Effects of couple stresses and variable suction/injection on the unsteady MHD flow of an Eyring Powell fluid between two parallel porous plates

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Abstract: The unsteady flow of an incompressible Eyring Powell fluid between two parallel porous plates with variable injection/suction velocity under the action of couple stresses and a uniform magnetic field is analyzed. The first order approximate solution is obtained using the Mathematica software while finite difference scheme is employed for the second order approximate solution of the resulting nonlinear partial differential equation. Damping behavior of fluid flow with increasing effect of couple stresses is found. It is predicted that in an electrically conducting polar fluids, couple stress effects may also be large. The effects of various non-dimensional parameters emerging in the model are discussed and presented graphically.

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1. Introduction

Channel flows of an electrically conducting viscous fluid under the action of transverse magnetic field have been studied by many workers because of its applications in many devices such as accelerators, magnetohydrodynamic (MHD) pumps, MHD power generators, electrostatic precipitation, petroleum industry, electrostatic precipitation, purification of crude oil, aerodynamics heating and fluid droplets sprays. Various workers [1–3] have examined the channel flows of a viscous fluid under the action of a transversely applied magnetic field. These results are important for designing duct wall and cooling arrangement. Because of growing use of non-Newtonian fluids in various manufacturing and processing industries, considerable efforts have been made towards understanding their flows. However, there is not a single governing equation which exhibits all the properties of non-Newtonian fluids and, therefore, many constitutive equations for non-Newtonian fluids have been proposed. Non-Newtonian fluids differ from Newtonian fluids in that the relationship between the shear stress and the flow field is more complicated. Soaps, food products, polymer solutions, glues, inks, etc. are few examples of such fluids. Rana et al. [4-6] analyzed non-Newtonian fluids in various geometries.

The flow through porous plates is of great importance both in technological as well as biophysical point of view. Examples of such flows are found in soil mechanics, transpiration cooling, food preservation, cosmetic industry, blood flow and artificial dialysis. A large number of theoretical investigations dealing with unsteady incompressible flow with either injection or suction had been carried out. Eldabe et al. [7] analyzed an unsteady MHD flow of Eyring Powell model [8] between two parallel porous plates. Assuming time dependent pressure gradient, exact solution for velocity distribution was obtained in the first approximation while in the second approximation a numerical solution was obtain taking constant pressure gradient. Effect of couple stresses on the MHD unsteady flow of Eyring Powell model between two porous plates is presented by Eldabe et al. [9]. Assuming a pulsatile pressure gradient, in the first approximation the solution is obtained using symbolic program-Mathematica whereas in the second order approximation the resulting nonlinear partial differential equation is solved numerically. Very interesting results regarding pulsatile pressure gradient and couple stresses are presented.

The existence of couple stresses in materials was originally postulated by Voigt [10]. However, Cosserat and Cosserat [11] were the first to develop a mathematical model to analyze materials with couple stresses. This idea was generalized much later by Toupin [12], Mindlin and Tiersten [13], Koiter [14] and the others. Couple stresses in fluid theory were developed by Stokes [15].

In the present paper the unsteady flow of an incompressible electrically conducting Eyring Powell fluid between two parallel porous plates with variable injection/suction is considered. A uniform magnetic field normal to the flow is applied. The effect of couple stresses is also taken into account. The solutions for the first and second-order approximations are obtained using symbolic program-Mathematica and finite difference procedure respectively. Damping behavior of fluid flow with increasing effect of couple stresses is observed. The possible applications of this work are the flow of blood through arteries where the boundaries are porous and movement of underground oil, where there is a natural magnetic field and the earth is considered as a porous boundary. The Eyring Powell model can be employed in some cases to depict the viscous behavior of polymer solutions and viscoelastic suspensions over a wide range of shear rates.

