Topological Structure of Generalized Rough Multisets

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Abstract: Rough set theory is a powerful mathematical tool for dealing with inexact, uncertain or vague information. The core concept of rough set theory are information systems and approximation operators of approximation spaces. In this paper, we study the relationships between mset relations and mset topology. Moreover, this paper concerns generalized mset approximation spaces via topological methods and studies topological properties of rough msets. Classical separation axioms for M-topological spaces are extended to generalized mset approximation spaces. Relationships among separation axioms for generalized mset approximation spaces and relationships between M-topological spaces and their induced generalized mset approximation spaces are investigated.

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1. Introduction

In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset or bag, for short). Thus, a multiset differs from a set in the sense that each element has a multiplicity – a natural number not necessarily one – that indicates how many times it is a member of the multiset. One of the most natural and simplest examples is the multiset of prime factors of a positive integer 'n'. The number 504 has the factorization $504 = 2^3\ 3^2\ 7^1$ which gives the multiset {2, 2, 2, 3, 3, 7}.

In any information system, some situations may occur, where the respective counts objects in the universe of discourse are not single. In such situations we replace its universe of discourse by multisets called rough multisets. The motivation to use rough multisets has come from the need to represent sub multisets of a multiset in terms of m-equivalence classes of a partition of that multiset (universe). The mset equivalence relation and mset partitions are explained in (Girish, *et al.,* 2009). The mset partition characterizes an M-topological space, called an approximation mset space (M, R) where M is an mset called the universe and R is an equivalence mset relation. The m- equivalence classes of R are also known as granules with repetition or elementary msets or blocks. $[m/x] \subseteq M$ is used to denote the m-equivalence class containing m/x in M. In the approximation mset, there are two operators, the upper mset approximation and lower mset approximation of submsets.

The concept of rough multisets and related properties with the help of lower mset approximation and upper mset approximations are important frameworks for certain types of information multisystems (Chan, 2004; Chang, 2010).

An interesting and natural research topic in rough set theory is to study rough set theory via topology. Indeed, (Polkowski, 2002) pointed: ''topological aspects of rough set theory were recognized early in the framework of topology of partitions". (Skowron, 1988) and (Wiweger, 1988) separately discussed this topic for classical rough set theory. (Polkowski, 2001) constructed and characterized topological spaces from rough sets based on information systems. (Pawlak, 1991) and (Polkowski, 2002) summarized related work respectively. (Kortelainen, 1994) considered relationships between modified sets, topological spaces and rough sets based on a pre-order (also see Jarvinen *et al.,* 2007). (Lin, 1992) continued to discuss this topic, and established a connection between fuzzy rough sets and topology. Furthermore, using topology and neighborhood systems (Lin, 1998) established a model for granular computing. Some authors discussed relationships between generalized rough sets and topology from different viewpoints. (Skowron, *et al.,* 1996; Skowron *et al.,* 2005) generalized the classical approximation spaces to tolerance approximation spaces, and discussed the problems of attribute reduction in these spaces. (Lashin *et al.,* 2005) introduced the topology generated by a subbase, also defined a topological rough membership function by the subbase of the topology. Other papers on this topic we refer to (Kondo, 2006; Zhu, 2007; Thuan,2009; Pei *et al.,* 2011; Yang *et al.,* 2011; Li *et al.,* 2012; Li *et al.,* 2012;). In addition, connections between fuzzy rough set theory and fuzzy topology were also investigated (see ; Qin *et al.,* 2005; Li, *et al.,* 2008).

We first introduce and study in Section 2 some properties of msets, mset relations and rough mset theory. Any reflexive binary mset relation determines an M-topology, also, if R is a reflexive and symmetric mset relation on a mset M, then $\tau = \{ A \subseteq M : \underline{R}(A) = A \}$ is a M-topology such that A is open if and only if it is closed, finally, for every topological space (M, τ) satisfying the condition that A is open if and only if it is closed, there exists a reflexive and symmetric relation θ such that $\tau = \{ A \subseteq M : \underline{R}(A) = A \}$ are investigated in Section 3. Moreover, this paper concerns generalized mset approximation spaces via topological methods and studies topological properties of rough msets. Classical separation axioms for M-topological spaces are extended to generalized mset approximation spaces. Relationships among separation axioms for generalized mset approximation spaces and relationships between M-topological spaces and their induced generalized mset approximation spaces are investigated in two Sections 4 and 5. At last, some conclusion is presented in section 6.

