A new inequality of Wilker-type

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Abstract: In this paper, we deduced the following new Wilker-type inequality: $x^a\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>1+\left(\frac{2}{\pi}\right)^{b+1}x^b\tan x,\quad 0<x<\frac{\pi}{2};\ \frac{\pi^2}{4}-1<a\le b$, where the constant $(2/\pi)^{b+1}$ is the best possible.

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1. Introduction

Wilker [12] proposed the following two open problems:

Problem 1: If $0<x<\pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2 \qquad (1)$$

Problem 2: For $0<x<\pi/2$, there exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2+c\,x^3\tan x \qquad (2)$$

In [10], the sharp constant c in (2) was found and it also proved that

$$2+\left(\frac{2}{\pi}\right)^4x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<2+\left(\frac{8}{45}\right)x^3\tan x \qquad (3)$$

where the constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ are the best possible.

In [14], the inequality (3) was refined as

$$2+\sum_{k=0}^{n}\frac{(-1)^k2^{2k+4}[1-(4k+10)B_{2k+4}]}{(2k+5)!}x^{2k+3}\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<2+\sum_{k=0}^{n-1}\frac{(-1)^k2^{2k+4}[1-(4k+10)B_{2k+4}]}{(2k+5)!}x^{2k+3}\tan x+\left(\frac{2}{\pi}\right)^{2n+4}\left(1-\sum_{k=0}^{n-1}\frac{(-1)^k\pi^{2k+4}[1-(4k+10)B_{2k+4}]}{(2k+5)!}\right)x^{2n+3}\tan x,\qquad 0<x<\frac{\pi}{2} \qquad (4)$$

where B_m denotes the Bernoulli number of order m, $m\in N$. A weighted and exponential generalization of Wilker's inequality (1) presented [13] as

$$\frac{\lambda}{\lambda+\mu}\left(\frac{\sin x}{x}\right)^p+\frac{\mu}{\lambda+\mu}\left(\frac{\tan x}{x}\right)^q>1\qquad \lambda,\mu>0;\ 0<x<\frac{\pi}{2}, \qquad (5)$$

where $q>0$ or $q<\min\left\{\frac{-\lambda}{\mu},-1\right\}$ and $p<\frac{2q\mu}{\lambda}$.

In[4], Wilker's inequality (1) established for inverse trigonometric and inverse hyperbolic functions by

$$2+\frac{17}{45}x^3\tan^{-1}x<\left(\frac{\sin^{-1}x}{x}\right)^2+\frac{\tan^{-1}x}{x},\quad 0<x<1 \qquad (6)$$

and

$$2+\frac{17}{45}x^3\sinh^{-1}x<\left(\frac{\sinh^{-1}x}{x}\right)^2+\frac{\tanh^{-1}x}{x},\quad 0<x<1 \qquad (7)$$

The constants in (6) and (7) are the best possible. Many mathematicians were interested in Wilker's inequality (1) and they presented different proofs, various generalizations and improvements, see [5]-[9], [11], [15]-[18].

In order to attain our aim we need the following power series expansion [1]:

$$\cot x=\sum_{n=0}^{\infty}(-1)^n2^{2n}B_{2n}\frac{x^{2n-1}}{2n!},\qquad 0<|x|<\pi \qquad (8)$$

and consequently we get

$$\csc^2x=-\frac{d}{dx}\cot x$$

$$= \frac{1}{x^2} + \sum_{n=0}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|}{2n!} x^{2n-2},$$

$$0 < |x| < \pi \qquad (9)$$

The purpose of this paper is to present another type of the Wilker inequality and prove it by power series expansions of some trigonometric functions.

2. Main Results.

We begin with an interesting result of Biernacki and Krzyż [3](see also[2]), which will play an important role in the sequel.

