On an implicit iterative splitting scheme for the problems of free thermal convection

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Abstract. This paper considers some implicit iterative splitting algorithms for the difference analogues of the system of free convection steady-state equations in variables “velocity vector and pressure”, written to shifted grids with symmetric approximation. The problems of stability of the difference problems according to the initial data and the right member, convergence and estimate of the linear algorithm degree of convergence were studied.
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Introduction

Sufficient number of papers [1-9] is devoted to numerical study of thermal convection differential problems described by equations (24) - (26). Computational algorithms to study the convective currents of incompressible liquid in a broad range of medium parameters were developed. However, there is no mathematical justification of the applicability of algorithms used in practice.

Works [10], [11] considered iterative schemes based on the approximation of the convective sums by Samarskiy formula [6] for the numerical solution of difference equations of heat convection corresponding to difference analog of a differential problem (24) - (27), and [7] there were also investigated the questions of sustainability and numerical implementation. It’s rather difficult to study the convergence of iterative algorithms proposed in works [10], [11], in view of the fact that the coefficient of system viscosity nonlinearly depends on the values of the velocity components.

The main part

When $0<t\leq T<\infty$ the cube of $D=\{0\leq x_\alpha \leq 1,\ \alpha=1,2,3\}$ let's consider thermal convection equations in the Boussinesq approximation, written to non-dimensional variables.

$$\frac{\partial \vec{u}}{\partial t}+(\vec{u}\ \nabla)\vec{u}+gradp=\Delta\vec{u}-\frac{Gr\vec{g}}{|\vec{g}|}\theta+\vec{f}(t,x), \quad (1)$$

$$div\vec{u}=0, \quad (2)$$

$$\frac{\partial\theta}{\partial t}+(\vec{u}\nabla)\theta=\frac{1}{Pr}\cdot\Delta\theta+g(t,x), \quad (3)$$

where $x=(x_1,x_2,x_3)$, $\vec{f}(t,x)$, $g(t,x)$ - prescribed functions, Gr – Grashof number, $\Pr$ - Prandtl number, $\theta(t,x)$ – temperature, $\vec{u}=(u,v,w)$ - velocity vector ($u=u(t,x), v=v(t,x), w=w(t,x)$) and $p(t,x)$ – pressure, $\vec{g}$ - gravitation force.

Boundary conditions: $\vec{u}=\theta=0$ при $x\in\partial D$ (4)

Initial conditions:

$$\vec{u}(0,x_1,x_2,x_3)=\vec{u}^0(x_1,x_2,x_3),\quad \theta(0,x_1,x_2,x_3)=\theta^0(x_1,x_2,x_3), \quad (5)$$

To define a grid difference:

$$t_n=\tau n,\quad n=0,...,\ \frac{T}{\tau}=M,$$

$$D_h=\left\{(kh,lh,mh),\ k,l,m=\overline{0,N},\ Nh=1\right\}$$

$$D_{hu}=\left\{((k+1/2)h,lh,mh),\ k=\overline{0,N-1},\ l,m=\overline{0,N}\right\}$$

$$D_{hv}=\left\{(kh,(l+1/2)h,mh),\ k,m=\overline{0,N},\ l=\overline{0,N-1}\right\}$$

$$D_{hw}=\left\{(kh,lh,(m+1/2)h),\ k,l=\overline{0,N},\ m=\overline{0,N-1}\right\}$$

Approximation values of pressure and temperature we determine in D_h grid nodes, and the components of the difference velocity vector

$$\vec{u}=(u_{k+1/2\,lm},\ v_{kl+1/2\,m},\ w_{klm+1/2})'$$

we find in D_{hu}, D_{hv}, D_{hw} grid nodes respectively.

Boundary values for the standard components of the velocity vector we suppose to be equal to zero on a half-step from the domain wall, id est:

$$u_{\frac{1}{2}lm} = u_{N-\frac{1}{2}lm} = 0, \quad l,m = 0,1,...,\ N,$$

$$v_{k\frac{1}{2}m} = v_{kN-\frac{1}{2}m} = 0, \quad k,m = 0,1,...,\ N,$$

$$w_{kl\frac{1}{2}} = w_{klN-\frac{1}{2}} = 0, \quad k,l = 0,1,...,\ N.$$

Let's note that the change of boundary conditions on the border, considering the continuity equation, corresponds to the second order of approximation.