2. M-relations and M-topology

In this subsection, a brief survey of the notion of sets introduced by (Yager, 1986), the different types of collections of msets and the basic definitions and notions of relations in mset context introduced by (Girish *et al.,* 2009; 2012) are presented.

Definition 2.1. (Jena *et al.,* 2001). An mset *M* drawn from the set *X* is represented by a function Count *M* or C_M defined as $C_M : X \rightarrow N$ where *N* represents the set of non negative integers.

In Definition 2.1, $C_M(x)$ is the number of occurrences of the element *x* in the mset. *M*. However those elements which are not included in the mset *M* have zero count. Let M_1 and M_2 be two msets drawn from a set *X*. Then the following are defined (see Jena, et al., 2001):

(1) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.

(2) $M \subseteq N$ if $C_M(X) \leq C_N(x)\ \forall x \in X$.

(3) $P = M \cup N$ if $C_P(x) = max\{ C_M(x), C_N(x)\}$ for all $x \in X$.

(4) $P = M \cap N$ if $C_P(x) = min\{ C_M(x), C_N(x)\}$ for all $x \in X$.

Definition 2.2 (Jena *et al.,* 2001). A domain *X*, is defined as a set of elements from which msets are the mset space $[X]^m$ is the set of all msets whose elements are *X* such that no element in the mset occurs more than *m* times.

If $X = \{x_1, x_2, \ldots, x_k\}$ then $[X]^m = \{\{m_1/x_1, m_2/x_2, \ldots, m_k/x_k\}$: for $i = 1, 2, \ldots, k$; $m_i \in \{0, 1, 2, \ldots, m\}$. Henceforth *M* stands for a multiset drawn from the multiset space $[X]^m$.

Definition 2.3 (Jena *et al.,* 2001). Let *M* be an mset drawn from a set *X*. The support set of *M* denoted by M^* is a subset of *X* and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., M^* is an ordinary set and it is also called root set.

Definition 2.4 (Jena *et al.,* 2001). Let *X* be a support set and $[X]^m$ be the mset space defined over *X*. Then for any mset $M \in [X]^m$, the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that $C_M^c(x) = m - C_M(x)$ for all $x \in X$.

Let *M* be an mset from $X = \{x_1, x_2, \ldots, x_n\}$ with *x* appearing *n* times in *M*. It is denoted by $x \in^n M$. The mset $M = \{\{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ drawn from *X* means that *M* is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. A new notation can be introduced for the purpose of defining Cartesian product of two multisets,

relation on multisets and its domain and co-domain. The entry of the form $(m/x, n/y)/k$ denotes that *x* is repeated *m*-times, *y* is repeated *n*-times and the pair (x, y) is repeated *k*-times. The counts of the members of the domain and co-domain vary in relation to the counts of the *x* co-ordinate and *y* co-ordinate in $(m/x, n/y)/k$. The notation $C_1(x, y)$ and $C_2(x, y)$ is therefore introduced. $C_1(x, y)$ denotes the count of the first co-ordinate in the ordered pair (x, y) and $C_2(x, y)$ denotes the count of the second co-ordinate in the ordered pair (x, y).

Definition 2.5 (Blizard, 1989). Let $M \in [X]^m$ be an mset. The power mset *P(M)* of *M* is the set of all the submsets of *M*. i.e., $N \in P(X)$ if and only if $N \subseteq M$. If $N = \phi$ then $N \in^1 P(M)$; and if $N \neq \phi$, then $N \in^k P(M)$ where $k = \Pi_z \left(\begin{smallmatrix} |[M]_z| \\ |[N]_z| \end{smallmatrix} \right)$, the product Π_z is taken over by distinct elements of *z* of the mset *N* and $|[M]_z| = m$ iff $z \in^m M$, $|[N]_z| = n$ iff $z \in^n N$, then $\Pi_z \left(\begin{smallmatrix} |[M]_z| \\ |[N]_z| \end{smallmatrix} \right) = \binom{m}{n} = \frac{m!}{n!(m-n)!}$.

