Boundary value problem for B-parabolic equation with integral condition of the first kind

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Abstract. The article deals with the boundary value problem with integral condition of the first kind for a parabolic equation with Bessel operator, and proves its equivalence to the boundary value problem for the same equation with usual local boundary condition. Uniqueness of solution of the problem is proven. The said solution is made as a sum of series by eigenfunctions of the corresponding eigenvalue and eigenfunction problem.

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Introduction

In recent years, much attention has been paid to the study of nonlocal problems with integral conditions. Such problems arise in course of studying various physical phenomena, when the border of process behavior is unavailable for direct measurement. As an example, let's consider problems arising in course of studying heat propagation [1], [2], particles diffusion in turbulent plasma [3], moisture transfer process in capillary-porous media [4]. Also, such problems arise in course of mathematical modeling of technological process external gettering that is used for cleaning impurities from silicon slices [5], [6].

One of the first works dedicated to the study of problems with integral conditions for partial differential equations was the work of Cannon J.R. [1] and Kamynin L.I. [7].

Research of parabolic problems with integral conditions was continued in works of Ionkin N.I. [2], Yurchuk N.I. [8], Muravei L.A., Filinovskii A.V. [5], [6], Bouziani A. and Mesloub S. [9], [10], [11].

Problems with nonlocal integral conditions for elliptic equations were considered in works of Skubachevskii A.L. [12], Guschin A.K. and Mikhailov V.P. [13].

Mixed problems with integral conditions for hyperbolic equations were studied in works of Gordeziani D.G. and Avalishvili G.A. [14], Pulkina L.S. [15] – [17], Beilin S.A. [18], [19]. In works [16], [17] Pulkina L.S. introduced the term "conditions of the first kind" and proved lemmas on equivalence of kind I and kind II conditions. Such problems were also considered in works of Bouziani A. [20] and Mesloub S. [21].

In this work, we consider a parabolic equation with Bessel operator

$$L_B u \equiv u_t - B_x u = 0, \qquad (1)$$

in rectangular area $G_T = \{(x,t): 0 < x < l,\ 0 < t \le T\}$ of the coordinate plane Oxt. Here $B_x = x^{-k}\frac{\partial}{\partial x}\left(x^k \frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial x^2} + \frac{k}{x}\frac{\partial}{\partial x}$ is the Bessel operator, $k > 0$.

Equation (1) was called B-parabolic by Kipriyanov I.A. [22], where a parity type condition is stated on the discriminating part of the border, i.e., the border condition

$$u_x(0,t) = 0.$$

Problem description

Let us consider the problem: find $u(x,t)$ function that satisfies conditions

$$u(x,t) \in C^{1,0}_{x,t}(\overline{G}_T) \cap C^{2,1}_{x,t}(G_T), \qquad (2)$$

$$L_B u = 0, \quad (x,t) \in G_T, \qquad (3)$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \qquad (4)$$

$$u_x(0,t) = 0, \quad 0 \le t \le T, \qquad (5)$$

$$\int_0^l u(x,t)x^k dx = 0, \quad 0 \le x \le T, \qquad (6)$$

where function $\varphi(x)$ is defined, and matching condition is observed

$$\int_0^l \varphi(x)x^k dx = 0. \qquad (7)$$

Lemma 1. If matching condition (7) is observed, then problems (2) - (6) and (2) - (5),

$$u_x(l,t) = 0, \quad 0 \le t \le T \qquad (8)$$

are equivalent.

Proof. Let $u(x,t)$ be the solution of problem (2) - (6). Then this solution satisfies condition (6). By differentiating this condition once by t, we obtain

$$\int_0^l u_t(x,t)x^k dx = 0. \qquad (9)$$

Substituting subintegral function $u_t(x,t)$ in equation (9) by its value from (1), we obtain

$$\int_0^l u_t(x,t)x^k dx = \int_0^1 B_x u(x,t)x^k dx = \int_0^l x^{-k}\frac{\partial}{\partial x}\left(x^k\frac{\partial u}{\partial x}\right)x^k dx =$$

$$= \int_0^l \frac{\partial}{\partial x}\left(x^k\frac{\partial u}{\partial x}\right)dx = \left(x^k\frac{\partial u}{\partial x}\right)\Bigg|_0^l = l^k u_x(l,t).$$

From here and from equality (9) it follows that $u_x(l,t) = 0$.