To approximate the system of equations (1) - (3), let's consider the following splitting scheme

$$\frac{\vec{u}^{n+\frac{1}{2}} - \vec{u}^n}{\tau} + L_{h,\vec{u}}\vec{u}^{n+\frac{1}{2}} + \overline{grad}_h\, p^n = \tilde{\Delta}_h \vec{u}^{n+\frac{1}{2}} - \frac{Gr\vec{g}}{|\vec{g}|}\theta^{n+1} + \vec{f}^{\,n}(x), \qquad (6)$$

$$\frac{\vec{u}^{n+1} - \vec{u}^{n+\frac{1}{2}}}{\tau} + \overline{grad}_h\left(p^{n+1} - p^n\right) = 0, \qquad (7)$$

$$\underline{div}_h \vec{u}^{n+1} = 0, \qquad (8)$$

$$\frac{\theta^{n+1} - \theta^n}{\tau} + L_{h,\theta}\theta^{n+1} = \frac{1}{\Pr}\tilde{\Delta}_h\theta^{n+1} + g^n(x), \qquad (9)$$

with appropriate initial and uniform boundary conditions, where $\overline{grad}_h p = \{p_{x_1}, p_{x_2}, p_{x_3}\}$, that is, to approximate pressure derivatives there were used "right-hand" difference formulas; $\underline{div}_h \vec{u} = u_{\bar{x}_1} + v_{\bar{x}_2} + w_{\bar{x}_3}$, that is, to approximate the divergence operator there used the "left-hand" difference formulas.

$\tilde{\Delta}_h$ - corresponds to Laplace difference operator when approximating the convective sums by Samarskiy formula, $L_{h,\vec{u}}$, $L_{h,\theta}$ - difference operators related to the approximation of the convective sums and meeting the conditions of energy neutrality, id est

$$(L_{h,\vec{u}}\vec{u},\vec{u}) = (L_{h,\theta}\theta,\theta) = 0. \qquad (10)$$

We show that the solutions of the difference problems (6) - (9) are stable in the sense of the initial data and according to the right member.

Multiplying scalarly both members (6) by $2\tau\vec{u}^{n+\frac{1}{2}}$ and taking into account (10), we get

$$\left\|\vec{u}^{n+\frac{1}{2}}\right\|^2 - \left\|\vec{u}\right\|^2 + \left\|\vec{u}^{n+\frac{1}{2}} - \vec{u}^n\right\|^2 + 2\tau\left(\overline{grad}_h\, p^n, \vec{u}^{n+\frac{1}{2}}\right) +$$

$$2\tau\left\|\nabla_h \vec{u}^{n+\frac{1}{2}}\right\|^2 = -2\tau\frac{Gr}{|\vec{g}|}\left(\vec{g}\theta^{n+1}, \vec{u}^{n+\frac{1}{2}}\right) + 2\tau\left(\vec{f}^{\,n}, \vec{u}^{n+\frac{1}{2}}\right) \qquad (11)$$

Here and below $\|\cdot\|$ - norm function in L_2.

Multiplying (7) by $\vec{u}^{n+1} + \vec{u}^{n+\frac{1}{2}} + \tau^2\left(\overline{grad}_h\, p^{n+1} + \overline{grad}_h\, p^n\right)$ and taking account (8) we get

$$\left\|\vec{u}^{n+1}\right\|^2 + \tau^2\left\|\overline{grad}_h\, p^{n+1}\right\|^2 - 2\tau(\vec{u}^{n+\frac{1}{2}}, \overline{grad}_h\, p^n) = \left\|\vec{u}^{n+\frac{1}{2}}\right\|^2 + \tau^2\left\|\overline{grad}_h\, p^n\right\|^2 \qquad (12)$$

Summing up partially (12) and (11) and applying Cauchy-Bunyakovskiy estimate for scalar products we have

$$\left\|\vec{u}^{n+1}\right\|^2 + \tau^2\left\|\overline{grad}_h p^{n+1}\right\|^2 + \left\|\vec{u}^{n+\frac{1}{2}} - \vec{u}^n\right\|^2 + 2\tau\left\|\nabla_{h,\vec{u}}\vec{u}^{n+\frac{1}{2}}\right\|^2 \le$$

$$\left\|\vec{u}^n\right\|^2 + \tau^2\left\|\overline{grad}_h p^n\right\|^2 + 2\tau|Gr|\left\|\theta^{n+1}\right\|\left\|\vec{u}^{n+\frac{1}{2}}\right\| + 2\tau\left\|\vec{f}^{\,n}\right\|\left\|\vec{u}^{n+\frac{1}{2}}\right\|.$$