2. Basic Equations

The fundamental equations governing the flow are

$$\nabla.\mathbf{V} = \mathbf{0}, \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \mathbf{T} + \mathbf{J} \times \mathbf{B} - \eta \nabla^4 \mathbf{V}, \tag{2}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{3}$$

with

$$\mathbf{B} = \mathbf{B_0} + \mathbf{b}, \tag{4}$$

The constitutive equation of an Erying Powell model is

$$\mathbf{T} = -p\mathbf{I} + \left\{ \mu + \frac{1}{\beta} \frac{\sinh^{-1} \frac{\sqrt{tr(\mathbf{A}_1)^2}}{2c}}{\frac{\sqrt{tr(\mathbf{A}_1)^2}}{2}} \right\} \mathbf{A}_1, \tag{5}$$

while

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{6}$$

$$\mathbf{L} = grad\mathbf{V}, \tag{7}$$

where ρ is fluid density, d/dt is the material derivative, ∇ is the gradient operator, $\mathbf{T}$ is stress tensor, $\mathbf{J}$ is electric current density, σ electric conductivity of the fluid, $\mathbf{E}$ is electric field, $\mathbf{V}$ is the velocity field, $\mathbf{B}$ is total magnetic field, $\mathbf{B_0}$ is the external magnetic field and $\mathbf{b}$ is induced magnetic field, $\eta\nabla^4\mathbf{V}$ gives the effect of couple stresses, μ, β and c are fluid parameter, $\mathbf{L}^T$ denotes the transpose of $\mathbf{L}$.

3. Formulation of the problem

Let us consider the unsteady flow of an incompressible electrically conducting Eyring Powell fluid between two parallel porous plates. The fluid is injected into the lower wall at $y = 0$ and is sucked through upper wall at $y = L$ with the variable velocity, where x and y denotes the space coordinates measured parallel and normal to the surface, u and v are the velocity of the fluid in x and y direction. Since the plates are infinite along the x direction and at the time of observation plates starts their oscillation and produce some disturbance in the y direction so all physical quantities will be the function of the independent variable y. The velocity field is given by

$$\mathbf{V} = [u(y,t), v(y,t), 0]. \tag{8}$$

Then the Eq. (1) becomes

$$\frac{\partial v}{\partial y} = 0,$$

which gives $v = f(t)$.

The function $f(t)$ is taken as a variable velocity which represents the velocity of injection/suction through the plates. We take

$$v = V_0\left(1 + \varepsilon A e^{iwt}\right), \tag{9}$$

where V_0 is a non-zero constant mean suction/injection velocity, ε is small, A is real positive constant such that $\varepsilon A \le 1$. A uniform magnetic field $\mathbf{B_0}$ perpendicular to the plates is applied. Under the low magnetic Reynolds number approximation the induced magnetic field $\mathbf{b}$ is neglected and electric field is assumed to be zero, then equation of motion (2) gives

$$\frac{\partial u}{\partial t} + V_0\left(1 + \varepsilon A e^{iwt}\right)\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(T_{xy}\right) - \frac{\sigma B_0^2}{\rho}u - \frac{\eta}{\rho}\frac{\partial^4 u}{\partial y^4}, \tag{10}$$

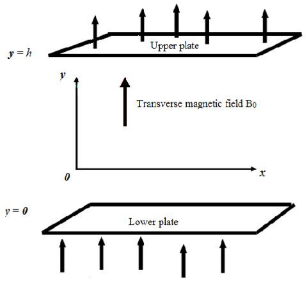

Fig. 1. Geometry of the problem.

