Some systems of second-order rational difference equations*

M. N. Qureshi, A. Q. Khan, Q. Din

Department of Mathematics, University of Azad Jammu & Kashmir, Muzaffarabad, Pakistan
nqureshi@ajku.edu.pk, abdulqadeerkhan1@gmail.com, qamar.sms@gmail.com

Abstract: In this paper, we study the qualitative behavior of following two systems of second-order rational difference equations:

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma y_n^2}, y_{n+1}=\frac{\alpha_1 y_{n-1}}{\beta_1+\gamma_1 x_n^2}, n=0,1,\cdots,$$

and

$$x_{n+1}=\frac{a y_{n-1}}{b+c x_n^2}, y_{n+1}=\frac{a_1 x_{n-1}}{b_1+c_1 y_n^2}, n=0,1,\cdots,$$

where the parameters $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1,a,b,c,a_1,b_1,c_1$ *and initial conditions* x_0,x_{-1},y_0,y_{-1} *are* positive real numbers. More precisely, we study the local asymptotic stability and instability of equilibrium points, global character of equilibrium points and rate of convergence of these systems. Some numerical examples are given to verify our theoretical results.

[M. N. Qureshi, A. Q. Khan, Q. Din. **Some systems of second-order rational difference equations.** *Life Sci J* 2014;11(6s):43-50]. (ISSN:1097-8135). http://www.lifesciencesite.com. 7

Keywords and phrases: Rational difference equations, stability, global character, rate of convergence
2010 AMS Mathematics subject classifications: 39A10, 40A05.

1. Introduction

The study of difference equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. For basic theory of difference equations, we refer interested readers to [1-4]. Moreover, in [5-14], dynamics of some difference equations is given. In Refs. [15-18], qualitative behavior of some biological models is discussed. Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example:

C. H. Gibbons *et al.* [5] investigated the global asymptotic stability of the difference equation:

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{\gamma+x_n}$$

where $\beta>0,\alpha,\gamma\geq 0$ and initial conditions x_0,x_{-1} are positive real numbers.

A.E. Hamza *et al.* [6] studied the global asymptotic behavior of the difference equation:

$$x_{n+1}=\frac{Ax_{n-1}}{B+Cx_n^2}$$

where A,B,C and initial conditions x_0,x_{-1} are positive real numbers.

To be motivated by the above studies, our aim in this paper is to investigate the qualitative behavior of following two systems of second-order rational difference equations

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma y_n^2}, y_{n+1}=\frac{\alpha_1 y_{n-1}}{\beta_1+\gamma_1 x_n^2}, n=0,1,\cdots, \quad (1)$$

and

$$x_{n+1}=\frac{a y_{n-1}}{b+c x_n^2}, y_{n+1}=\frac{a_1 x_{n-1}}{b_1+c_1 y_n^2}, n=0,1,\cdots \quad (2)$$

where $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1,a,b,c,a_1,b_1,c_1$ and initial conditions x_0,x_{-1},y_0,y_{-1} are positive real numbers.

Let us consider four-dimensional discrete dynamical system of the form:

$$\left.\begin{aligned} x_{n+1}&=f(x_n,x_{n-1},y_n,y_{n-1}),\\ y_{n+1}&=g(x_n,x_{n-1},y_n,y_{n-1}), n=0,1,\cdots,\end{aligned}\right\} \quad (3)$$

where $f:I^2\times J^2\rightarrow I$ and $g:I^2\times J^2\rightarrow J$ are continuously differentiable functions and I,J are some intervals of real numbers. Furthermore, a solution $\{(x_n,y_n)\}_{n=-1}^{\infty}$ of system (3) is uniquely determined by initial conditions $(x_i,y_i)\in I\times J$ for $i\in\{-1,0\}$. Along with the system (3) we consider the corresponding vector map $F=(f,x_n,g,y_n)$. An equilibrium point of system (3) is a point $(\bar{x},\bar{y})$ that satisfies

$$\bar{x}=f(\bar{x},\bar{x},\bar{y},\bar{y})$$
$$\bar{y}=g(\bar{x},\bar{x},\bar{y},\bar{y})$$

The point $(\bar{x},\bar{y})$ is also called a fixed point of the vector map F.

Definition 1. Let $(\bar{x},\bar{y})$ be an equilibrium point of the system (3).

(i) An equilibrium point $(\bar{x},\bar{y})$ is said to be stable if for every $\varepsilon>0$ there exists $\delta>0$ such that for every initial conditions $(x_i,y_i), i\in\{-1,0\}$ if

$\|\sum_{i=-1}^{0}(x_i, y_i) - (\bar{x}, \bar{y})\| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\|\cdot\|$ is usual Euclidian norm in $\mathbb{R}^2$.
(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.
(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\|\sum_{i=-1}^{0}(x_i, y_i) - (\bar{x}, \bar{y})\| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.
(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.
(v) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2. Let $(\bar{x}, \bar{y})$ be an equilibrium point of a map $F = (f, x_n, g, y_n)$ where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (3) about the equilibrium point $(\bar{x}, \bar{y})$ is given by

$$X_{n+1} = F(X_n) = F_J X_n,$$

Where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and F_J is Jacobean matrix of the system (3) about the equilibrium point $(\bar{x}, \bar{y})$.

Lemma 1. [3] For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$, of difference equations such that $\bar{X}$ be a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.