Using the known inequations

$$\delta_0\left\|\vec{u}\right\|^2 \le \left\|\nabla\vec{u}\right\|^2 \text{ и}$$

$$a \cdot b \le \varepsilon a^2 + \frac{b^2}{4\varepsilon},$$

where δ_0 - minimum eigenvalue of Laplace difference operator, ε - certain positive number, we obtain

$$\left\|\vec{u}^{n+1}\right\|^2 + \tau^2\left\|\overline{grad}_h\, p^{n+1}\right\|^2 + \left\|\vec{u}^{n+\frac{1}{2}} - \vec{u}^n\right\|^2 + 2\tau\left(1 - \frac{\varepsilon_1 Gr}{\delta_0} - \frac{\varepsilon_2}{\sqrt{\delta_0}}\right)\left\|\nabla_h\vec{u}^{n+\frac{1}{2}}\right\|^2 \le$$

$$\le \left\|\vec{u}^n\right\|^2 + \tau^2\left\|\overline{grad}_h\, p^n\right\|^2 + \frac{\tau Gr}{2\varepsilon_1\delta_0}\left\|\nabla_h\theta^{n+1}\right\|^2 + \frac{\tau}{2\varepsilon_2\sqrt{\delta_0}}\left\|\vec{f}^{\,n}\right\|^2, \qquad (13)$$

where ε_1, ε_2 - certain positive numbers.

Multiplying scalarly both members (9) by $2\tau\theta^{n+1}$ and keeping analogous transformations we have

$$\left\|\theta^{n+1}\right\|^2 - \left\|\theta^n\right\|^2 + \left\|\theta^{n+1} - \theta^n\right\|^2 + \frac{2\tau}{\Pr}(1-\varepsilon_3)\left\|\tilde{\nabla}_{h,\theta}\theta^{n+1}\right\|^2 \le \frac{\tau\Pr}{2\delta_0\varepsilon_3}\left\|\vec{g}^{\,n}\right\|^2, \qquad (14)$$

where ε_3 - certain positive number.

Multiplying both members (14) by $\frac{Ra}{4\varepsilon_1(1-\varepsilon_3)\delta_0}$, taking $0 < \varepsilon_3 < 1$, where $Ra = Gr\Pr$ - Rayleigh number, adding obtained inequation partially to (13), in which the following representation is applied

$$\frac{\tau Gr}{2\varepsilon_1\delta_0}\left\|\tilde{\nabla}_h\theta^{n+1}\right\|^2 = \frac{2\tau}{\Pr}(1-\varepsilon_3)\frac{Gr\Pr}{4\varepsilon_1(1-\varepsilon_3)\delta_0}\left\|\tilde{\nabla}_{h,\theta}\theta^{n+1}\right\|^2,$$

and introducing the notation

$$E^n = \left\|\vec{u}^n\right\|^2 + \tau^2\left\|\overline{grad}_h\, p^n\right\|^2 + \frac{Ra}{4\varepsilon_1(1-\varepsilon_3)\delta_0}\left\|\theta^n\right\|^2$$

we obtain

$$E^{n+1}+2\tau\left(1-\frac{\varepsilon_1 Gr}{\delta_0}-\frac{\varepsilon_2}{\sqrt{\delta_0}}\right)\left\|\tilde{\nabla}_h \vec{u}^{\,n+1/2}\right\|^2+\left\|\vec{u}^{\,n+1/2}-\vec{u}^{\,n}\right\|^2+\frac{Ra}{4\varepsilon_1(1-\varepsilon_3)\delta_0}\left\|\theta^{n+1}-\theta^n\right\|^2 \le$$
$$\le E^n+\frac{\tau}{2\sqrt{\delta_0}\varepsilon_2}\left\|\vec{f}^{\,n}\right\|^2+\frac{Ra\Pr\tau}{8\varepsilon_1(1-\varepsilon_3)\delta_0^2\varepsilon_3}\left\|\vec{g}^{\,n}\right\|^2, \quad (15)$$

where $\varepsilon_1, \varepsilon_2$ - certain positive numbers, $0<\varepsilon_3<1$. Let's choose $\varepsilon_1, \varepsilon_2$ from conditions:

$$1-\frac{\varepsilon_1 Gr}{\delta_0}-\frac{\varepsilon_2}{\sqrt{\delta_0}} \ge \delta > 0 .$$

Then from (15)

$$E^n \le E^0+\frac{n\tau}{2\sqrt{\delta_0}\varepsilon_2}\|\vec{f}\|_*^2+\frac{Ra\ \Pr}{8\varepsilon_1(1-\varepsilon_3)\delta_0^2\varepsilon_3}n\tau\|g\|_*^2$$

If it is remembered that by the construction of grid $n\tau \le T$, then from the last inequation

$$E^n \le E^0+\frac{T}{2\sqrt{\delta_0}\varepsilon_2}\|\vec{f}\|_*^2+\frac{Ra\ \Pr}{8\varepsilon_1(1-\varepsilon_3)\delta_0^2\varepsilon_3}T\|g\|_*^2,$$

where

$\left\|\vec{f}\right\|_* = \max_{0\le n\le M}\left\|\vec{f}^{\,n}\right\|$, $\|g\|_* = \max_{0\le n\le M}\left\|g^n\right\|$. By this means, we have established the theorem stating stability of problem solution (6) - (9) by initial and by right member.