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = V_0 iAwe^{iwt}. \quad (11)$$

This means that $\partial p/\partial y$ is small which can be neglected. Also assuming that $U(t) = V_0\left(1 + \varepsilon e^{iwt}\right)$ is free stream velocity of the form

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{dU}{dt}.$$

From Eqs. (10) and (11), we get

$$\frac{\partial u}{\partial t} + V_0\left(1 + \varepsilon Ae^{iwt}\right)\frac{\partial u}{\partial y} = \frac{dU}{dt} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(T_{xy}\right) - \frac{\sigma B_0^2}{\rho}u - \frac{\eta}{\rho}\frac{\partial^4 u}{\partial y^4}. \quad (12)$$

The stress tensor for the Erying Powell model is of the form

$$\tau_{xy} = \mu\frac{\partial u}{\partial y} + \frac{1}{\beta}\sinh^{-1}\left[\left(\frac{1}{c}\frac{\partial u}{\partial y}\right)\right], \quad (13)$$

where μ is the coefficient of viscosity, β and c are the characteristics of the Eyring Powell fluid model. We take the first and second order approximation of the $\sinh^{-1}$ function as

$$\sinh^{-1}\left(\frac{1}{c}\frac{\partial u}{\partial y}\right) \cong \frac{1}{c}\frac{\partial u}{\partial y} - \frac{1}{6}\left(\frac{1}{c}\frac{\partial u}{\partial y}\right)^3, \quad \left|\frac{1}{c}\frac{\partial u}{\partial y}\right| < 1, \quad (14)$$

Then Eq. (12) will be reduced to

$$\frac{\partial u}{\partial t} + V_0\left(1 + \varepsilon Ae^{iwt}\right)\frac{\partial u}{\partial y} = \frac{dU}{dt} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} - \frac{1}{2\rho\beta c^3}\left(\frac{\partial u}{\partial t}\right)^2\frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\rho}B_0^2 u - \frac{\eta}{\rho}\frac{\partial^4 u}{\partial y^4}. \quad (15)$$

The appropriate conditions are

$$\left.\begin{aligned} &u = 0, u'' = 0 \quad \text{at } y = 0,\\ &u = 0, u'' = 0 \quad \text{at } y = L,\\ &u = V_0\sin\left(\frac{\pi y}{L}\right), 0 < y < 1, t \le 0.\end{aligned}\right\}, \quad (16)$$

where a dash means differentiation with respect to independent variable y. Introducing non-dimensional parameters

$$x^* = \frac{x}{L},\ y^* = \frac{y}{L},\ t^* = \frac{V_0}{L}t,\ u^* = \frac{u}{V_0}. \quad (17)$$

Substituting values from Eq. (17) into Eqs (15) and (16) and dropping " $*$ " for convenience, we obtain

$$\frac{\partial u}{\partial t} + \left(1 + \varepsilon Ae^{iwt}\right)\frac{\partial u}{\partial y} = \varepsilon iwe^{iwt} + \frac{N^*}{R_e}\frac{\partial^2 u}{\partial y^2} - D^*\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} - \frac{H_a^2}{R_e}u - \frac{1}{a^2 R_e}\frac{\partial^4 u}{\partial y^4}, \quad (18)$$

subject to the boundary conditions

$$\left.\begin{aligned} &u = 0, u'' = 0 \quad \text{at } y = 0,\\ &u = 0, u'' = 0 \quad \text{at } y = 1,\\ &u = \sin(\pi y),\ 0 < y < 1, t \le 0.\end{aligned}\right\} \quad (19)$$

with

$$\left.\begin{aligned} &R_e = \frac{V_0 L}{\nu} \equiv \text{Reynolds number},\\ &H_a = B_0 L\sqrt{\frac{\sigma}{\rho\nu}} \equiv \text{Hartmann number},\\ &\nu = \frac{\mu}{\rho},\ N^* = 1 + M,\ M = \frac{1}{\mu c\rho},\\ &D^* = \frac{V_0}{2\rho\beta c^3 L^3},\ a^2 = \frac{L^2}{h^2},\ h^2 = \frac{\eta}{\mu},\end{aligned}\right\} \quad (20)$$

where M and D^* represent non-Newtonian effects.

4. Analytic solutions

The solution in the first approximation is obtained using symbolic computational program "Mathematica". In the second approximation a numerical solution of the non-linear partial differential is obtained using finite difference method.

