

Numerical Solutions of Quadratic Integral Equations

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Abstract: In this paper, some numerical methods apply for solving quadratic integral equations. A comparative between the methods present from some examples to show the efficiency of each method.

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1. Introduction

Integral equations of the Hammerstein type can be an important tool for modeling the various applied problems arising in engineering and science [1-4]. Many authors studied the existence of the solutions for several classes of quadratic integral equations [5-7]. It is worth mentioning that, there are a few numerical and analytical methods to estimate the solution of the quadratic integral equations [8, 9]. **El-sayed et. al.** [10], concerned with Picard and Adomian decomposition method (ADM). Adomian decomposition method is useful and powerful method for solving nonlinear functional equations. Since this method was first presented [11, 12] in 1980's, Adomian decomposition method has led to several modification[13, 14] in attempt to improve the accuracy of the original method. **Behiry et. al.** [15], introduced a discrete version of the (ADM) and applied it to nonlinear Fredholm integral equations. This method is called a discrete Adomian decomposition method (DADM). **Bakodah and Darwish** [16] proposed an improvement of (DADM) by using some identified Clenshaw-Curtis quadrature rules. This method is called new discrete Adomian decomposition method (NDADM). The aim of this paper is to concern with the application of ADM and some of its modification to approximate the solution of a quadratic integral equation of Hammerstain type of the form

$$x(t) = g(t) + f(t, x(t)) \int_a^b k(t, s) u(s, x(s)) ds, a \leq t, s \leq b \quad (1)$$

and a quadratic integral equation of Hammerstain-Volterra type

$$x(t) = g(t) + f(t, x(t)) \int_a^t k(t, s) u(s, x(s)) ds, a \leq t, s \leq T, \quad (2)$$

where $x(t)$ is an unknown function, $g(t)$, $f(t, x)$ and $u(t, x)$ are given functions.

Quadratic integral equations of the forms (1) and (2) create generalization of several kinds of quadratic integral equations. The necessary and sufficient conditions for existence and uniqueness of solution of Eq. (1) could be found by Argyros, Banaś and Darwish [5, 6, 7].

2- Description of the numerical methods.

In this section we consider the quadratic integral equation (1). Recently, a great deal of interest has been focused on the applications of Adomian decomposition method. In this method the solution is considered as an infinite series which usually converges rapidly to the exact solutions. Applying ADM, the solution $x(t)$ of equation (1) is given by the following series form

$$x(t) = \sum_{k=0}^{\infty} x_k(t) \quad (3)$$

where the components $x_k(t)$, $k \geq 0$, can be computed later on. We represent the nonlinear term $u(x(t), t)$ by the Adomian polynomials, $A_k(t)$, as follows

$$u[x(t), t] = \sum_{k=0}^{\infty} A_k[x_0(t), \dots, x_k(t)] \quad (4)$$

where $A_k(t)$ can be evaluated by the following formula [12]

$$A_k[x_0(t), \dots, x_k(t)] = \frac{1}{k!} \frac{d^k}{dx^k} \left[u \left\{ \sum_{k=0}^{\infty} \alpha^k x_k \right\} \right]_{\alpha=0} \quad (5)$$

By substituting from (3) and (4) into (1) we obtain

$$\sum_{k=0}^{\infty} x_k(t) = g(t) + f(t, \sum_{k=0}^{\infty} x_k(t)) \sum_{k=0}^{\infty} \int_a^b k(t,s) u(s, A_k(s)) ds, a \leq t, s \leq b \quad (6)$$

Now, we can compute the components $x_k(t)$, $k \geq 0$, by the following recursive relations [13].

$$\begin{aligned} x_0(t) &= g(t), \\ x_{k+1}(t) &= f(t, x_k(t)) \int_a^b k(t,s) u(s, A_k(s)) ds, k \geq 0. \end{aligned} \quad (7)$$

This method was modified by Wazwaz [13], the modified form was established based on the divided into two parts, namely \mathcal{G}_0 and \mathcal{G}_1 . Using Wazwaz's modified method, the recursive relation (7) takes the form

$$\begin{aligned} x_0(t) &= g_0(t), \\ x_1(t) &= g_1(t) + f(t, x_0(t)) \int_a^b k(t,s) u(s, A_0(s)) ds, \\ x_{k+2}(t) &= f(t, x_{k+1}(t)) \int_a^b k(t,s) u(s, A_{k+1}(s)) ds, k \geq 0. \end{aligned} \quad (8)$$