Lemma 2. [4] Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$, is a system of difference equations and $\bar{X}$ is the equilibrium point of this system. The characteristic polynomial of this system about the equilibrium point $\bar{X}$ is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $P(\lambda)$ lies inside the open unit disk $|\lambda|$ if and only if $\Delta_k > 0$ for $k = 0, 1, \cdots$, where Δ_k is the principal minor of order k of the $n \times n$ matrix

$$\Delta_n = \begin{pmatrix} a_1 & a_3 & \cdots & 0 \\ a_0 & a_2 & & a_4 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \quad (4)$$

The following results give the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = (A + B(n))X_n, \quad (5)$$

Where X_n is an m –dimensional vector, $A \in C^{m \times m}$ is a constant matrix, and $B: Z^+ \to C^{m \times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \quad (6)$$

as $n \to \infty$, where $\|.\|$ denotes any matrix norm which is associated with the vector norm

$$\|B(n)\| = \sqrt{x^2 + y^2}$$

Proposition 1. (Perron's theorem) [13] Suppose that condition (6) holds. If X_n is a solution of (5), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty}(\|X_n\|)^{\frac{1}{n}} \quad (7)$$

exist and is equal to the modulus of one the eigenvalues of matrix A.

Proposition 2. (Perron's theorem) [13] Suppose that condition (6) holds. If X_n is a solution of (5), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty}\frac{\|X_{n+1}\|}{\|X_n\|} \quad (8)$$

exist and is equal to the modulus of one the eigenvalues of matrix A.

2. On the system $x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma y_n^2}, y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 + \gamma_1 x_n^2}$

In this section, we shall investigate the qualitative behavior of the system (1). Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1), then for $\alpha > \beta$ and $\alpha_1 > \beta_1$ system (1) has following two positive equilibrium points $P_0 = (0,0)$ and $P_1 = \left(\sqrt{\frac{\alpha_1 - \beta_1}{\gamma_1}}, \sqrt{\frac{\alpha - \beta}{\gamma}}\right)$.

To construct corresponding linearized form of the system (1) we consider the following transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \to (f, f_1, g, g_1) \quad (9)$$

where $f = \frac{\alpha x_{n-1}}{\beta + \gamma y_n^2}$, $f_1 = x_n$, $g = \frac{\alpha_1 y_{n-1}}{\beta_1 + \gamma_1 x_n^2}$, $g_1 = y_n$.

The Jacobian matric about the fixed point $(\bar{x}, \bar{y})$ under the transformations (9) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & A & B & 0 \\ 1 & 0 & 0 & 0 \\ C & 0 & 0 & D \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Where

$A = \frac{\alpha}{\beta + \gamma \bar{y}^2}$, $B = \frac{-2\alpha\gamma\bar{x}\bar{y}}{(\beta + \gamma\bar{y}^2)^2}$, $C = \frac{-2\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1 + \gamma_1\bar{x}^2)^2}$, $D = \frac{\alpha_1}{\beta_1 + \gamma_1\bar{x}^2}$. The characteristic polynomial of $F_J(\bar{x}, \bar{y})$ about equilibrium point $(\bar{x}, \bar{y})$ is given by

$$P(\lambda) = \lambda^4 - \left(\frac{\alpha_1}{\beta_1 + \gamma_1\bar{x}^2} + \frac{4\alpha\alpha_1\gamma\gamma_1\bar{x}^2\bar{y}^2}{(\beta + \gamma\bar{y}^2)^2(\beta_1 + \gamma_1\bar{x}^2)^2} + \frac{\alpha}{\beta + \gamma\bar{y}^2}\right)\lambda^2 + \frac{\alpha\alpha_1}{(\beta + \gamma\bar{y}^2)(\beta_1 + \gamma_1\bar{x}^2)} \quad (10)$$

2.1 Main Results

Theorem 1. Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then every solution $\{(x_n, y_n)\}$ of the system (1) is bounded.

Proof. It is easy to verify that

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{m+1} x_{-1}, \quad n = 2m + 1,$$

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{m+1} x_0, \quad n = 2m + 2,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-1}, \quad n = 2m+1,$$
$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_0, \quad n = 2m+2,$$

Taking $\delta_1 = \max\{x_{-1}, x_0\}$ and $\delta_2 = \max\{y_{-1}, y_0\}$. Then, $0 \le x_n \le \delta_1$ and $0 \le y_n \le \delta_2$ for all $n = 0,1,\cdots$. ■

Theorem 2. For the equilibrium point P_0 of the system (1) the following results hold:

(i) If $\alpha < \beta$ and $\alpha_1 < \beta_1$, then equilibrium point P_0 of the system (1) is locally asymptotically stable.

(ii) If $\alpha > \beta$ or $\alpha_1 > \beta_1$, then equilibrium point P_0 of the system (1) is unstable.

Proof (i). The linearized system of (1) about the equilibrium point P_0 is given by

$$X_{n+1} = F_J(P_0)X_n,$$

where $= \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and $F_J(0,0) = \begin{pmatrix} 0 & \frac{\alpha}{\beta} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(0,0)$ is given by

$$P(\lambda) = \lambda^4 - \left(\frac{\alpha}{\beta} + \frac{\alpha_1}{\beta_1}\right)\lambda^2 + \frac{\alpha\alpha_1}{\beta\beta_1} \qquad (11)$$

The roots of $P(\lambda)$ are $\pm\sqrt{\frac{\alpha}{\beta}}$ and $\pm\sqrt{\frac{\alpha_1}{\beta_1}}$. Since all eigenvalues of Jacobian matrix $F_J(0,0)$ about $(0,0)$ lie in open unit disk $|\lambda| < 1$. Hence, the equilibrium point $(0,0)$ is locally asymptotically stable.

Proof (ii). It is easy to see that if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then there exist at least one root λ of equation (11) such that $|\lambda| > 1$. Hence by Lemma 1 if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then P_0 is unstable ■

Theorem 3. The equilibrium point $P_1 = \left(\sqrt{\frac{\alpha_1-\beta_1}{\gamma_1}}, \sqrt{\frac{\alpha-\beta}{\gamma}}\right)$ of the system (1) is unstable.