4.1 First approximation

In view of first approximation of Eq. (14), the Eq. (18) reduces to

$$\frac{\partial u}{\partial t}+\left(1+\varepsilon A e^{iwt}\right)\frac{\partial u}{\partial y}=\varepsilon i w e^{iwt}+\frac{N^*}{R_e}\frac{\partial^2 u}{\partial y^2}-\frac{H_a^2}{R_e}u-\frac{1}{a^2 R_e}\frac{\partial^4 u}{\partial y^4}. \quad (21)$$

Consider the perturbation

$$u=u_0+u_1 e^{iwt}, \quad (22)$$

then the Eq. (21) yields

$$\frac{1}{a^2 R_e}\frac{d^4 u_0}{dy^4}-\frac{N^*}{R_e}\frac{d^2 u_0}{dy^2}+\frac{du_0}{dy}+\frac{H_a^2}{R_e}u_0=0, \quad (23)$$

and

$$\frac{1}{a^2 R_e}\frac{d^4 u_1}{dy^4}-\frac{N^*}{R_e}\frac{d^2 u_1}{dy^2}+\frac{du_1}{dy}+\left(\frac{H_a^2}{R_e}+iw\right)u_1=Aiw. \quad (24)$$

The boundary conditions are

$$\left.\begin{aligned} u_0=u_1=0, u_0''=u_1''=0 \text{ at } y=0,\\ u_0=u_1=0, u_0''=u_1''=0 \text{ at } y=1.\end{aligned}\right\} \quad (25)$$

The Eqs. (23) and (24) subject to boundary conditions (25) are solved using symbolic software "Mathematica" for various values of dimensionless parameters a, R_e, H_a, M. Only graphical representation of these solutions is given in Figs. 2-5, and the detail is omitted just for space saving.

4.2 Second approximation

From Eq. (18) we obtain

$$\frac{\partial u}{\partial t}+\left(1+\varepsilon A\cos wt\right)\frac{\partial u}{\partial y}=-\varepsilon\sin wt-\frac{H_a^2}{R_e}u+\left(\frac{N^*}{R_e}-D^*\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial y^2}-\frac{1}{a^2 R_e}\frac{\partial^4 u}{\partial y^4}. \quad (26)$$

The Eq. (26) is highly nonlinear partial differential equation and therefore its analytic solution is impossible. The finite difference procedure will be employed for its numerical solution. Moreover, we note that the Eq. (26) includes non-Newtonian effects M and D^* which died out in first approximation (Newtonian flows). Let

$$u_{ij}=u\left(y_i,t_j\right),$$

then

$$\frac{\partial u_{ij}}{\partial y}=\frac{u_{i+1,j}-u_{ij}}{\Delta y},\ \frac{\partial u_{ij}}{\partial t}=\frac{u_{i,j+1}-u_{ij}}{\Delta t}, \quad (27)$$

$$\frac{\partial^2 u_{ij}}{\partial y^2}=\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{\left(\Delta y\right)^2}, \quad (28)$$

$$\frac{\partial^4 u_{ij}}{\partial y^4}=\frac{4}{3}\left[\begin{pmatrix} u_{i+2,j}-4u_{i+1,j}+6u_{ij}\\ -4u_{i-1,j}+u_{i-2,j}\end{pmatrix}/\left(\Delta y\right)^4\right]. \quad (29)$$

Replacing the finite difference approximations of partial derivatives involved in the Eq. (26) and defined by Eqs. (27)-(29), we obtain finite-difference equation after simplification given by