Now let $u(x,t)$ be the solution for problem (2) - (5), (8). Let us write equation (3) as

$$u_t = x^{-k}\frac{\partial}{\partial x}\left(x^k\frac{\partial u}{\partial x}\right). \qquad (10)$$

Multiplying equation (10) by x^k, and integrating by x on the interval $[0,l]$, we obtain

$$\int_0^l u_t x^k dx = \int_0^l \frac{\partial}{\partial x}\left(x^k\frac{\partial u}{\partial x}\right)dx = \left(x^k\frac{\partial u}{\partial x}\right)\Big|_0^l = l^k u_x(l,t).$$

From here and from condition (8) it follows that

$$\int_0^l u_t(x,t)x^k dx = 0. \qquad (11)$$

By integrating equality (11) by t, we obtain

$$\int_0^l u(x,t)x^k dx = c.$$

Assuming here that $t = 0$, taking into account conditions of coordination (7), we obtain $c = 0$ and, consequently, the condition (6) is observed. Equivalence has been proven.

Uniqueness of solution

Theorem 1. Problem (2) - (6) cannot have more than one solution.

Proof. Let there be two solutions $u_1(x,t)$ and $u_2(x,t)$ of problem (2) - (7). Then their difference $v(x,t) = u_1(x,t) - u_2(x,t)$ shall be the solution of boundary value problem: find a function $v(x,t)$ that satisfies conditions

$$v(x,t) \in C_{x,t}^{1,0}(\overline{G}_T) \cap C_{x,t}^{2,1}(G_T), \qquad (12)$$

$$L_B v = 0, \quad (x,t) \in G_T, \qquad (13)$$

$$v(x,0) = 0, \quad 0 \le x \le l, \qquad (14)$$

$$v_x(0,t) = 0, \quad 0 \le t \le T, \qquad (15)$$

$$\int_0^l v(x,t)x^k dx = 0, \quad 0 \le t \le T. \qquad (16)$$

Let us write equation (13) as

$$v_t = x^{-k}\frac{\partial}{\partial x}\left(x^k\frac{\partial v}{\partial x}\right).$$

By multiplying this equation by $2x^k v$ and noting that $2vv_t = \frac{\partial}{\partial t}(v^2)$, we obtain

$$x^k\frac{\partial}{\partial \tau}\left(v^2(x,\tau)\right) = 2v(x,\tau)\frac{\partial}{\partial x}\left(x^k\frac{\partial}{\partial x}(v(x,\tau))\right).$$

From equality of functions equality of definite integrals follows

$$\int_0^l\int_0^t x^k\frac{\partial}{\partial \tau}\left(v^2(x,\tau)\right)d\tau dx = 2\int_0^l\int_0^t v(x,\tau)\frac{\partial}{\partial x}\left(x^k\frac{\partial}{\partial x}(v(x,\tau))\right)d\tau dx.$$

Let us change the order of integration in the right part

$$\int_0^l\int_0^t x^k\frac{\partial}{\partial \tau}\left(v^2(x,\tau)\right)d\tau dx = 2\int_0^t\left[\int_0^l v(x,\tau)\frac{\partial}{\partial x}\left(x^k\frac{\partial}{\partial x}(v(x,\tau))\right)dx\right]d\tau. \qquad (17)$$

From initial condition (14) it follows that

$$\int_0^l\int_0^t x^k\frac{\partial}{\partial \tau}\left(v^2(x,\tau)\right)d\tau dx = \int_0^l x^k v^2(x,t)dx.$$