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$. Power mset is an mset but its support set is an ordinary set whose elements are msets.

Definition 2.6 (Girish *et al.,* 2009). A submset R of $M_1 \times M_2$ is said to be an mset relation on M if every member $(m/x,\ n/y)$ of R has a count, the product of $C_1(x, y)$ and $C_2(x, y)$. m/x related to n/y is denoted by $(m/x)\ R\ (n/y)$.

Definition 2.7 (Girish *et al.,* 2009). Let M be an mset in $[X]^m$. Then the following are defined.

(1) An mset relation R on an mset M is reflexive if and only if $(m/x)\ R\ (m/x)$ for all m/x in M, irreflexive if and only if $(m/x)\ R\ (m/x)$ never holds.

(2) An mset relation R on an mset M is symmetric if and only if $(m/x)\ R\ (n/y)$ implies $(n/y)\ R\ (m/x)$, antisymmetric if and only if $(m/x)\ R\ (n/y)$ and $(n/y)\ R\ (m/x)$ implies m/x and n/y are equal.

(3) An mset relation R on an mset M is transitive if $(m/x)\ R\ (n/y)$, $(n/y)\ R\ (k/z)$, then $(m/x)\ R\ (k/z)$.

A mset relation R on a mset M is called an equivalence mset relation if it is reflexive, symmetric and transitive. A mset relation R on a mset M is called a partial ordered mset relation if it is reflexive, antisymmetric and transitive. A mset relation R on a mset M is called a preorder relation if it is reflexive and transitive.

Definition 2.8 (Girish *et al.,* 2012). Let $M \in [X]^m$ and $P^*(M)$. Then τ is called a multiset topology if τ satisfies the following properties.

(1) ϕ and M are in τ.

(2) The union of the elements of any sub collection of τ is in τ.

(3) The intersection of the elements of any finite sub collection of τ is in τ.

Mathematically, a multiset topological space is an ordered pair $(M;\ \tau)$ consisting of an mset $M \in [X]^m$ and a multiset topology $\tau \subseteq P^*(M)$ on M. Note that τ is an ordinary set whose elements are msets and the multiset topology is abbreviated as an M-topology. A submset U of an M-topological space M is an open mset of M if U belongs to the M-topology. Also, a submset U of an M-topological space M is called closed if U^C is open (Girish *et al.,* 2012).

Dfinition 2.9. (Girish *et al.,* 2012) Let R be an mset relation on M. The successor-set of $x \in^m M$ is defined as $R_s(m/x) = \{n/y : \exists$ some k with $(k/x)\ R\ (n/y)\}$ and the predecessor-set

of $x \in^m M$ is defined as $R_p(m/x) = \{n/y : \exists$ some k with $(n/y)\ R\ (k/x)\}$.

Theorem 2.1. (Girish *et al.,* 2012). If R is an mset relation on M, then the successor class $=\{R_s(m/x) : x \in^m M\}$ form a sub M-base for an M-topology τ on M and the predecessor class $= \{R_p(m/x) : x \in^m M\}$ form a sub M-base for a dual M-topology of τ on M.

Definition 2.10. (Girish *et al.,* 2012). If M is an mset, an M-basis for an M-topology on M is a

collection **B** of partial whole submsets of M (called M-basis element) such that

(i) For each $x \in^m M$, for some $m > 0$, there is at least one M-basis element $B \in$ **B** containing m/x. i.e., for each mset in **B** there is at least one element with full multiplicity as in M.

(ii) If $m{=}x$ belongs to the intersection of two M-basis elements M_1 and M_2, then there is an M-basis element M_3 containing m/x such that $M_3 \subseteq M_1 \cap M_2$. i.e., there is an M-basis element M_3 containing an element with full multiplicity as in M and that element must be in M_1 and M_2 also.