Proof. The linearized system of (1) about the equilibrium point P_1 is given by

$$X_{n+1} = F_J(P_1)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and $F_J(P_1) = \begin{pmatrix} 0 & 1 & L & 0 \\ 1 & 0 & 0 & 0 \\ M & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(P_1)$ is given by

$$P(\lambda) = \lambda^4 - (AB+2)\lambda + 1 \qquad (12)$$

Where

$$L = \frac{-2}{\alpha}\sqrt{\frac{\gamma(\alpha-\beta)(\alpha_1-\beta_1)}{\gamma_1}} \text{ and } M = \frac{-2}{\alpha_1}\sqrt{\frac{\gamma_1(\alpha-\beta)(\alpha_1-\beta_1)}{\lambda}}.$$

It is clear that not all of $\Delta_k > 0$ for $k = 1,2,3,4$. Hence by Lemma 2, the positive equilibrium point $\left(\sqrt{\frac{\alpha_1-\beta_1}{\gamma_1}}, \sqrt{\frac{\alpha-\beta}{\gamma}}\right)$ is unstable.■

Theorem 4. Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then equilibrium point P_0 of the system (1) is globally asymptotically stable.

Proof. For $\alpha < \beta$ and $\alpha_1 < \beta_1$, from Theorem 2 (0, 0) is locally asymptotically stable. From Theorem 1, every positive solution $\{(x_n, y_n)\}$ of the system (1) is bounded. Now, it is sufficient to prove that $\{(x_n, y_n)\}$ is decreasing. From system (1) one has

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma y_n^2}$$
$$\le \frac{\alpha}{\beta} x_{n-1} < x_{n-1}.$$

This implies that $x_{2n+1} < x_{2n-1}$ and $x_{2n+3} < x_{2n+1}$. Hence, the subsequences $\{x_{2n+1}\}, \{x_{2n+2}\}$ are decreasing $i.e.$ the sequence $\{x_n\}$ is decreasing. Similarly, one has

$$y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 + \gamma_1 x_n^2}$$
$$\le \frac{\alpha_1}{\beta_1} y_{n-1} < y_{n-1}.$$

This implies that $y_{2n+1} < y_{2n-1}$ and $y_{2n+3} < y_{2n+1}$. Hence, the subsequences $\{y_{2n+1}\}, \{y_{2n+2}\}$ are decreasing $i.e.$ the sequence $\{y_n\}$ is decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.■

2.2. Rate Of Convergence

In this section we investigate the rate of convergence of a solution that converges to the equilibrium point P_0 of the system (1).

Let $\{(x_n, y_n)\}$ be any solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} y_n = \bar{y}$. To find the error terms, one has from the system (1)

$$x_{n+1} - \bar{x} = \frac{\alpha x_{n-1}}{\beta+\gamma y_n^2} - \frac{\alpha \bar{x}}{\beta+\gamma \bar{y}^2},$$
$$= -\frac{\alpha\gamma\bar{x}(y_n^2 - \bar{y}^2)}{(\beta+\gamma y_n^2)(\beta+\gamma\bar{y}^2)(y_n-\bar{y})}(y_n - \bar{y}) + \frac{\alpha}{\beta+\gamma y_n^2}(x_{n-1} - \bar{x}),$$

$$y_{n+1} - \bar{y} = \frac{\alpha_1 y_{n-1}}{\beta_1+\gamma_1 x_n^2} - \frac{\alpha_1 \bar{y}}{\beta_1+\gamma_1 \bar{x}^2},$$
$$= -\frac{\alpha_1\gamma_1\bar{y}(x_n^2 - \bar{x}^2)}{(\beta_1+\gamma_1 x_n^2)(\beta_1+\gamma_1\bar{x}^2)(x_n-\bar{x})}(x_n - \bar{x}) + \frac{\alpha_1}{\beta_1+\gamma_1 x_n^2}(y_{n-1} - \bar{y}).$$

Set $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, one has

$$e_{n+1}^1 = a_n e_{n-1}^1 + b_n e_n^2,$$
$$e_{n+1}^2 = c_n e_n^1 + d_n e_{n-1}^2.$$

Where

$$a_n = \frac{\alpha}{\beta+\gamma y_n^2}, \; b_n = -\frac{\alpha\gamma\bar{x}(y_n^2-\bar{y}^2)}{(\beta+\gamma y_n^2)(\beta+\gamma\bar{y}^2)(y_n-\bar{y})},$$
$$c_n = -\frac{\alpha_1\gamma_1\bar{y}(x_n^2-\bar{x}^2)}{(\beta_1+\gamma_1 x_n^2)(\beta_1+\gamma_1\bar{x}^2)(x_n-\bar{x})}, \; d_n = \frac{\alpha_1}{\beta_1+\gamma_1 x_n^2}$$

Moreover,

$$\lim_{n\to\infty} a_n = \frac{\alpha}{\beta+\gamma\bar{y}^2}, \; \lim_{n\to\infty} b_n = \frac{-2\alpha\gamma\bar{x}\bar{y}}{(\beta+\gamma\bar{y}^2)^2}$$
$$\lim_{n\to\infty} c_n = \frac{-2\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1+\gamma_1\bar{x}^2)^2}, \; \lim_{n\to\infty} d_n = \frac{\alpha_1}{\beta_1+\gamma_1\bar{x}^2}$$

So, the limiting system of error terms can be written as

$E_{n+1} = KE_n,$

Where $E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix}$ and

$$K = \begin{pmatrix} 0 & \frac{\alpha}{\beta+\gamma\bar{y}^2} & \frac{-2\alpha\gamma\bar{x}\bar{y}}{(\beta+\gamma\bar{y}^2)^2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{-2\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1+\gamma_1\bar{x}^2)^2} & 0 & 0 & \frac{\alpha_1}{\beta_1+\gamma_1\bar{x}^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which is similar to linearized system of (1) about the equilibium point $(\bar{x},\bar{y}) = (0,0)$. Using the proposition (1), one has following result.