In the new modification by Wazwaz and El-Sayed [14], we can replace the process of dividing $g(t)$ into two component by a series of infinite

components, i.e., $g(t) = \sum_{k=0}^{\infty} g_k(t)$. A new recursive relationship expressed in the form

$$\begin{aligned} x_0(t) &= g_0(t), \\ x_{k+1}(t) &= g_{k+1}(t) + f(t, x_k(t)) \int_a^b k(t,s) u(s, A_k(s)) ds, k \geq 0. \end{aligned} \quad (9)$$

It is important to note that if $g(t)$ consists of one term only, then scheme (9) reduces to relation (9). Moreover if $g(t)$ consists of two terms, then relation (9) reduces to the modified relation (8). If the computation of the integral in equation (7) is very complicated we can consider the Taylor expansion of the integrand and consider a few first terms of the expansion. We can observe that algorithm (9) reduces the number of terms involved in each component, and hence the size of calculations is minimized compared to the standard Adomian decomposition method only. The new modification overcomes the difficulty of decomposing $x(t)$ and introduces an efficient algorithm that improves the performance of the standard Adomian decomposition method.

If the evaluation of integral in (7) is analytically impossible, the (ADM) cannot be applied. Discrete Adomian decomposition method (DADM) arises when quadrature methods are apply to compute the definite

integrals (7) which cannot be computed analytically to get a numerical solution at the quadrature rule

$$\int_a^b f(x) dx = \sum_{i=1}^{n+1} f(x_i) \frac{h}{3} \left(\frac{4}{3} f(x_{2i-1}) + \frac{2}{3} f(x_{2i}) + \frac{1}{3} f(x_{2i+1}) \right) \quad (10)$$

Thus

$$\begin{aligned} \int_a^b k(x,t) A_k(t) dt &\cong \frac{h}{3} (k(x, s_0) A(s_0)) + \frac{4h}{3} \sum_{i=1}^{\frac{n}{2}} (k(x, s_{2i-1}) A(s_{2i-1})) \\ &+ \frac{2h}{3} \sum_{i=1}^{\frac{n}{2}} (k(x, s_{2i}) A(s_{2i})) + \frac{h}{3} (k(x, s_n) A(s_n)) \end{aligned} \quad (11)$$

Approximating the definite integral in Eqs. (7), (8) and (9) by applying formula (11) to get the approximate solution of Eq. (1). To approximate solution of Eq. (2), we rewrite it in the form

$$x(t_i) = g(t_i) + f(t_i, x(t_i)) \int_a^{t_i} k(t_i, s) u(s, x(s)) ds \quad (12)$$

Let $s = t_i v$, we obtain

$$x(t_i) = g(t_i) + f(t_i, x(t_i)) \int_a^{t_i} k(t_i, v) u(v, x(v)) dv \quad (13)$$

Then we approximate the integral by applying formula (10). After discretize the independent variable at the nodes used for the quadrature rule in Eq.(10), the solution can be obtained by summing the approximate values to the component $x_k(t)$, $k \geq 1$ represented by one of the equations (7), (8) and (9) at the nodes $s_{n,i} = a + ih$, $i = 0, 1, 2, \dots, n$ and $h = \frac{b-a}{n}$.

As we know, when applying the Clenshaw-Curtis rule to compute definite integrals it gives better result than $\frac{1}{3}$ Simpson rule, [17]. The Clenshaw-Curtis method essentially approximates a function $f(x)$ over any interval $[-1, 1]$ by using the Chebyshev polynomials $T_r(x)$ of degree n , i.e.,

$$f(x) = \sum_{r=0}^n a_r T_r(x), -1 \leq x \leq 1,$$

where a_r are the expansion coefficients and \sum denotes a finite sum whose first term is to be halved

before beginning to sum. Collocating $f(x)$ at the

$n + 1$ points, $x_i = \cos \frac{i\pi}{n}$, $i = 1, 2, \dots, n$, one can

evaluate the expansion coefficients a_r . Thus

$$I(f) = \int_{-1}^1 w(x)f(x)dx \approx \frac{\pi}{2N} \left[f(-1) + f(1) + 2 \sum_{i=1}^{N-1} f\left(\cos \frac{i\pi}{N}\right) \right]. \quad (14)$$

In order to use this numerical method to compute definite integrals in any one of the equations (7), (8)

and (9), we transform the interval $[a, b]$ into the interval $[-1, 1]$ by using the transformation $\tau = \frac{1}{2}[(b - a)x + a + b]$.