Definition 2.11. (Girish *et al.,* 2012). Given a submset A of an M-topological space M, the closure of an mset A is defined as the intersection of all closed msets containing A and is denoted by $Cl(A)$; $i{:}e{:};Cl(A) = \cap\{K \subseteq M : K$ is a closed mset and $A \subseteq K\}$ and $C_{Cl(A)}(x) = Min\{C_K(x) : A \subseteq K\}$.

Definition 2.12. (Girish *et al.,* 2012). A closure mset space is a pair $(M;Cl)$ where M is any multiset and $Cl : P^*(M) \Rightarrow P^*(M)$ is a mapping with each element $A \subseteq M$, a submset $Cl(A) \subseteq M$, called the closure of A such that

(i) $Cl(\phi) = \phi$.
(ii) $A \subseteq Cl(A)$.
(iii) $Cl(Cl(M)) = Cl(M)$.
(iv) $Cl(A \cup B) = Cl(A) \cup Cl(B)$.

Definition 2.13. (Girish *et al.,* 2012). Given a submset A of an M-topological space M, the interior of an mset is defined as the union of all open msets contained in A and is denoted by $I(A)$,i.e., $I(A) = \cup\{G \subseteq M : G$ is an open mset and $G \subseteq A\}$ and $C_{I(A)}(x) = Max\{C_G(x) : G \subseteq A\}$.

Definition 2.14. (Girish *et al.,* 2012) The operator $I : P^*(M) \to P^*(M)$ is called an interior operator, if it satisfies the following rules. For every $A;B \subseteq M$,

(i) $I(A \cap B) = I(A) \cap I(B)$.
(ii) $I(A) \subseteq A$.
(iii) $I(M) = M$.
(iv) $I(I(M)) = I(M)$.

3. On the structure of generalized rough msets

Let M be a non-empty mset and R be a binary mset relation on M. By the support set of $P(M)$, $P^*(M)$, and for all $A \subseteq M$, we define operations $\underline{R}$, $\overline{R}$ from $P^*(M)$ to itself by

$$\underline{R}(A) = \{x \in^m M : R_s(m/x) \subseteq A\}$$

$$\overline{R}(A) = \{x \in^m M : R_s(m/x) \cap A \neq \phi\}$$

$\underline{R}(A)$ is called a lower mset approximation of A and $\overline{R}(A)$ an upper mset approximation of A. The mset A is called definable if $\underline{R}(A) = \overline{R}(A)$. The pair $(M;R)$ is called a generalized mset approximation space (GMAS) or generalized rough mset (Abo-Tabl, 2014a).

Proposition 3.1. (Abo-Tabl, 2014b). If R is reflexive, then we have for all $A \subseteq M$,

1- $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$;

2- If $A \subseteq B$, then $\underline{R}(A) \subseteq \underline{R}(B)$;

3- $\underline{R}(A^c) = \{\overline{R}(A)\}^c$.

Moreover we can induce a mset topology by a reflexive mset relation. Let R be a binary mset relation on M. We define $\tau = \{A \subseteq M : \underline{R}(A) = A\}$.

Proposition 3.2. If R is reflexive, then τ is an M-topology on M.

Proof. We only prove that $U_\lambda \in \tau$ for every $\lambda \in \Lambda$ imply $\cup_\lambda U_\lambda \in \tau$. Let $U_\lambda \in \tau$ for every $\lambda \in \Lambda$ and $x \in^m \cup_\lambda U_\lambda$. There exists $\mu \in \Lambda$ such that $x \in^m U_\mu = \underline{R}(U_\mu)$. For all n/y such that $m/x\ R\ n/y$, we have $y \in^n U_\mu \subseteq \cup_\lambda U_\lambda$. This means that $\cup_\lambda U_\lambda \in \underline{R}(\cup_\lambda U_\lambda)$, that is $\cup_\lambda U_\lambda = \underline{R}(\cup_\lambda U_\lambda)$. Therefore, $\tau = \{A \subseteq M : \underline{R}(A) = A\}$ is an M-topology on M.