$$u_{i,j+1}=\Delta t\left\{\begin{aligned}&\left(-\varepsilon\sin wt_j\right)+\begin{pmatrix}\frac{1}{\Delta t}+\frac{\left(1+\varepsilon A\cos wt_j\right)}{\Delta y}-\frac{H_a^2}{R_e}\\ -\frac{8}{a^2R_e\left(\Delta y\right)^4}-\frac{2N^*}{R_e\left(\Delta y\right)^2}\end{pmatrix}u_{ij}\\ &+\left(\frac{N^*}{R_e\left(\Delta y\right)^2}-\frac{\left(1+\varepsilon A\cos wt_j\right)}{\Delta y}+\frac{16}{3a^2R_e\left(\Delta y\right)^4}\right)u_{i+1,j}\\ &+\left(\frac{N^*}{R_e\left(\Delta y\right)^2}+\frac{16}{3a^2R_e\left(\Delta y\right)^4}\right)u_{i-1,j}\\ &-\frac{4}{3a^2R_e\left(\Delta y\right)^4}u_{i+2,j}+\frac{4}{3a^2R_e\left(\Delta y\right)^4}u_{i-2,j}\\ &-\begin{bmatrix}\left(u_{i+1,j}\right)^3-4u_{i,j}\left(u_{i+1,j}\right)^2-2\left(u_{i,j}\right)^3\\ -3\left(u_{i,j}\right)^2u_{i+1,j}+\left(u_{i+1,j}\right)^2u_{i-1,j}\\ +\left(u_{i,j}\right)^2u_{i-1,j}-2u_{i,j}u_{i+1,j}u_{i-1,j}\end{bmatrix}\frac{D^*}{\left(\Delta y\right)^4}\end{aligned}\right\}. \quad (30)$$

The initial and boundary conditions are

$$\left.\begin{aligned}0<y<1, u=\sin\pi y \quad \text{for } t\le 0\\ y=0,\ 1,\ u=0,\ u''=0 \quad \text{for } t>0\end{aligned}\right\} \quad (31)$$

This Eq. (30) subject to conditions (31) is solved numerically for different value of dimensionless parameters involved in the equation and graphs are sketched to visualize the effects of these parameters in Figs. 6-10

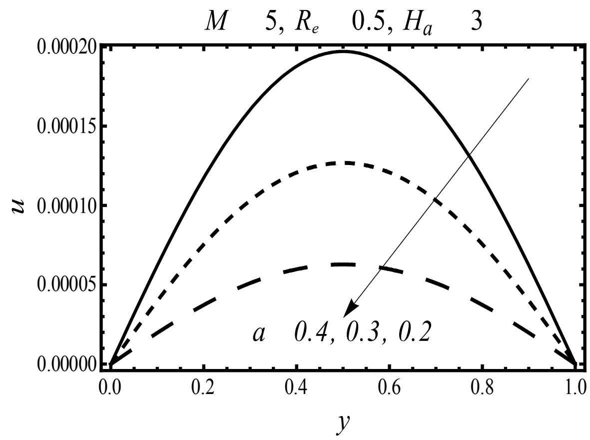

Fig. 2(a). Influence of a on velocity field at t = 12.5.

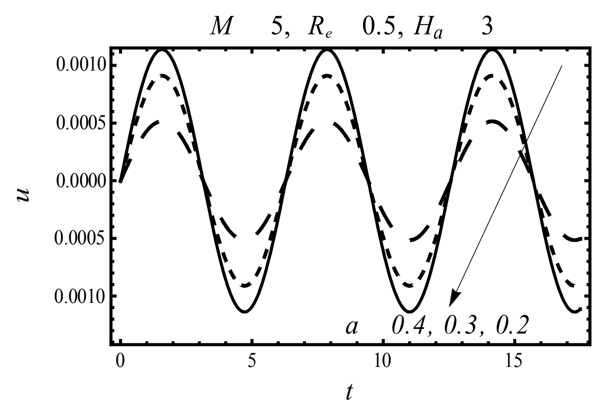

Fig. 2(b). Influence of a on velocity field at y = 1.5.