Lemma

Consider the power series $F(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ and $G(x) = \sum_{n=0}^{\infty} \beta_n x^n$ are convergent on $(-r, r)$, $r > 0$, where $\alpha_n \in R$ and $\beta_n > 0$ for all $n = 0, 1, \dots$. If the sequence $\left\{\frac{\alpha_n}{\beta_n}\right\}_{n=0}^{\infty}$ is increasing (decreasing, resp.), then the function $\frac{F(x)}{G(x)}$ is increasing (decreasing, resp.) too on $(-r, r)$.

We can easily see that the above lemma will be true in case of odd and even functions.

Theorem 1

$$x^a \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 1 + \left(\frac{2}{\pi}\right)^{b+1} x^b \tan x,$$

$$0 < x < \frac{\pi}{2}; \quad \frac{\pi^2}{4} - 1 < a \le b$$

where the constant $\left(\frac{2}{\pi}\right)^{b+1}$ is the best possible.

Proof

Consider the function

$$P(x) = \frac{\frac{1}{x} - \cot x}{x^a}, \quad 0 < x < \frac{\pi}{2}; \quad a \in R$$

then

$$x \frac{d}{dx} \ln P(x) = \frac{-1 - a + ax\cot x + x^2 \csc^2 x}{1 - \cot x}$$

Let

$A(x) = -1 - a + ax\cot x + x^2 \csc^2 x$

$B(x) = 1 - \cot x$

Using the relations (8) and (9), we obtain

$$A(x) = -1 - a + ax\cot x + x^2 \csc^2 x$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!} x^{2n} - \sum_{n=1}^{\infty} \frac{2^{2n} a |B_{2n}|}{(2n)!} x^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} (2n - 1 - a) x^{2n}$$

$$= \sum_{n=1}^{\infty} \alpha_n x^{2n}$$

where $\alpha_n = \frac{2^{2n}|B_{2n}|}{(2n)!}(2n - 1 - a) \in R$. Also,

$$B(x) = 1 - \cot x = \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n} = \sum_{n=1}^{\infty} \beta_n x^{2n},$$

where $\beta_n = \frac{2^{2n}|B_{2n}|}{(2n)!} > 0$. Now let

$$\gamma_n = \frac{\alpha_n}{\beta_n} = 2n - 1 - a,$$

which is increasing function in the variable n. Then γ_n is increasing and hence $\frac{A(x)}{B(x)}$ is also increasing. Using

$$\lim_{x\to 0} \frac{A(x)}{B(x)} = 1 - a$$

and

$$\lim_{x\to \pi/2} \frac{A(x)}{B(x)} = \frac{1}{4}(-4 - 4a + \pi^2),$$

we get

$$1 - a < x \frac{d}{dx} \ln P(x) < -1 - a + \frac{\pi^2}{4},$$

$$0 < x < \pi/2.$$

If $a > \pi^2/4 - 1$, then $P(x)$ is decreasing function for $0 < x < \pi/2$, where $P(x) > 0$. Now consider the function

$$\phi(x) = x^{a-b}\left[\frac{\sin(2x)}{2x^2} + P(x)\right].$$

The function x^{a-b} is positive decreasing function for $a < b$ and the function $\left[\frac{\sin 2x}{2x^2} + P(x)\right]$ is positive decreasing function for $0 < x < \pi/2$ and $a > \pi^2/4 - 1$. Then $\phi(x)$ is decreasing function for $0 < x < \pi/2$ and $\frac{\pi^2}{4} - 1 < a \le b$. Also,

$$\lim_{x\to \pi/2} \phi(x) = \left(\frac{2}{\pi}\right)^{b+1}.$$

Then

$$x^{a-b}\left[\frac{\sin(2x)}{2x^2} + \frac{\frac{1}{x} - \cot x}{x^a}\right] > \left(\frac{2}{\pi}\right)^{b+1}$$

$$0 < x < \frac{\pi}{2}; \quad \frac{\pi^2}{4} - 1 < a \le b$$

with a sharp bound and this complete the proof of inequality (10).

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