler's summation formula of the form

$$F(x) = g(x) + a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ...$$

satisfies for every fixed k, that

$$\lim_{x\to\infty} x^k \left[F(x) - \left(g(x) + a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_k}{x^k} \right) \right] = 0.$$

Then

$$\lim_{n\to\infty} n^{2m} \left[H_n - \ln n - \gamma - \frac{1}{2n} + \sum_{k=1}^{2m-1} \frac{B_{2k}}{2k\, n^{2k}} \right] = -\frac{B_{2m}}{2m}, \quad m = 1,2,3,... . \qquad (17)$$

If we have other constants $c_2, c_4, c_6, ...$ have the property that for all $n \in N$

$$-\frac{c_2}{n^2} - \frac{c_4}{n^4} - \frac{c_6}{n^6} < H_n - \ln n - \gamma - \frac{1}{2n} < -\frac{c_2}{n^2} - \frac{c_4}{n^4},$$

$$-\frac{c_2}{n^2} - \frac{c_4}{n^4} - \frac{c_6}{n^6} - \frac{c_8}{n^8} - \frac{c_{10}}{n^{10}} < H_n - \ln n - \gamma - \frac{1}{2n} < -\frac{c_2}{n^2} - \frac{c_4}{n^4} - \frac{c_6}{n^6} - \frac{c_8}{n^8},$$

$$-\frac{c_2}{n^2} - \frac{c_4}{n^4} - \frac{c_6}{n^6} - \frac{c_8}{n^8} - \frac{c_{10}}{n^{10}} - \frac{c_{12}}{n^{12}} - \frac{c_{14}}{n^{14}} < H_n - \ln n - \gamma - \frac{1}{2n} < -\frac{c_2}{n^2} - \frac{c_4}{n^4} - \frac{c_6}{n^6} - \frac{c_8}{n^8} - \frac{c_{10}}{n^{10}} - \frac{c_{12}}{n^{12}},$$

etc. These inequalities give us that

$$\left.\begin{aligned} &\lim_{n\to\infty} n^2 \left[H_n - \ln n - \gamma - \frac{1}{2n} \right] = -c_2 \\ &\lim_{n\to\infty} n^4 \left[H_n - \ln n - \gamma - \frac{1}{2n} + \frac{c_2}{n^2} \right] = -c_4 \\ &\lim_{n\to\infty} n^6 \left[H_n - \ln n - \gamma - \frac{1}{2n} + \frac{c_2}{n^2} + \frac{c_4}{n^4} \right] = -c_6 \end{aligned}\right\} \qquad (18)$$

etc. Comparing the relations (17) and (18), gives us that

$$c_{2l} = \frac{B_{2l}}{2l}, \quad \forall l \in N. \qquad (19)$$

Then the choice of the constants $\frac{B_{2k}}{2k}$ in the inequality (16) is the best one. To complete our results, we need to prove that the constant $1/2$ in the the sequence $H_n - \ln n - \gamma - \frac{1}{2n}$ can not be improved by any method whatsoever. Consider the sequence

$$a_n = H_n - \ln n - \gamma - \frac{A}{n},$$

then

$$a_n - a_{n+1} = \frac{-n - A}{n(n+1)} + \ln\left(\frac{n+1}{n}\right).$$

Now, let

$$V(x) = \frac{-x - A}{x(x+1)} + \ln\left(\frac{x+1}{x}\right); \quad x > 0$$

then

$$V'(x) = \frac{A(2x+1) - x}{x^2 (x+1)^2}.$$

The function $V(x)$ will be increasing if

$$A > \frac{x}{2x+1} = v(x)$$

and the function $v(x)$ is increasing function with $\lim_{x\to\infty} v(x) = 1/2$. So, the best choice of A is $1/2$. Also, the function $V(x)$ is increasing with limit tends to zero as $x \to \infty$, then

$$V(x) < 0.$$

Hence the sequence a_n is increasing with $a_n \to 0$ as $n \to \infty$, which give us that

$$a_n < 0 \quad or \quad H_n - \ln n - \gamma < \frac{1}{2n}$$

with sharp bound. Now, consider the sequence

$$b_n = H_n - \ln n - \gamma - \frac{1}{2n} + \frac{B}{n^2},$$

then

$$b_{n+1} - b_n = \frac{-2B(1+2n) + n + 3n^2 + 2n^3}{2n^2(n+1)^2} + \ln\left(\frac{n}{n+1}\right).$$

Let

$$T(x) = \frac{-2B(1+2x) + x + 3x^2 + 2x^3}{2x^2(x+1)^2} + \ln\left(\frac{x}{x+1}\right); \qquad x > 0$$

then

$$T'(x) = \frac{4B(1+3x+3x^2) - x(1+x)}{2x^3(x+1)^3}.$$

The function $T(x)$ will be increasing if

$$B > \frac{x(x+1)}{4(1+3x+3x^2)} = t(x)$$

and the function $t(x)$ is increasing function with $\lim_{x\to\infty} t(x) = 1/12$. So, the best choice of B is $1/12$. Also, the function $T(x)$ is increasing with limit tends to zero as $x \to \infty$, then

$$T(x) < 0.$$

Hence the sequence b_n is decreasing with $b_n \to 0$ as $n \to \infty$, which give us that

$$b_n > 0 \quad or \quad H_n - \ln n - \gamma > \frac{1}{2n} - \frac{1}{12n^2}.$$

Hence

$$\frac{1}{2n} - \frac{B_2}{12n^2} < H_n - \ln n - \gamma < \frac{1}{2n}$$

with sharp bounds.

By direct calculations, we can see that the inequality (16) give us a superiority over the inequalities (6) and (8) at some values of the integer m, since

$$\frac{1}{2n + \dfrac{2\gamma - 1}{1-\gamma}} < \frac{1}{2n} - \frac{B_2}{12n^2}, \qquad n > 1,$$

$$\frac{1}{2n} - \sum_{k=1}^{2} \frac{B_{2k}}{2k\, n^{2k}} < \frac{1}{2n + 1/3}, \qquad n \ge 1$$

$$\frac{1}{2n} - \frac{1}{12n^2 + \dfrac{2(7-12\gamma)}{2\gamma - 1}} < \frac{1}{2n} - \sum_{k=1}^{5} \frac{B_{2k}}{2k\, n^{2k}}, \qquad n > 1$$

$$\frac{1}{2n} - \sum_{k=1}^{4} \frac{B_{2k}}{2k\, n^{2k}} < \frac{1}{2n} - \frac{1}{12n^2 + 6/5}, \quad n > 1$$

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