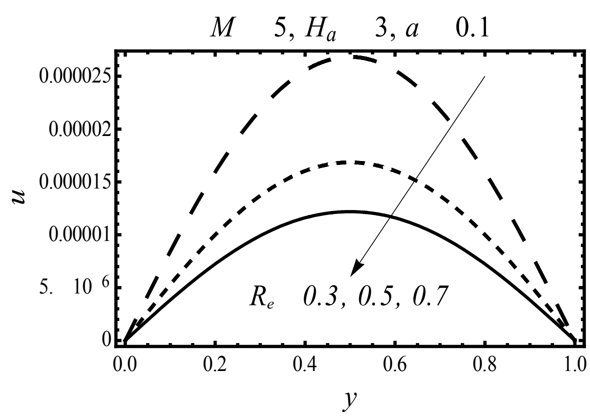

Fig. 3(a). Influence of R_e on velocity field at t =12.5.

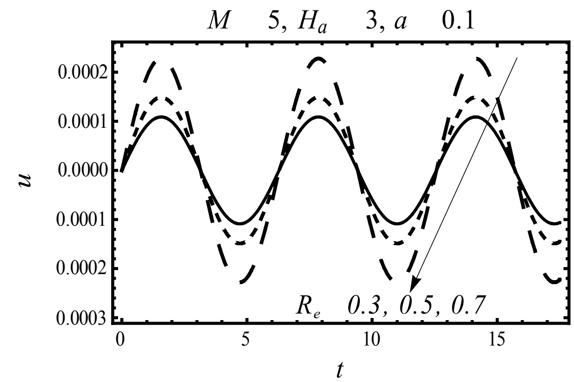

Fig. 3(b). Influence of R_e on velocity field at y =1.5.

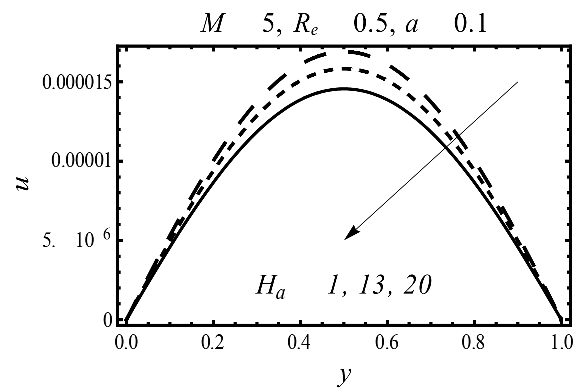

Fig. 4(a). Influence of H_a on velocity field at t =12.5.

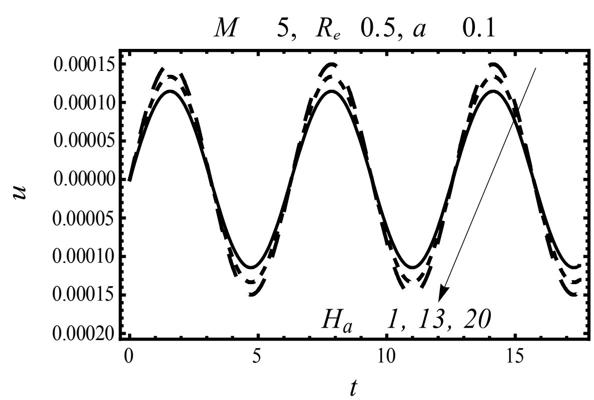

Fig. 4(b). Influence of H_a on velocity field at y = 1.5.

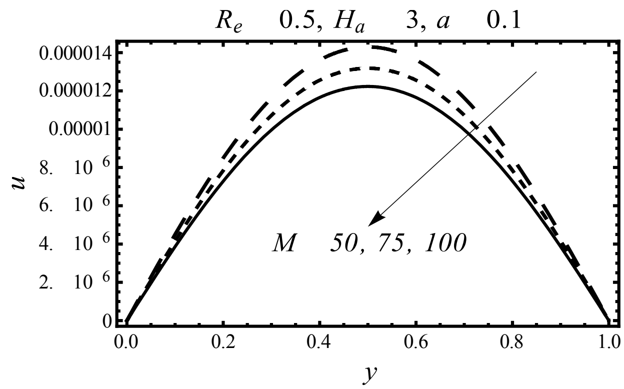

Fig. 5(a). Influence of *M* on velocity field at *t* = 12.5.