Theorem 5 Assume that $\{(x_n,y_n)\}$ be a positive solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$ where $(\bar{x},\bar{y}) = (0,0)$. Then, the error term $E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix}$ of every solution of (1) satisfies both of the following asymptotic relations

$\lim_{n\to\infty} (\|E_n\|)^{\frac{1}{n}} = |\lambda F_j(\bar{x},\bar{y})|,$

$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda F_j(\bar{x},\bar{y})|,$

where $\lambda F_j(\bar{x},\bar{y})$ are the characteristic roots of Jacobian matrix $F_j(\bar{x},\bar{y})$ about $(0,0)$.

3. On the system $x_{n+1} = \frac{a y_{n-1}}{b+c x_n^2}, y_{n+1} = \frac{a_1 x_{n-1}}{b_1+c_1 y_n^2}$

In this section, we shall investigate the qualitative behavior of the system (2). Let $(\bar{x},\bar{y})$ be an equilibrium point of system (2), then system (2) has a unique positive equilibrium point $(0,0)$. To construct corresponding linearized form of the system (1) we consider the following transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \to (f, f_1, g, g_1) \qquad (13)$$

where $f = \frac{a y_{n-1}}{b+c x_n^2}, f_1 = x_n, g = \frac{a_1 x_{n-1}}{b_1+c_1 y_n^2}, g_1 = y_n$.

The Jacobian matric $F_j(\bar{x},\bar{y})$ about the fixed point $(\bar{x},\bar{y})$ under the transformations (13) is given by

$$F_j(\bar{x},\bar{y}) = \begin{pmatrix} E & 0 & 0 & F \\ 1 & 0 & 0 & 0 \\ 0 & G & H & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Where

$E = \frac{-2ac\bar{x}\bar{y}}{(b+c\bar{x}^2)^2}, \quad F = \frac{a}{b+c\bar{x}^2}, G = \frac{a_1}{b_1+c_1\bar{y}^2},$

$H = \frac{-2a_1c_1\bar{x}\bar{y}}{(b_1+c_1\bar{y}^2)^2}.$

The characteristic polynomial of $F_j(\bar{x},\bar{y})$ about equilibrium point $(\bar{x},\bar{y})$ is given by

$$P(\lambda) = \lambda^4 + \left(\frac{2ac\bar{x}\bar{y}}{(b+c\bar{x}^2)^2} + \frac{2a_1c_1\bar{x}\bar{y}}{(b_1+c_1\bar{y}^2)^2}\right)\lambda^3 + \frac{4a\,a_1cc_1\bar{x}^2\bar{y}^2}{(b+c\bar{x}^2)^2(b_1+c_1\bar{y}^2)^2}\lambda^2 - \frac{aa_1}{(b+c\bar{x}^2)(b_1+c_1\bar{y}^2)} \qquad (14)$$

3.1 Main Results

Theorem 6. Let $\{(x_n,y_n)\}$ be a positive solution of the system (2), then for every $m \geq 0$ the following result hold:

$$(i)\ 0 \le x_n \le \begin{cases} \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^m y_{-1}, & n = 4m+1, \\ \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^m y_0, & n = 4m+2, \\ \left(\frac{aa_1}{bb_1}\right)^{m+1} x_{-1}, & n = 4m+3, \\ \left(\frac{aa_1}{bb_1}\right)^{m+1} x_0, & n = 4m+4, \end{cases}$$

$$(ii)\ 0 \le y_n \le \begin{cases} \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} x_{-1}, & n = 4m+1, \\ \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} x_0, & n = 4m+2, \\ \left(\frac{aa_1}{bb_1}\right)^{m+1} y_{-1}, & n = 4m+3, \\ \left(\frac{aa_1}{bb_1}\right)^{m+1} y_0, & n = 4m+4, \end{cases}$$

Theorem 7. For the equilibrium point P_0 of the system (2) the following results hold:

(i) If $a < b$ and $a_1 < b_1$, then equilibrium point P_0 is locally asymptotically stable.

(ii) If $a > b$ or $a_1 > b_1$, then equilibrium point P_0 is unstable.

Proof (i). The linearized system of (2) about the equilibrium point P_0 is given by

$X_{n+1} = F_j(0,0)X_n,$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix} \text{ and } F_j(0,0) = \begin{pmatrix} 0 & 0 & 0 & \frac{a}{b} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{a_1}{b_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $F_j(0,0)$ is given by

$$P(\lambda) = \lambda^4 - \frac{\alpha\alpha_1}{\beta\beta_1} \qquad (15)$$

The roots of $P(\lambda)$ are $= \pm\left(\frac{aa_1}{bb_1}\right)^{\frac{1}{4}}, \ \lambda = \pm i\left(\frac{aa_1}{bb_1}\right)^{\frac{1}{4}}$. Since all eigenvalues of Jacobian matrix $F_j(0,0)$ about $(0,0)$ lie in open unit disk $|\lambda| < 1$. Hence, the equilibrium point $(0,0)$ is locally asymptotically stable.

Proof (ii). It is easy to see that if $a > b$ or $a_1 > b_1$, then there exist at least one root λ of equation (15) such that $|\lambda| > 1$. Hence by Lemma 1 if $a > b$ or $a_1 > b_1$, then P_0 is unstable.■

Theorem 8. Let $a < b$ and $a_1 < b_1$, then equilibrium point P_0 of the system (2) is globally asymptotically stable.