Theorem 1. If $L_{h,\vec{u}}$, $L_{h,\theta}$ are energetically neutral, then there are positive constants M_1, M_2, which are independent of h and τ, and whereby the following estimate is valid

$$E^n \le E^0+M_1\|\vec{f}\|_{(*)}^2+M_2\|g\|_{(*)}^2,$$

where

$$E^n=\left\|\vec{u}^n\right\|^2+\tau^2\left\|\overline{grad}_h p^n\right\|^2+c\left\|\theta^n\right\|^2,\quad \|\vec{f}\|_*=\max_{0\le n\le M}\left\|\vec{f}^n\right\|,\quad \|g\|_*=\max_{0\le n\le M}\left\|g^n\right\|,$$

c - positive constant.

If $\vec{f}=0$ and $g=0$, then the inequation (15) takes the form

$$E^{n+1}+\left\|\vec{u}^{\,n+1/2}-\vec{u}^{\,n}\right\|^2+\tau^2\left\|\overline{grad}_h(p^{n+1}-p^n)\right\|^2+$$
$$2\tau\left(1-\frac{\varepsilon_1 Gr}{\delta_0}-\frac{\varepsilon_2}{\sqrt{\delta_0}}\right)\left\|\tilde{\nabla}_h\vec{u}^{\,n+1/2}\right\|^2+\frac{Ra}{4\varepsilon_1(1-\varepsilon_3)\delta_0}\left\|\theta^{n+1}-\theta^n\right\|^2\le E^n. \quad (16)$$

Passing into (16) to the limit we obtain that the solution of (6) - (9) with uniform right members converges to zero stationary solution for any given initial data from $L_2(D_h)$.

Next, we consider the stationary problem. For stationary difference problem

$$\begin{cases} L_{h,\vec{u}}\vec{u}+\overline{grad}_h\, p=\tilde{\Delta}_h\vec{u}-\dfrac{Gr\vec{g}}{|\vec{g}|}\theta+\vec{f}(x), & (17)\\ \underline{div}\,\vec{u}=0 & (18)\\ L_{h\theta}\theta=\dfrac{1}{Pr}\cdot\tilde{\Delta}\theta+g(x),\ x\in D_h & (19)\end{cases}$$

(17), (18),(19)

$$\vec{u}\,|_{\partial D_h}=\theta\,|_{\partial D}=0 \quad (20)$$

the following theorem is valid.

Theorem 2. For solution of $\vec{u}$ and θ of the problems (17) - (20) prior estimates are valid

$$\left\|\tilde{\nabla}_h\theta\right\| \le \frac{\Pr\|g\|}{\sqrt{\delta_0}}, \quad (21)$$

$$\left\|\tilde{\nabla}_h\vec{u}\right\| \le \frac{Ra}{\delta_0\sqrt{\delta_0}}\|g\|+\frac{1}{\sqrt{\delta_0}}\|\vec{f}\|. \quad (22)$$

The proof:

Multiplying scalarly by $\vec{u}$ (17), taking into account (18) equalities, $(\overline{grad}_h\, p,\vec{u})=-(p,div_h\vec{u})$ and $(L_{h,u}\vec{u},\vec{u})=0$, we obtain

$$(\tilde{\nabla}_h\vec{u},\tilde{\nabla}_h\vec{u})=-\left(\frac{Gr\vec{g}}{|\vec{g}|}\theta,\vec{u}\right)+\left(\vec{f},\vec{u}\right)$$

Using Cauchy-Bunyakovskiy inequalities and $\sqrt{\delta_0}\|\vec{u}\|\le\|\tilde{\nabla}\vec{u}\|$, we have

$$\left\|\tilde{\nabla}_h\vec{u}\right\| \le \frac{Gr}{\delta_0}\left\|\tilde{\nabla}_h\theta\right\|+\frac{1}{\sqrt{\delta_0}}\left\|\vec{f}\right\|. \quad (23)$$

Multiplying (19) scalarly by θ taking into account the equality (10) we obtain

$$\left\|\tilde{\nabla}_h\theta\right\|^2=Pr(g,\theta).$$

Therefore, applying Cauchy-Bunyakovskiy inequalities and $\|\theta\|\le\dfrac{1}{\sqrt{\delta_0}}\left\|\tilde{\nabla}_h\theta\right\|$ we arrive at the inequality (21).