To calculate inner integral in the right part of (17), let us apply partial integration formula

$$\int_0^l v(x,\tau)\frac{\partial}{\partial x}\left(x^k\frac{\partial}{\partial x}(v(x,\tau))\right)dx = v(x,\tau)x^k\frac{\partial}{\partial x}(v(x,\tau))\Big|_0^l - \int_0^l x^k\left(\frac{\partial}{\partial x}(v(x,\tau))\right)^2 dx$$

From the boundary conditions (15), (16) and on the basis of Lemma 1, we obtain

$$v(x,\tau)x^k\frac{\partial}{\partial x}(v(x,\tau))\Big|_0^l = 0.$$

Thus, equation (17) takes the form

$$\int_0^l x^k v^2(x,t)dx + 2\int_0^t\int_0^l x^k\left(\frac{\partial}{\partial x}(v(x,\tau))\right)^2 dx d\tau = 0.$$

Each summand of the last equality is non-negative. Therefore, they are equal to zero. Since function $v(x,t)$ is continuous, from equality $\int_0^l x^k v^2(x,t)dx = 0,$ it follows that $v(x,t) \equiv 0.$

From here we obtain that $u_1(x,t) \equiv u_2(x,t).$

Solution existence. To prove existence of solution to problem (2) - (6) it is sufficient to prove existence of solution to problem (2) - (5), (8).

According to Fourier method, specific solutions to equation (3) can be found in form

$$u(x,t) = X(x)T(t), \qquad (18)$$

whereas $X(x)$ and $T(t)$ are yet indeterminate functions. By inserting function (18) into equation (3), we obtain

$$T' + \lambda^2 T = 0, \qquad (19)$$

$$X'' + \frac{k}{x}X' + \lambda^2 X = 0. \qquad (20)$$

For the specific solution (18) that is different from null equation to satisfy boundary conditions (5) and (8), the following conditions should be observed

$$X'(0) = 0, \quad X'(l) = 0. \qquad (21)$$

It is known [23] that equation (20), by replacing variables in formulas

$$X = \left(\frac{z}{\lambda}\right)^{\frac{1-k}{2}} Z, \quad x = \frac{z}{\lambda}, \qquad (22)$$

can be reduced to Bessel equation

$$z^2 Z'' + zZ' + \left(z^2 - \frac{(k-1)^2}{4}\right)Z = 0, \qquad (23)$$

where general solution is function

$$Z = C_1 J_{\frac{k-1}{2}}(z) + C_2 Y_{\frac{k-1}{2}}(z), \qquad (24)$$

where $J_{\frac{k-1}{2}}(z)$, $Y_{\frac{k-1}{2}}(z)$ are Bessel functions of first and second kind of order $\frac{k-1}{2}$. Returning to original variables in function (24), taking into account formulas (22), we obtain

$$X = C_1 x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}(\lambda x) + C_2 x^{-\frac{k-1}{2}} Y_{\frac{k-1}{2}}(\lambda x), \qquad (25)$$

whereas C_1, C_2, λ are arbitrary constants. They can be found from the requirement that the general solution (25) satisfies conditions (21). To do so, let us insert it into these conditions. As a result, we shall obtain

$$X' = C_1\left(x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}(\lambda x)\right)' + C_2\left(x^{-\frac{k-1}{2}} Y_{\frac{k-1}{2}}(\lambda x)\right)' =$$

$$= -C_1 \lambda x^{-\frac{k-1}{2}} J_{\frac{k+1}{2}}(\lambda x) - C_2 \lambda x^{-\frac{k-1}{2}} Y_{\frac{k+1}{2}}(\lambda x).$$

Since owing to a known asymptotic formula in Bessel function [23] $x^{-\frac{k-1}{2}} J_{\frac{k+1}{2}}(\lambda x) \to 0$, $x^{-\frac{k-1}{2}} Y_{\frac{k+1}{2}}(\lambda x) \to \infty$ with $x \to 0$, then to satisfy the first boundary condition (21), there should be $C_2 = 0$. Then from the second boundary condition (21), we obtain

$$J_{\frac{k+1}{2}}(\lambda l) = 0. \qquad (26)$$

It is known [24] that equation (26) has an infinite number of real roots. Let us state equation roots $J_{\frac{k+1}{2}}(\mu) = 0$ as μ_n, $n = 1,2,3,\ldots$.