Proposition 3.3. If R is reflexive and transitive, then $\underline{R}$ is an interior operator and $\overline{R}$ a closure one.

Proposition 3.4. If R is reflexive and symmetric, then $(M; \tau)$ is the M-topological space such that it satisfies the condition (sym) $B^c \subseteq \underline{R}(\underline{R}(B))^c$ for all $B \subseteq M$, where $\tau = \{A \subseteq M : \underline{R}(A) = A\}$.

Proof. It is obvious that τ is an M-topology. We have to show that τ satisfies the condition (sym). If $x \notin^m \underline{R}(\underline{R}(B))^c$, then there exists n/y such that $m/x\ R\ n/y$ but $y \notin^n (\underline{R}(B))^c$. Since R is symmetric and $y \in^n \underline{R}(B)$, we have $x \in^m B$, that is, $x \notin^m B^c$. This implies that $B^c \subseteq \underline{R}(\underline{R}(B))^c$.

Proposition 3.5. Let R be reflexive and symmetric binary mset relation. For all $A \subseteq M$, $\underline{R}(A) = A \Leftrightarrow \underline{R}(A^c) = A^c$.

Proof. Assume that $\underline{R}(A) = A$. It is sufficient to show that $A^c \subseteq \underline{R}(A^c)$. If $x \in^m \underline{R}(A^c)$, then there exists n/y such that $m/x\ R\ n/y$ but $y \in^n A = \underline{R}(A)$. Since R is symmetric, we get $n/y\ R\ m/x$ and hence x $x \in^m A$, that is, $x \notin^m A^c$. This means that $A^c \subseteq \underline{R}(A^c)$. The converse can be proved similarly.

From Proposition 3.5 we have the following proposition.

Proposition 3.6. If R is a reflexive and symmetric mset relation on M, then the M-topological space $(M; \tau)$ has a property that (clop): A is M-open if and only if A is M-closed.

Proof. A is M-open if and only if $A \in \tau$ if and only if $\underline{R}(A) = A$ if and only if $A^c = \underline{R}(A^c)$ if and only if $A^c \in \tau$ if and only if A^c is M-open if and only if A is M-closed.

From Propositions 3.4 and 3.5 we have

Proposition 3.7. If R is a reflexive and symmetric mset relation on M. Then the two conditions are equivalent to each other:

(sym) : $B^c \subseteq \underline{R}(\underline{R}(B))^c$ for all $B \subseteq M$,

(clop) : B is M-open if and only if B is M-closed for all $B \subseteq M$.

Thus we have the following corollary.

Corollary 3.1. If R is an equivalence mset relation on M, then $(M; \tau)$ is the M-topological space induced by the interior operator R such that it satisfies the conditions:

(sym) : $B^c \subseteq \underline{R}(\underline{R}(B))^c$ for all $B \subseteq M$,

(clop) : B is M-open if and only if B is M-closed for all $B \subseteq M$.

Where $\tau = \{A \subseteq M : \underline{R}(A) = A\}$.

In particular.

1. If we take $R = \{(m/x, m/x) : x \in^m M\}$, then $\tau = P^*(M)$, that is, τ is the discrete M-topology.

2. If we take $R = \{(m/x, n/y) : m/x, n/y \in M\}$, then $\tau = \{M; \phi\}$, that is, τ is the indiscrete M-topology.

Summing up, we have established that

(i) If R is a reflexive and transitive mset relation on M, then $(M; \tau)$ is an M-topological space induced by an interior operator R.

(ii) If R is a reflexive and symmetric mset relation on M, then the M-topological space $(M; \tau)$ has a property that A is M-open if and only if A is M-closed for all $A \subseteq M$.

Now we ask whether the converse of above hold, and the answer in the following theorems.

Theorem 3.1. Let $(M;\tau)$ be an M- topological space satisfying the condition (clop). Then there exists a reflexive and symmetric mset relation θ on M such that $\tau = \{A \subseteq M : \underline{\theta}(A) = A\}$.