Estimate (22) follows from (23) and (21). The theorem is proved.

$$(\vec{u}\nabla)\vec{u}+gradp=\Delta\vec{u}-\frac{\vec{g}}{|\vec{g}|}\cdot Gr\,\theta+\vec{f}(x), \quad (24)$$

$$div\ \vec{u}=0, \quad (25)$$

$$(\vec{u}\,\nabla)\theta=\frac{1}{Pr}\cdot\Delta\theta+g(x), \quad (26)$$

$$\vec{u}\,|_{\partial D}=\theta\big|_{\partial D}=0. \quad (27)$$

Next, we consider the question of convergence of the iterative splitting algorithm.

$$\frac{\vec{u}^{n+1/2}-\vec{u}^{n}}{\tau}+L_{h,\vec{u}}\vec{u}^{n+1/2}+\overline{grad}_{h}\,p^{n}=\Delta_{h}\vec{u}^{n+1/2}-\frac{Gr\vec{g}}{|\vec{g}|}\theta^{n+1}+\vec{f}(x) \quad (28)$$

$$\frac{\vec{u}^{n+1}-\vec{u}^{n+1/2}}{\tau}+\overline{grad}_{h}(p^{n+1}-p^{n})=0 \quad (29)$$

$$div_{h}\vec{u}^{n+1}=0, \quad (30)$$

$$\frac{\theta^{n+1}-\theta^{n}}{\tau}+L_{h,\theta}\theta^{n+1}=\frac{1}{\Pr}\Delta_{h}\theta^{n+1}+g(x), \quad (31)$$

with initial uniform zero boundary conditions to the solution of the difference analogue of the stationary problem.

Introducing iteration errors

$$\vec{z}^{n+1/2}=\vec{u}^{n+1/2}-\vec{u},\quad \vec{z}^{n}=\vec{u}^{n}-\vec{u},\quad \pi^{n}=p^{n}-p,\quad T^{n}=\theta^{n}-\theta$$

where $\vec{u}$, p, θ - solutions of corresponding grid stationary difference problem.

Let's first analyze linear situation.

$$\frac{\vec{u}^{n+1/2}-\vec{u}^{n}}{\tau}+\overline{grad}_{h}\,p^{n}=\Delta_{h}\vec{u}^{n+1/2}-\frac{Gr\vec{g}}{|\vec{g}|}\theta^{n+1}+\vec{f}(x) \quad (32)$$

$$\frac{\vec{u}^{n+1}-\vec{u}^{n+1/2}}{\tau}+\overline{grad}_{h}(p^{n+1}-p^{n})=0 \quad (33)$$

$$div_{h}\vec{u}^{n+1}=0, \quad (34)$$

$$\frac{\theta^{n+1}-\theta^{n}}{\tau}=\frac{1}{\Pr}\Delta_{h}\theta^{n+1}+g(x) \quad (35)$$

The following theorem is valid.

Theorem 3. Iterative algorithm (32) - (35) with uniform boundary conditions converges to the solution of the corresponding stationary difference problem with geometrical progression velocity, in this connection there is the estimate

$$F^{n+1}\le qF^{n},\quad 0<q<1,$$

where

$$F^{n}=c_{1}\|\vec{z}^{n}\|^{2}+\tau^{2}\|grad_{h}\pi^{n}\|^{2}+(c_{2}+c_{3}\tau)\|T^{n}\|^{2}$$

c_1, c_2, c_3 - positive constants.

The proof:

In linear situation the equations for error are given by

$$\frac{\vec{z}^{n+1/2}-\vec{z}^{n}}{\tau}+\overline{grad}_{h}\pi^{n}=\Delta_{h}\vec{z}^{n+1/2}-\frac{Gr}{|\vec{g}|}\vec{g}T^{n+1}, \quad (36)$$

$$\frac{\vec{z}^{n+1}-\vec{z}^{n+1/2}}{\tau}+\overline{grad}_{h}(\pi^{n+1}-\pi^{n})=0 \quad (37)$$

$$div_{h}\vec{z}^{n+1}=0, \quad (38)$$

$$\frac{T^{n+1}-T^{n}}{\tau}=\frac{1}{\Pr}\Delta_{h}T^{n+1} \quad (39)$$

with uniform boundary conditions for the component of the velocity and temperature.

At first we examine temperature error.