Then the eigenvalues

$$X_n = x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right), \quad n = 1,2,3,\ldots, \qquad (27)$$

shall correspond to eigenfunctions $\lambda_n = \frac{\mu_n}{l}$, $n = 1,2,3,\ldots$

Lemma 2. Function (27) is orthogonal with the weight x^k and form a complete system.

Proof. Orthogonality with the weight is obvious

$$\int_0^l X_n(x)X_m(x)x^k dx = \int_0^l x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right) x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_m}{l}x\right) x^k dx =$$

$$= \int_0^l J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right) J_{\frac{k-1}{2}}\left(\frac{\mu_m}{l}x\right) x dx = \begin{cases} 0, & n \neq m; \\ \frac{l^2}{2}\left(1 - \frac{(k-1)^2}{4\mu_n^2}\right) J^2_{\frac{k-1}{2}}(\mu_n), & n = m. \end{cases} \qquad (28)$$

Let us prove completeness of this system. Assume that there is a function $v(x)$ that is different from null equation and orthogonal to all functions (27)

$$\int_0^l x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right)\nu(x)dx = 0.$$

It is known [23] that system $\left\{J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right)\right\}$ is complete in $L_2(0,l)$ and, consequently,

$$x^{-\frac{k-1}{2}}\nu(x) = 0,$$

which is feasible only for $\nu(x)$ that is equal to zero almost everywhere on $(0,l)$. This proves the completeness of the system (27).

Let us assume that that the function $f(x)$ can be represented as a series

$$f(x) = \sum_{n=1}^{\infty} a_n x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right). \qquad (29)$$

By multiplying both parts of factorization (29) by $x^{\frac{k+1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right)$ and integrating by interval $[0,l]$, taking into account formula (28), we obtain

$$a_n = \frac{2}{l^2\left(1-\frac{(k-1)^2}{4\mu_n^2}\right)J^2_{\frac{k-1}{2}}(\mu_n)}\int_0^l f(x)x^{\frac{k+1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right)dx, \quad n = 1,2,3,\ldots$$

The following solutions of equation (19) shall correspond to values of parameter $\lambda = \lambda_n = \frac{\mu_n}{l}$

$$T = A_n e^{-\left(\frac{\mu_n}{l}\right)^2 t},$$

whereas A_n are arbitrary constants. So, all functions

$$u_n(x,t) = X_n(x)T_n(t) = A_n x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right)e^{-\left(\frac{\mu_n}{l}\right)^2 t}, \qquad (30)$$

satisfy equation (3) and boundary conditions (5) and (8) for any constants A_n.

Let us make series

$$u(x,t) = \sum_{n=1}^{\infty} A_n x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right)e^{-\left(\frac{\mu_n}{l}\right)^2 t}. \qquad (31)$$

Requiring satisfaction of initial condition (4), we obtain

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} A_n x^{-\frac{k-1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right). \qquad (32)$$

The written series is the factorization of the given function $\varphi(x)$ into a series of Bessel functions in interval $(0,l)$. Coefficients of factorization (32) are defined by formulas

$$A_n = \frac{2}{l^2\left(1-\frac{(k-1)^2}{4\mu_n^2}\right)J^2_{\frac{k-1}{2}}(\mu_n)}\int_0^l \varphi(x)x^{\frac{k+1}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n}{l}x\right)dx, \quad n = 1,2,3,\ldots \qquad (33)$$

Conclusions

Theorem 2. If function $\varphi(x) \in C^2[0,l]$ and $\varphi'(l) = 0$, $\varphi(0) = \varphi'(0) = 0$, then there exists a unique solution to problem (2) - (6), and it is defined as the sum of series (31), coefficients of which are calculated according to formulas (33).

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