Proof. For $a < b$ and $a_1 < b_1$, from Theorem 7, (0, 0) is locally asymptotically stable. From Theorem 6, it is easy to show that every positive solution $\{(x_n, y_n)\}$ of the system (2) is bounded. Now, it is sufficient to prove that $\{(x_n, y_n)\}$ is decreasing. From system (2) one has

$$x_{n+1} = \frac{a y_{n-1}}{b + c x_n^2}$$
$$\le \frac{a}{b} y_{n-1} < y_{n-1}.$$

This implies that $x_{4n+1} < y_{4n-1}$ and $x_{4n+5} < y_{4n+3}$. Also

$$y_{n+1} = \frac{a_1 x_{n-1}}{b_1 + c_1 y_n^2}$$
$$\le \frac{a_1}{b_1} x_{n-1} < x_{n-1}.$$

This implies that $y_{4n+1} < x_{4n-1}$ and $y_{4n+5} < x_{4n+3}$. Hence, $x_{4n+5} < y_{4n+3} < x_{4n+1}$ and $y_{4n+5} < x_{4n+3} < y_{4n+1}$. Hence, the subsequences

$\{x_{4n+1}\}, \{x_{4n+2}\}, \{x_{4n+3}\}, \{x_{4n+4}\}$ and $\{y_{4n+1}\}, \{y_{4n+2}\}, \{y_{4n+3}\}, \{y_{4n+4}\}$

are decreasing. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0. \blacksquare$$

3.2. Rate Of Convergence

In this section we investigate the rate of convergence of a solution that converges to the equilibrium point P_0 of the system (2).

Let $\{(x_n, y_n)\}$ be any solution of the system (2) such that $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} y_n = \bar{y}$. To find the error terms, one has from the system (2)

$$x_{n+1} - \bar{x} = \frac{a y_{n-1}}{b + c x_n^2} - \frac{a\bar{y}}{b + c\bar{x}^2},$$
$$= -\frac{ac\bar{y}(x_n^2 - \bar{x}^2)}{(b + c x_n^2)(b + c\bar{x}^2)(x_n - \bar{x})}(x_n - \bar{x})$$
$$+ \frac{a}{b + c x_n^2}(y_{n-1} - \bar{x}),$$

$$y_{n+1} - \bar{y} = \frac{a_1 y_{n-1}}{\beta_1 + \gamma_1 x_n^2} - \frac{a_1 \bar{y}}{\beta_1 + \gamma_1 \bar{x}^2},$$
$$= \frac{a_1}{b_1 + c_1 y_n^2}(x_{n-1} - \bar{x}) - \frac{a_1 c_1 \bar{x}(y_n^2 - \bar{y}^2)}{(b_1 + c_1 y_n^2)(b_1 + c_1 \bar{y}^2)(y_n - \bar{y})}(y_n - \bar{y}).$$

Set $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, one has

$$e_{n+1}^1 = a_n e_n^1 + b_n e_{n-1}^2,$$
$$e_{n+1}^2 = c_n e_{n-1}^1 + d_n e_n^2.$$

Where

$$a_n = -\frac{ac\bar{y}(x_n^2 - \bar{x}^2)}{(b + c x_n^2)(b + c\bar{x}^2)(x_n - \bar{x})},\ b_n = \frac{a}{b + c x_n^2},$$
$$c_n = \frac{a_1}{b_1 + c_1 y_n^2},\ d_n = -\frac{a_1 c_1 \bar{x}(y_n^2 - \bar{y}^2)}{(b_1 + c_1 y_n^2)(b_1 + c_1 \bar{y}^2)(y_n - \bar{y})}.$$

Moreover,

$$\lim_{n\to\infty} a_n = \frac{-2ac\bar{x}\bar{y}}{(b + c\bar{x}^2)^2},\ \lim_{n\to\infty} b_n = \frac{a}{b + c\bar{x}^2}$$
$$\lim_{n\to\infty} c_n = \frac{a_1}{b_1 + c_1\bar{y}^2},\ \lim_{n\to\infty} d_n = \frac{-2a_1 c_1 \bar{x}\bar{y}}{(b_1 + c_1\bar{y}^2)^2}.$$

So, the limiting system of error terms can be written as

$$E_{n+1} = H E_n,$$

Where

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix}$$ and

$$H = \begin{pmatrix} \frac{-2ac\bar{x}\bar{y}}{(b+c\bar{x}^2)^2} & 0 & 0 & \frac{a}{b+c\bar{x}^2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{a_1}{b_1+c_1\bar{y}^2} & \frac{-2a_1c_1\bar{x}\bar{y}}{(b_1+c_1\bar{y}^2)^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which is similar to linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y}) = (0, 0)$. Using the proposition (1), one has following result.

Theorem 9 Assume that $\{(x_n, y_n)\}$ be a positive solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$ where $(\bar{x}, \bar{y}) = (0,0)$. Then, the error term $E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix}$ of every solution of (1) satisfies both of the following asymptotic relations

$$\lim_{n\to\infty} (\|E_n\|)^{\frac{1}{n}} = |\lambda F_j(\bar{x}, \bar{y})|,$$
$$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda F_j(\bar{x}, \bar{y})|,$$

where $\lambda F_j(\bar{x}, \bar{y})$ are the characteristic roots of Jacobian matrix $F_j(\bar{x}, \bar{y})$ about $(0,0)$.

4. Examples

In order to verify our theoretical results we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the systems of nonlinear difference equations (1) and (2). All plots in this section are drawn with mathematica.