Multiplying both members (39) by $2\tau T^{n+1}$ scalarly we obtain

$$\|T^{n+1}\|^{2}+\|T^{n+1}-T^{n}\|^{2}+\frac{2\tau}{\Pr}\|\nabla_{h}T^{n+1}\|^{2}=\|T^{n}\|^{2} \quad (40)$$

Multiplying (36) scalarly by $2\tau\vec{z}^{n+1/2}$ we obtain the following inequation

$$\|\vec{z}^{n+1/2}\|^{2}-\|\vec{z}^{n}\|^{2}+\|\vec{z}^{n+1/2}-\vec{z}^{n}\|^{2}+2\tau(\overline{grad}_{h}\pi^{n},\vec{z}^{n+1/2})+2\tau\|\nabla_{h}\vec{z}^{n+1/2}\|^{2}\le 2\tau Gr\left|\left(\frac{\vec{g}}{|\vec{g}|}T^{n+1},\vec{z}^{n+1/2}\right)\right|. \quad (41)$$

From equalities $\vec{z}^{n+1/2}=\vec{z}^{n+1}+\tau\,\overline{grad}_{h}(\pi^{n+1}-\pi^{n})$, got from (37), and $(\vec{z}^{n+1},\overline{grad}_{h}\pi^{n+1})=-(div\vec{z}^{n+1},\pi^{n+1})=0,$ got from (38), the next equations are followed

$$\|\vec{z}^{n+1/2}\|^{2}=\|\vec{z}^{n+1}\|^{2}+\tau^{2}\|\overline{grad}_{h}\pi^{n+1}\|^{2}+\tau^{2}\|\overline{grad}_{h}\pi^{n}\|^{2}-2\tau(\vec{z}^{n+1},\overline{grad}_{h}\pi^{n})-2\tau^{2}(\overline{grad}_{h}\pi^{n+1},\overline{grad}_{h}\pi^{n});$$

$$2\tau(\overline{grad}_{h}\pi^{n},\vec{z}^{n+1/2})=2\tau(\overline{grad}_{h}\pi^{n},\vec{z}^{n+1})+2\tau^{2}(\overline{grad}_{h}\pi^{n},\overline{grad}_{h}\pi^{n+1})-2\tau^{2}\|\overline{grad}_{h}\pi^{n}\|^{2}$$

Substituting these values into (41) inequation we obtain

$$\|\vec{z}^{n+1}\|^{2}+\tau^{2}\|\overline{grad}_{h}\pi^{n+1}\|^{2}+\|\vec{z}^{n+1/2}-\vec{z}^{n}\|^{2}+2\tau\|\nabla_{h}\vec{z}^{n+1/2}\|^{2}\le \|\vec{z}^{n}\|^{2}+\tau^{2}\|\overline{grad}_{h}\pi^{n}\|^{2}+2\tau\frac{Gr}{|\vec{g}|}\left|(\vec{g}T^{n+1},\vec{z}^{n+1/2})\right|. \quad (42)$$

Applying the already known estimates for (42) we obtain

$$\|\vec{z}^{n+1}\|^{2}+\tau^{2}\|\overline{grad}_{h}\pi^{n+1}\|^{2}+\|\vec{z}^{n+1/2}-\vec{z}^{n}\|^{2}+$$

$$2\tau\left(1-\frac{\varepsilon_1 Gr}{\delta_0}\right)\left\|\nabla_h z^{n+1/2}\right\|^2 \le \left\|z^n\right\|^2+\tau^2\left\|grad_h \pi^n\right\|^2+\frac{\tau Gr}{2\delta_0\varepsilon_1}\left\|\nabla_h T^{n+1}\right\|^2, \quad (43)$$

where ε_1 - certain positive number.

Write over the equality (40) as

$$\left\|T^{n+1}\right\|^2+\left\|T^{n+1}-T^n\right\|^2+\frac{2\tau}{\Pr}(1-\varepsilon_2)\left\|\nabla_h T^{n+1}\right\|^2+\frac{2\tau}{\Pr}\varepsilon_2\left\|\nabla_h T^{n+1}\right\|^2=\left\|T^n\right\|^2,$$

with certain $\varepsilon_2 \in (0;1]$.

Multiplying both members of the equality by $\frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}$, where $Ra = Gr\Pr$ - Rayleigh number, we obtain

$$\frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\left\|T^{n+1}\right\|^2+\frac{\tau Gr}{2\delta_0\varepsilon_1\varepsilon_2}(1-\varepsilon_2)\left\|\nabla_h T^{n+1}\right\|^2+\frac{\tau Gr}{2\delta_0\varepsilon_1}\left\|\nabla_h T^{n+1}\right\|^2 \le \frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\left\|T^n\right\|^2 \quad (44)$$

Adding (44) and (43) we obtain the inequation

$$E^{n+1}+\left\|z^{n+1/2}-z^n\right\|^2+2\tau\left(1-\frac{\varepsilon_1 Gr}{\delta_0}\right)\left\|\nabla_h z^{n+1/2}\right\|^2+\frac{\tau Gr(1-\varepsilon_2)}{2\delta_0\varepsilon_1\varepsilon_2}\left\|\nabla_h T^{n+1}\right\|^2 \le E^n, \quad (45)$$

where

$$E^n=\left\|z^n\right\|^2+\tau^2\left\|grad_h \pi^n\right\|^2+\frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\left\|T^n\right\|^2.$$

Therefore, setting $0<\varepsilon_1<\frac{\delta_0}{Gr}$ we obtain $0<E^{n+1}\le E^n$ for any n, id est the convergence of sequence E^n.