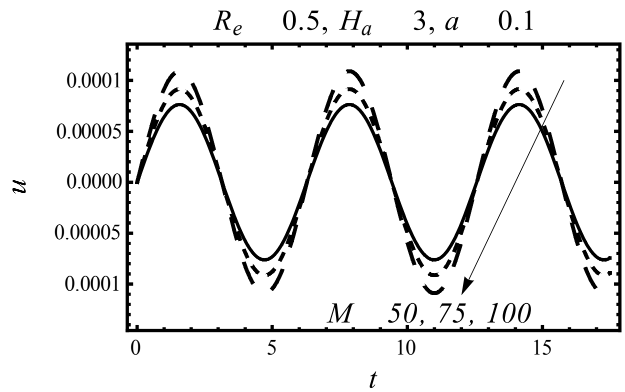

Fig. 5(b). Influence of *M* on velocity field at *y* = 1.5.

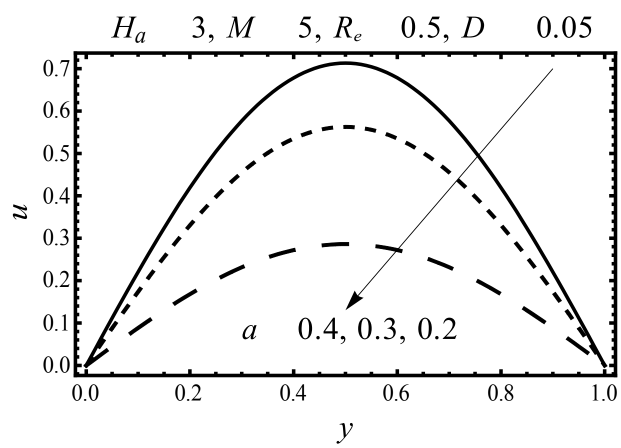

Fig. 6. Variation of velocity for different values of *a*-2nd approximation.

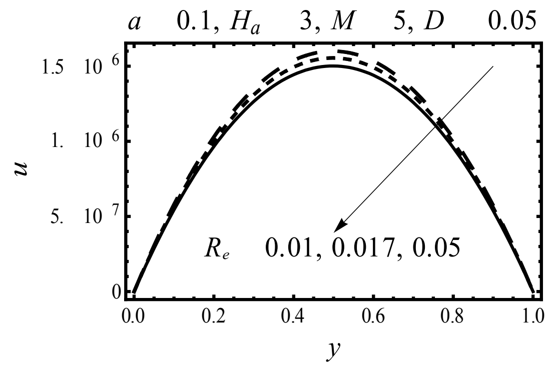

Fig. 7. Variation of velocity for different values of R_e -2nd approximation.

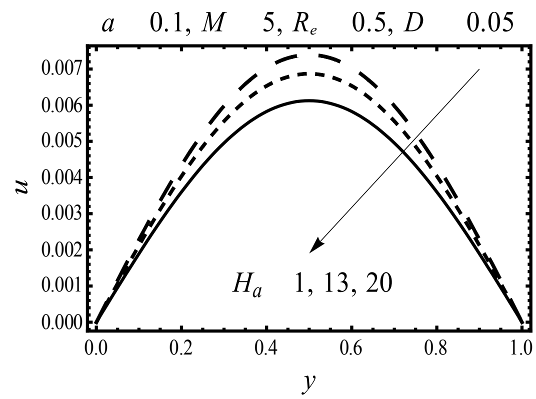

Fig. 8. Variation of velocity for different values of H_a-2nd approximation.