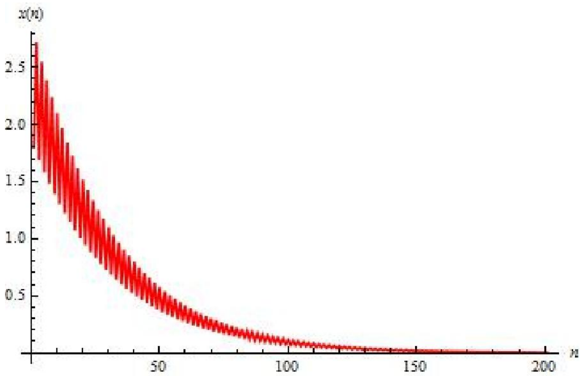
Example 1. Consider the system (1) with conditions $x_{-1} = 2.9, x_0 = 1.8, y_{-1} = 0.5, y_0 = 0.99$. Moreover, choosing the parameters $\alpha = 30,\ \beta = 32, \gamma = 0.009, \alpha_1 = 21, \beta_1 = 22, \gamma_1 = 0.008$

Then, the system (1) can be written as

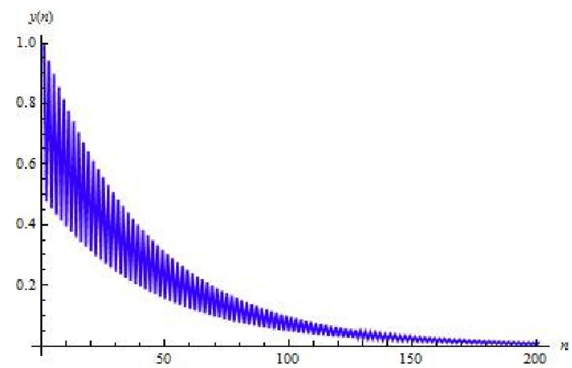
$$x_{n+1} = \frac{30 x_{n-1}}{32 + 0.009 y_n^2},\ y_{n+1} = \frac{21 y_{n-1}}{22 + 0.008 x_n^2},\ n = 0, 1, \cdots, \quad (16)$$

and with conditions $x_{-1} = 2.9, x_0 = 1.8, y_{-1} = 0.5, y_0 = 0.99$.

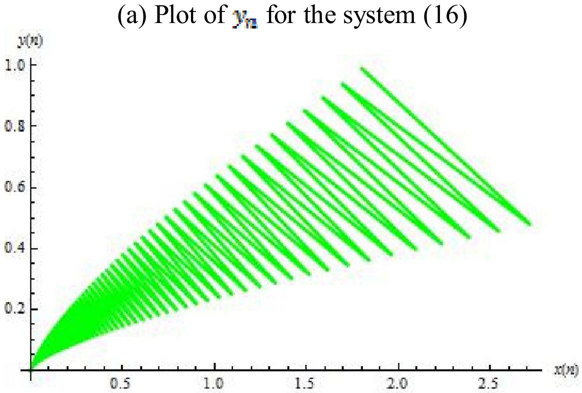
Moreover, in Fig. 1 the plot of x_n is shown in Fig. 1a, the plot of y_n is shown in Fig. 1b and an attractor of the system (16) is shown in Fig. 1c.



(a) Plot of x_n for the system (16)



(a) Plot of y_n for the system (16)



(c) An attractor for the system (16)

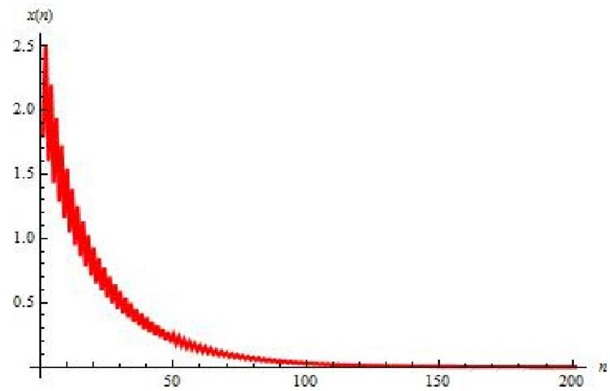
Figure 1: Plots for the system (16)

Example 2. Consider the system (1) with initial conditions $x_{-1} = 2.9, x_0 = 0.8, y_{-1} = 1.8, y_0 = 0.99$. Moreover, choosing the parameters $\alpha = 12,\ \beta = 13, \gamma = 0.9, \alpha_1 = 19, \beta_1 = 20, \gamma_1 = 0.2$. Then, the system (1) can be written as

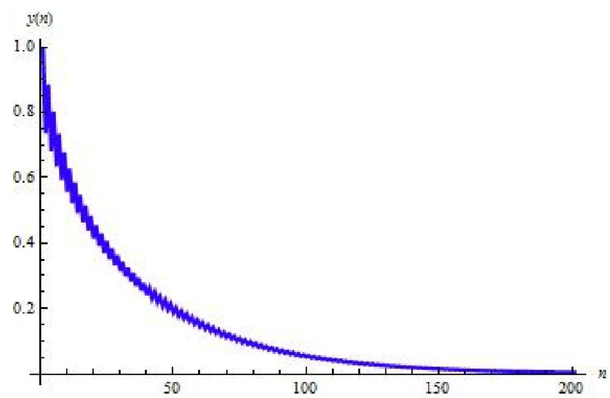
$$x_{n+1} = \frac{12x_{n-1}}{13+0.9y_n^2}, y_{n+1} = \frac{19y_{n-1}}{20+0.2x_n^2}, n = 0,1,\cdots, \qquad (17)$$

and with conditions $x_{-1} = 2.9, x_0 = 0.8, y_{-1} = 1.8, y_0 = 0.99$.