Expressing $\tau\, grad_h \pi^n$ from (36) and carrying on the known estimates we get

$$\tau\left\|grad_h \pi^n\right\| \le \tau\left\|\Delta_h z^{n+1/2}\right\|+\frac{\tau Gr}{\sqrt{\delta_0}}\left\|\nabla_h T^{n+1}\right\|+\left\|z^{n+1/2}-z^n\right\|$$

Applying estimates

$$\left\|\Delta_h z^{n+1/2}\right\| \le \frac{c_0}{h}\left\|\nabla_h z^{n+1/2}\right\|,$$

where c_0 invariable which is independent of $\vec{z}$ and h, and $\left\|\pi^n\right\| \le \frac{1}{\sqrt{\delta_0}}\left\|grad_h \pi^n\right\|$, we arrive at the inequality

$$\tau\left\|grad_h \pi^n\right\| \le \frac{c_0\tau}{h}\left\|\nabla_h z^{n+1/2}\right\|+\frac{\tau Gr}{\sqrt{\delta_0}}\left\|\nabla_h T^{n+1}\right\|+\left\|z^{n+1/2}-z^n\right\|. \quad (46)$$

Squaring both members (46) and applying inequation $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ we obtain

$$\tau^2\left\|grad_h \pi^n\right\|^2 \le 3\left(\frac{c_0^2\tau^2}{A h^2}A\left\|\nabla_h z^{n+1/2}\right\|^2+\frac{\tau^2 Gr^2}{B\delta_0^2}B\left\|\nabla_h T^{n+1}\right\|^2+\left\|z^{n+1/2}-z^n\right\|^2\right) \le$$
$$\le L\left(\left\|z^{n+1/2}-z^n\right\|^2+\tau A\left\|\nabla_h z^{n+1/2}\right\|^2+\tau B\left\|\nabla_h T^{n+1}\right\|^2\right),$$

where A, B - certain positive numbers, $L=3\left(1+\frac{c_0^2\tau}{A h^2}+\frac{\tau Gr^2}{B\delta_0^2}\right)$.

Multiplying the last inequation by certain positive number α and adding to (45) we get

$$E^{n+1}+\left\|z^{n+1/2}-z^n\right\|^2+2\tau\left(1-\frac{\varepsilon_1 Gr}{\delta_0}\right)\left\|\nabla_h z^{n+1/2}\right\|^2+\frac{\tau Gr(1-\varepsilon_2)}{2\delta_0\varepsilon_1\varepsilon_2}\left\|\nabla_h T^{n+1}\right\|^2 \le$$
$$\le E^n+\alpha L\left(\left\|z^{n+1/2}-z^n\right\|^2+\tau A\left\|\nabla_h z^{n+1/2}\right\|^2+\tau B\left\|\nabla_h T^{n+1}\right\|^2\right)$$

Setting $A=2\left(1-\frac{\varepsilon_1 Gr}{\delta_0}\right)$, $B=\frac{Gr(1-\varepsilon_2)}{2\delta_0\varepsilon_1\varepsilon_2}$ and choosing ε_1, α from conditions $0<\varepsilon_1<1$, $1-\alpha L \ge \alpha_0>0$, we obtain

$$E^{n+1}+\alpha_0\left\|z^{n+1/2}-z^n\right\|^2+\tau A\alpha_0\left\|\nabla_h z^{n+1/2}\right\|^2+\alpha\tau^2\left\|grad_h \pi^n\right\|^2+\tau B\alpha_0\left\|\nabla_h T^{n+1}\right\|^2 \le E^n$$