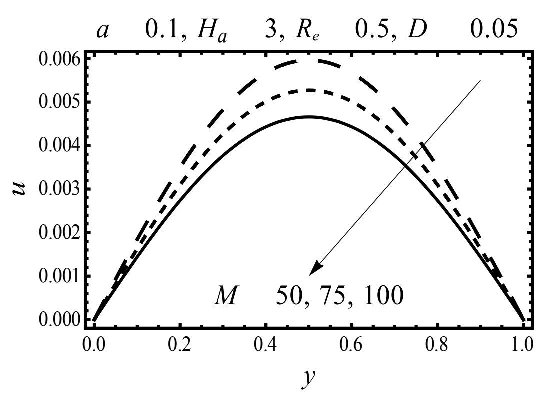

Fig. 9. Variation of velocity for different values of *M*-2nd approximation.

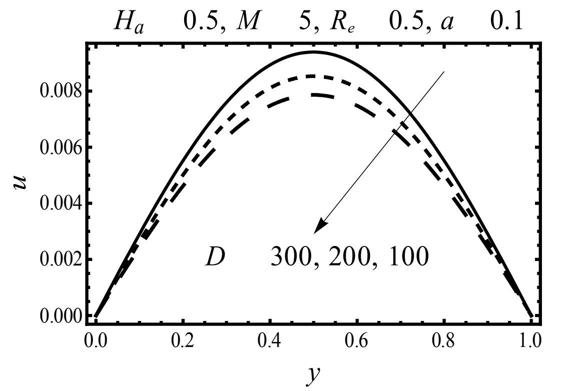

Fig. 10. Variation of velocity for different values of D^* -2nd approximation.

5. Results and discussion

The unsteady flow of an incompressible MHD flow of Eyring Powell fluid between two parallel porous plates with time dependent suction/injection is modeled and analyzed numerically. The effects of couple stresses, Reynolds number Re, Hartmann number Ha, non-Newtonian material parameters M and D^* on velocity field are studied graphically in Figs. 2-10. The Figs. 2-5 demonstrate velocity profiles for the first approximation whereas Figs. 6-10 exhibit the same ones for the second approximation.

The effect of parameter a^2 (which is the inverse of couple stresses) is shown in Fig. 2 (in the case of first approximation), and in Fig. 6 (in the case of second approximation). It is obvious that velocity increases with an increase in a^2. It is well known that couple stresses for the model like the Eyring Powel depends on the vorticity gradients. As vorticity gradients are known to be large in hydromagnetic flows of nonpolar fluids causing a remarkable reduction in the flow of the fluid.

The Figs. 3 and 7 describe the influence of Reynolds number over the velocity field in the first and second approximations respectively. It is observed that velocity decreases as the Reynolds number increases. Physically it shows that viscous forces are dominant over inertia forces. The effect of Hartmann parameter Ha on velocity profile is presented in Figs. 4 and 8 for the first and second approximations respectively. It is noted that an increase in Ha reduces the velocity due to the magnetic force effect against the flow.

For the first and second approximations, the effects of non-Newtonian parameter M on velocity field are depicted in Figs. 5 and 9 respectively. It is noted that velocity decreases with an increase in non-Newtonian parameter M. This is physically justified.

The effect of second approximation parameter D^* is exposed in Fig 10. It reveals that velocity increases with an increase in D^*.

6. Concluding remarks

In this study we have investigated an unsteady flow of Eyring Powell fluid flow between two parallel porous plates under the action of a uniform external magnetic field applied normal to the velocity field. The finding of the present study may be summarized as follows:

- Flow is damping with increasing effect of couple stresses. In many cases, this result may be very useful for the discussion of some diseases of the blood
- Hydromagnetic flows of nonpolar fluids causes incredible degradation in the flow of the fluid, which may be useful tool in studying the blood flow in the arteries, particularly the defective ones
- Hartmann number provides a useful mechanism for decreasing the flow which leads to decrease the blood pressure and any related diseases
- In electrically conducting polar fluids, couple stress effects may also be expected to be large
- It is noted that viscous forces are dominant over the inertia forces

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