Now, we will make use of the following quadrature rules [16]. For example Clenshaw-Curtis five point rule

$$I(f) = \int_{-1}^1 w(x)f(x)dx \approx \frac{1}{15} \left[f(1) + 8f(1/\sqrt{2}) + 12f(0) + 8f(-1/\sqrt{2}) + f(-1) \right] \quad (15)$$

and

$$I(f) = \int_{-1}^1 w(x)f(x)dx \approx \frac{1}{315} \left[9f(-1) + 89f(-\sqrt{3}/2) + 144f(-1/2) + 164f(0) + 144f(1/2) + 89f(\sqrt{3}/2) + 9f(1) \right] \quad (16)$$

To illustrate the description above and to show the efficiency of each method for solving Eq. (1), we introduce some examples with known analytical solutions.

3. Illustrative Examples

In this section, some numerical examples are studied and applying the numerical methods, then comparing the results with exact solutions. The computations associated with the examples were performed using Mathematica 7.0 software.

Example 1: Consider the nonlinear Volterra quadratic integral equation

$$x(t) = t^2 - \frac{t^{10}}{35} + \frac{1}{5}t \int_0^t s^2 x^2(s) ds,$$

where its exact solution is given by $x(t) = t^2$.

By applying the first modification on ADM introduced by Wazwaz, Eq. (8), and comparing the results with standard ADM, we have

$$x_0(t) = t^2,$$

$$x_1(t) = \frac{-t^{10}}{35} + \frac{1}{5}t \int_0^t s^2 B_0(s) ds,$$

$$x_{n+2}(t) = \frac{1}{5}t \int_a^b s^2 B_{n+1}(s) ds, n \geq 0.$$

The first few Adomian polynomials for B_n can be given by

$$B_0 = x_0^2(t)$$

$$B_1 = 2x_0(t)x_1(t)$$

$$B_2 = \frac{1}{2}(2x_1^2(t) + 4x_0(t)x_2(t))$$

$$B_3 = \frac{1}{6}(2x_1(t)x_2(t) + 12x_0(t)x_3(t))$$

⋮

and so on.

In Table 1, the absolute error $|e_m| = |x_{exact} - x_{approx}|$, where m is the number of components, is given.

Table 1: comparison of the absolute error between ADM and the first modification, Eq.(8).

t	The modified ADM	Standard ADM
	m = 5	m = 5
0.2	2.92175 × 10 ⁻²³	4.13724 × 10 ⁻¹⁶
0.4	1.96074 × 10 ⁻¹⁵	1.08453 × 10 ⁻¹⁰
0.6	7.42611 × 10 ⁻¹¹	1.60199 × 10 ⁻⁷
0.8	1.31479 × 10 ⁻⁷	7.35011 × 10 ⁻⁴
1.0	0.0000433099	6.84836 × 10 ⁻³

Example 2: Consider the following nonlinear quadratic integral equation

$$x(t) = t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110} + \frac{1}{10}t^3 x^2(t) \int_0^t (s+1)x^3(s) ds$$

with exact solution $x(t) = t^3$.

By applying new modified ADM, we get

$$x_0(t) = t^3$$

$$x_1(t) = \frac{-t^{19}}{100} + \frac{1}{10}t^3 A_0(t) \int_0^t (s+1)B_0(s) ds$$

$$x_2(t) = -\frac{t^{20}}{110} + \frac{1}{10}t^3 A_1(t) \int_a^b s^2 B_1(s) ds$$

$$x_{n+3}(t) = \frac{1}{10}t^3 A_{n+2}(t) \int_a^b s^2 B_{n+2}(s) ds, n \geq 0$$

The first few Adomian polynomials for A_n and B_n are given as

$$A_0 = x_0^2(t)$$

$$A_1 = 2x_0(t)x_1(t)$$

$$A_2 = \frac{1}{2}(2x_1^2(t) + 4x_0(t)x_2(t))$$

$$A_3 = \frac{1}{6}(2x_1(t)x_2(t) + 12x_0(t)x_3(t))$$

⋮

and so on, and

$$B_0 = x_0^3(t)$$

$$B_1 = 3x_0^2(t)x_1(t)$$

$$B_2 = \frac{1}{2}(6x_0(t)x_1^2(t) + 6x_0^2(t)x_2(t))$$

$$B_3 = \frac{1}{6}(6x_1^3(t) + 36x_0(t)x_1(t)x_2(t) + 18x_0^2(t)x_3(t))$$

⋮

and so on.