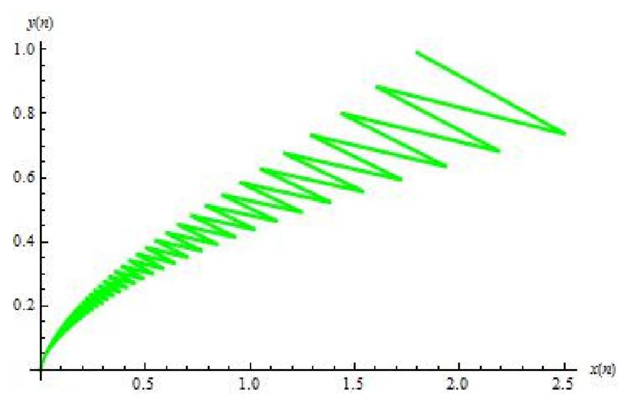
Moreover, in Fig. 2 the plot of x_n is shown in Fig. 2a, the plot of y_n is shown in Fig. 2b and an attractor of the system (17) is shown in Fig. 2c.



(a) Plot of x_n for the system (17)



(b) Plot of y_n for the system (17)



(c) An attractor of the system (17)

Figure 2: Plots for the system (17)

Example 3. Consider the system (2) with initial conditions $x_{-1} = 1.9, x_0 = 1.98, y_{-1} = 1.6, y_0 = 0.77$. Moreover, choosing the parameters

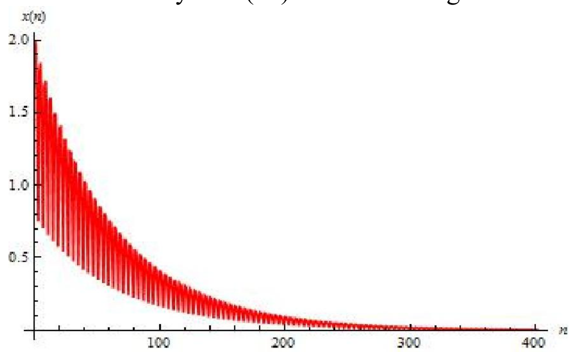
$a = 111, b = 113, c = 0.3, a_1 = 115, b_1 = 120, c_1 = 0.1$

. Then, the system (2) can be written as

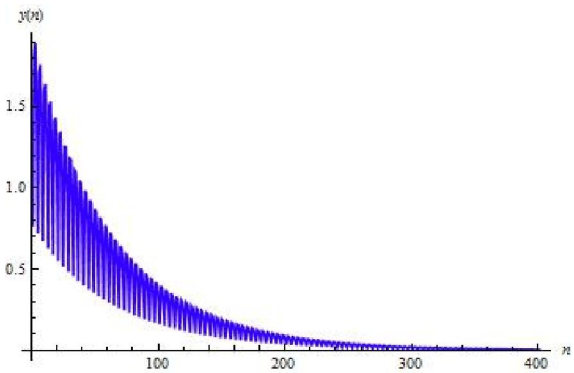
$$x_{n+1} = \frac{111 y_{n-1}}{113 + 0.3 x_n^2}, y_{n+1} = \frac{115 x_{n-1}}{120 + 0.1 y_n^2}, n = 0, 1, \cdots, \quad (18)$$

and with conditions $x_{-1} = 1.9, x_0 = 1.98, y_{-1} = 1.6, y_0 = 0.77$.

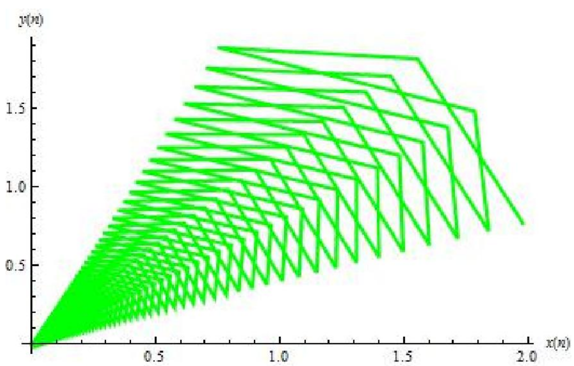
Moreover, in Fig. 3 the plot of x_n is shown in Fig. 3a, the plot of y_n is shown in Fig. 3b and an attractor of the system (18) is shown in Fig. 3c.



(a) Plot of x_n for the system (18)



(a) Plot of y_n for the system (18)



(c) An attractor for the system (18)

Figure 3: Plots for the system (18)

Example 4. Consider the system (2) with initial conditions $x_{-1} = 0.9, x_0 = 1.9, y_{-1} = 1.8, y_0 = 0.5$.

Moreover, choosing the parameters.

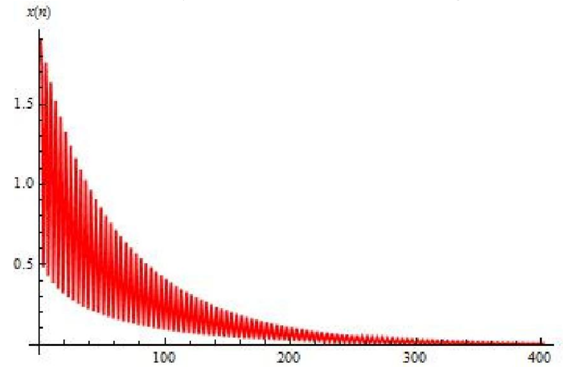
$a = 150, b = 151, c = 1.4, a_1 = 125, b_1 = 131, c_1 = 2.9$

Then, the system (2) can be written as

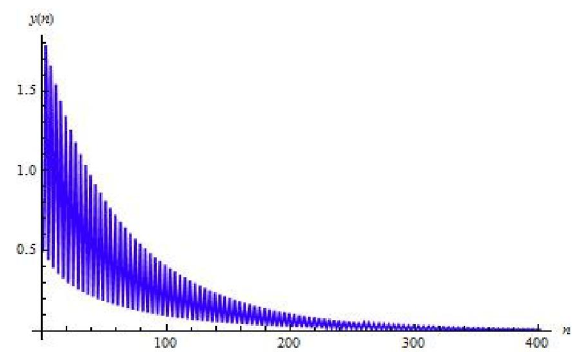
$$x_{n+1} = \frac{150 y_{n-1}}{151 + 1.4 x_n^2}, y_{n+1} = \frac{125 x_{n-1}}{131 + 2.9 y_n^2}, n = 0, 1, \cdots, \quad (19)$$

and with conditions $x_{-1} = 0.9, x_0 = 1.9, y_{-1} = 1.8, y_0 = 0.5$.