Applying the estimate $\delta_0\left\|\vec{z}^{n+1}\right\|^2 \le \left\|\nabla_h \vec{z}^{n+1/2}\right\|^2$ and substituting the values E^n, E^{n+1} we get

$$(1+\tau A\alpha_0\delta_0)\left\|\vec{z}^{n+1}\right\|^2+\tau^2\left\|grad_h \pi^{n+1}\right\|^2+\frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\left(1+\frac{4\tau B\alpha_0\delta_0\varepsilon_1\varepsilon_2}{Ra}\right)\left\|T^{n+1}\right\|^2$$

$$(1+\tau A\alpha_0\delta_0)\frac{1}{1+\tau A\alpha_0\delta_0}\left\|\vec{z}^n\right\|^2+$$

$$+\tau^2(1-\alpha)\left\|grad_h \pi^n\right\|^2+$$

$$\frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\left(1+\frac{4\tau B\alpha_0\delta_0\varepsilon_1\varepsilon_2}{Ra}\right)\frac{1}{1+\frac{4\tau B\alpha_0\delta_0\varepsilon_1\varepsilon_2}{Ra}}\left\|T^n\right\|^2.$$

Introducing the notation

$$F^n=(1+\tau A\alpha_0\delta_0)\left\|z^n\right\|^2+\tau^2\left\|grad_h \pi^n\right\|^2+\frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\left(1+\frac{4\tau B\alpha_0\delta_0\varepsilon_1\varepsilon_2}{Ra}\right)\left\|T^n\right\|^2$$

we obtain

$$F^{n+1} \le qF^n \quad (47)$$

at any n, where

$$q=\max\left\{\frac{1}{1+\tau A\alpha_0\delta_0},\ 1-\alpha,\ \frac{1}{1+\frac{4\tau B\alpha_0\delta_0\varepsilon_1\varepsilon_2}{Ra}}\right\}.$$

Since, $0 < q < 1$, then (47) implies the convergence to zero $\|\vec{z}^{\,n}\|$, $\|\pi^n\|$ and $\|T^n\|$ with geometric progression velocity. The theorem is proved.

Note that, if you choose $\alpha \approx O(h)$, then the fulfillment of the condition $1-\alpha L \geq \alpha_0 > 0$ is equally matched to the choice of the iteration parameter τ complying the formula $\tau \leq Lh$, where $L > 0$ is an equibounded constant and the degree of convergence will have the first order in extensive difference interval.

Conclusion

The results of this paper have theoretical and practical significance. The methodology of getting of prior estimates can be useful in studies of difference schemes for the numerical solution of nonlinear equations of hydrodynamic by finite - difference methods, and they make a significant contribution to the further development of the theory of numerical solution of mathematical physics problems.

Conclusions

The implicit difference splitting schemes for thermal convection problems in "velocity vector, pressure" variables were investigated in this paper and there were also determined: stability of difference problem in L_2 according to the initial data and the right member; convergence of uniform problem solutions to a zero stationary solution; prior estimates, illustrating the boundedness of solutions of stationary problems; convergence and estimate of convergence degree of the linear algorithm written to displaced grids with symmetric approximation, to solve linear steady difference problem corresponding to a linear scheme of steady-state equations of free convection.

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References

1. Tarunin, Y.L., 1977. Convection in a closed cavity, heated below, under violation of balance state conditions. Proceedings of USSR AS. MZhG, 2: 203-207.
2. Zhumagulov, B.T., Sh. Smagulov and N.M. Temirbekov, 1989. Numerical methods of Navier-Stokes equations in intricate regains. III international seminar on flame structure, pp: 8-18.
3. Bahvalov, N.S., 2005. Numerical methods to solve the problems mathematical physics. Today problems of computational mathematics and mathematical modelling. Moscow. Nauka, pp: 310.
4. Batchelor, G.K., 1954. Heat transfer by free convection across a closed cavity between vertical boundaries at different temperatures. Quart. Appl. Math. Vol. 12, 3: 209-233.
5. Berkowsky, B.M. and V.K. Polevikov, 1977. Numerical study of problems of high-intensive free convection. Heat Transfer and Turbulent Buoyant Convection. Washington. Hemisphere Publishing, 1(15): 443-445.
6. Samarskiy, A.A., 1977. The theory of difference schemes. Moscow. Nauka, pp: 656.
7. Samarskiy, A.A., and P.N. Vabischevich, 2003. Computational heat transfer. Moscow. Editorial URSS, pp: 784.
8. Danaev, N.T., 2006. On solution of Nafier-Stokes auxiliary grid equations for incompreeible fluids. Notes on numerical fluid mechanics and multidisciplinary design. Springer-Verlag Berlin. Heidelberg, Vol. 91: 55-65.
9. Catton, I., 1978. Natural convection in enclosures. Heat Transfer. Washington Hemisphere Publishing, Vol. 3: pp: 13-31.
10. Beisebay, P.B. and N.T. Danaev, 2007. One difference scheme for the equations of heat convection. Journal Vesnik KazNPU, 1(17): 56-61.
11. Beisebay, P.B. and N.T. Danaev, 2007. About the numerical decision free convection carried out at warning sideways. Journal Vesnik KazNU, 1(52): 71-80.