Table 2 shows the absolute error $|e_m| = |x_{exact} - x_{approx}|$, where m is the number of components.

Table 2: comparison of the absolute error between ADM and the second modification, Eq.(9).

t	The new modified ADM	Standard ADM
	$m = 5$	$m = 5$
0.2	2.98711×10^{-33}	9.58466×10^{-29}
0.4	7.26011×10^{-27}	4.39063×10^{-18}
0.6	3.06194×10^{-23}	8.21177×10^{-12}
0.8	8.48642×10^{-15}	2.42074×10^{-7}
1.0	4.97888×10^{-8}	0.000731679

Example 3: Consider the quadratic integral equation

$$x(t) = f(t) + \frac{t x(t)}{\sin(t) + e^t} \int_0^1 s^2 \tanh(t) x^2(s) ds, t, s \in [0,1]$$

where $f(t)$ is chosen such that the exact solution is given by $x(t) = e^t \sin t$.

In this example we used Simpson's rule to approximate the integrals, then comparing the results with exact solution. Table 3 shows the computed absolute error $|e_m^n| = |x_{exact} - x_{approx}|$ where m is the number of components and $n = 2, 4, 8$ are the number of the nodes of the quadrature rule.

Example 4: Consider the quadratic integral equation

$$x(t) = \frac{1+t}{10} + e^{-t} \frac{x^2(t)}{30} + \frac{x(t)}{5} \int_0^1 \frac{s}{1+t+s^2} \cosh(x(t)) ds, t \in [0,1].$$

Table 3: The absolute error of DADM, Eq. (11) with $n = 2, 4, 8$.

t	The DADM 1/3 Simpson rule		
	$n=2$	$n=4$	$n=8$
0.2	0.000881111	0.000059901	3.764990×10^{-6}
0.4	0.006147350	0.000416844	0.0000261963
0.6	0.018244400	0.001233880	0.0000775313
0.8	0.037941300	0.002560360	0.0001608610
1.0	0.064537800	0.004348130	0.0002731580

In [18] Salem proved that, the above equation has at least one continuous solution $x(t)$ with $0.1 \leq x(t) \leq 0.3$. Using new discrete Adomian decomposition method to approximate the integrals eq.(15). Since the exact solution is not known, we

compute the absolute error between x_m and x_{m+1} . Table 4 shows the computed absolute error $|e_m^n| = |x_{m+1} - x_m|$ where m is the iterative number and n is the number of the nodes of the quadrature rule.

Table 4: the absolute error example 4.

t	The new DADM, n=5 [16]		
	$m = 1$	$m = 2$	$m = 3$
0.0	0.00216759	1.5701×10^{-7}	1.03796×10^{-15}
0.14644661	0.0137835	5.47716×10^{-6}	8.65306×10^{-13}
0.5	0.0171361	5.9443×10^{-6}	7.16267×10^{-13}
0.85355339	0.0100062	1.42369×10^{-6}	2.98254×10^{-14}
1.0	0.00161367	3.21519×10^{-8}	1.35733×10^{-16}

The approximate solutions $xa(t)$ at these value of t recorded in Table 5 and it verify the condition proved in [18].

Table 5: the approximate solution of example 4.

t	The new DADM, n=5 [16]	
	$xa(t)$	$0.1 \leq x(t) \leq 0.3$
0.0	0.102168	
0.14644661	0.128439	
0.5	0.167148	
0.85355339	0.195364	
1.0	0.201614	

4. Conclusion

Some practical problems lead to quadratic integral equations. These types of equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, using some modified numerical methods to approximate the solution after applying Adomian

decomposition method. (ADM) has many advantages such as simplicity, high accuracy and the solution when it exists is found in rapidly convergent series form. For cases that evaluation of integrals analytically is impossible or complicated the (ADM) cannot be applied. Thus, we used to convert the non-numerical (ADM) to a numerical results state that the method has good accuracy and remarkable performance. Also, approximate solution may be more accurate using larger n .

In this paper we proposed some numerical method for solving the quadratic integral equations. The obtained results showed that these modification of the Adomian decomposition method can be flexible to solve many different problem effectively.

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