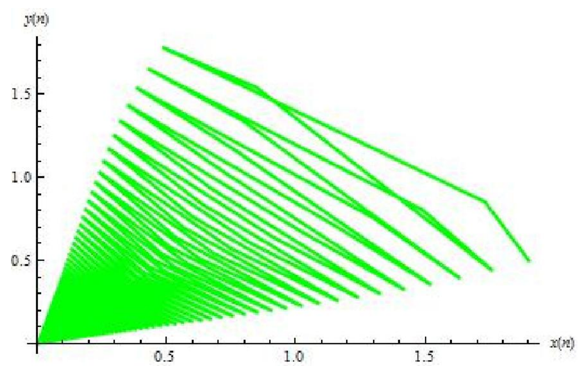
Moreover, in Fig. 4 the plot of x_n is shown in Fig. 4a, the plot of y_n is shown in Fig. 4b and an attractor of the system (19) is shown in Fig. 4c.



(a) Plot of x_n for the system (19)



(b) Plot of y_n for the system (19)



(c) An attractor of the system (19)

Figure 4: Plots for the system (19)

Conclusions

In the paper, we have investigated the qualitative behavior of two four-dimensional discrete dynamical systems. Each system has only one equilibrium point which is stable under some restriction to parameters. The most important finding here is that the unique equilibrium point $(0,0)$ can be a global asymptotic attractor for the systems (1) and (2). Moreover, we have determined the rate of convergence of a solution that converges to the equilibrium point $(0,0)$ of the systems (1) and (2). Some numerical examples are provided to support our theoretical results. These examples are experimental verifications of theoretical discussions.

Acknowledgement

This work was supported by the Higher Education Commission of Pakistan.

2010 AMS Mathematics subject classifications: 39A10, 40A05.

References

1. Agarwal RP, Difference Equations and Inequalities: Second Edition, Revised and Expended, Marcel Dekker, New York, 2000, pp. 980.
2. Grove E A, Ladas G, Periodicities in Nonlinear Difference Equations, Chapman and Hall/CRC Press, Boca Raton, (2004).
3. Sedaghat H, Nonlinear difference equations: Theory with applications to social science models, Kluwer Academic Publishers, Dordrecht, (2003).
4. Kocic VL, Ladas G, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic, Dordrecht (1993).
5. Gibbons CH, Kulenovic MR S, Ladas G, On the recurrence sequence $x_{n+1}=\frac{\alpha+\beta x_{n-1}}{\gamma+x_n}$, Math. sci. Res. Hot-Line 4(2) (2000)1-11.
6. Hamza AE, Khalaf-Allah R, dynamics of second order rational difference equation, BAMS, 23(1), (2008) Golden Jubilee Year Volume, 206-214.
7. Bajo I, Liz E, Global behaviour of a second-order nonlinear difference equation, J. Differ. Eq. Appl., 17(10), 1471-1486(2011).
8. Kalabusic S, Kulenovic MRS, Pilav E, Dynamics of a two-dimensional system of rational difference equations of Leslie--Gower type, Advances in Difference Equations,(2011), doi:10.1186/1687-1847-2011-29.
9. Kalabusic S, Kulenovic MRS, Pilav E, Global Dynamics of a Competitive System of Rational Difference Equations in the Plane, Advances in Difference Equations, (2009), Article ID 132802, 30 pages.
10. Kurbanli AS, Cinar C, Yalcinkaya I, On the behavior of positive solutions of the system of rational difference equations $x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1}, y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1}$, Mathematical and Computer Modelling, 53, 1261-1267(2011).
11. Zhang Q, Yang L, Liu J, Dynamics of a system of rational third order difference equation, Advances in Difference Equations, doi: 10.1186/1687-1847-2012-136, 8 pages.
12. Saadati MR, Adibi H, Stability and periodic character of a rational third order differenceequation, Chaos, Solitons and Fractals, 39(2009), 1203--1209.
13. Pituk M, More on Poincare's and Perron's theorems for difference equations, J. Diff. Eq. App., 8(2002), 201--216.
14. Din Q, Qureshi MN, Khan A Q, Dynamics of a fourth-order system of rational difference equations, Advances in Difference Equations, doi:10.1186/1687-1847-2012-216.
15. Khan AQ, Qureshi MN, Din Q, Global dynamics of some systems of higher-order rational difference equations, Ad. Differ. Equ., 1(2013), 354.
16. Din Q, T. Donchev, Global character of a host-parasite model, Chaos Soliton Fractals, 54(2013), 1--7.
17. Din Q, Dynamics of a discrete Lotka-Volterra model, Ad.Differ. Equ, doi: 10.1186/1687-1847-2013-95.
18. Din Q, Khan AQ, Qureshi MN, Qualitative behavior of a host-pathogen model, Ad. Differ. Equ., 1(2013) ,263.
19. Qureshi MN, Khan AQ, Din Q, Asymptotic behavior of a Nicholson-Bailey model, Ad. Diff. Eq., 1(2014), 62.

3/26/2014