Proof. Assume that $(M;\tau)$ is an M-topological space satisfying the condition (clop). At first, we take $IB = \cup\{O \in \tau : O \subseteq B\}$. For all $m/x,\ n/y \in M$, we define an mset relation θ by $(m/x, n/y) \in \theta \Leftrightarrow \forall B \subseteq M$ $(m/x \in IB \to n/y \in B)$. It follows from definition of I that for all $B;C \subseteq M$

(i) $IB \in \tau$

(ii) $IB \subseteq B$

(iii) $B \subseteq C \Rightarrow IB \subseteq IC$

(iv) $I(IB) = IB$

(v) $B \in \tau$ if and only if $B = IB$.

It is clear that θ is reflexive and symmetric mset relation. From (ii) above we can show that for all $A \subseteq M$, we have $IA \subseteq \underline{\theta}(A)$. Conversely, let $x \notin^m IA$. We take $\Gamma = \{B : x \in^m IB\} \cup \{A^c\}$. For this Γ, we can conclude $\cap\Gamma \neq \phi$. Otherwise we assume that $\cap\{B : x \in^m IB\} \cap A^c = \phi$. Since $IB \subseteq B$, we have $\cap\{IB : x \in^m IB\} \cap A^c = \phi$ and hence $\cap\{IB : x \in^m IB\} \subseteq A$. This implies that $I(\cap\{IB : x \in^m IB\}) \subseteq IA$. Since IB is M-open in $(M;\tau)$, it follows from (clop) that IB is M-closed, and hence $I(\cap\{IB : x \in^m IB\}) = \cap\{I(IB) : x \in^m IB\} = \cap\{IB : x \in^m IB\}$. This means that $x \in^m \cap\{IB : x \in^m IB\} \subseteq IA$. But this contradicts to $x \notin^m IA$. Hence we have $\cap\Gamma \neq \phi$. There exists $y \in^n \cap\Gamma$. For this element n/y, we have $(m/x, n/y) \in \tau$ and $y \in^n A^c$ that is, $y \notin^n A$. Hence $x \notin^m \underline{\theta}(A)$. This means that $\underline{\theta}(A) \subseteq IA$ and therefore $\underline{\theta}(A) = IA$.

We give the following condition to prove the next theorem.

(comp) : If for all $O_\lambda \in \tau$, $A \subseteq M$ and $\cap O_\lambda \cap A = \phi$, then there exists a finite submset $\{Oi\}$ $(i \leq n)$ of $\{O\lambda\}$ such that $O_1 \cap O_2 \cap \ldots \cap O_n \cap A = \phi$.

Theorem 3.2. Let $(M;\tau)$ be an M-topological space induced by an interior operator I. If $(M;\tau)$ satisfying the condition (comp), Then there exists a reflexive and transitive mset relation θ on M such that $\underline{\theta}(A) = IA$ for all $A \subseteq M$.

Proof. Assume that $(M;\tau)$ is an M-topological space induced by an interior operator I satisfying the condition (comp). We define an mset relation θ on M as above, that is, for all $m/x,\ n/y \in M$, $(m/x, n/y) \in \theta \Leftrightarrow \forall B \subseteq M(x \in^m IB \to y \in^n B)$. It is clear that θ is reflexive and symmetric mset relation. We shall show that $\underline{\theta}(A) = IA$ for all $A \subseteq M$. Since it is clear that $IA \subseteq \underline{\theta}(A)$, we only show that $\underline{\theta}(A) \subseteq IA$. Assume that $x \notin^m IA$. As in the proof above we take $\Gamma = \{B : x \in^m IB\} \cup \{A^c\}$. For this Γ, we can conclude $\cap\Gamma \neq \phi$ under the condition (comp). Otherwise we may assume that $\cap\{B : x \in^m IB\} \cap A^c = \phi$. Since $IB \subseteq B$, we have $\cap\{IB : x \in^m IB\} \cap A^c = \phi$. Since IB is M-open (i.e., $IB \in \tau$), it follows from (comp) that there exists a finite sub-mset $\{IBi\}$ $(i \leq n)$ of $\{IB : x \in^m IB\}$ such that $IB_1 \cap IB_2 \cap \ldots \cap IB_n \cap A^c = \phi$, and hence that $IB_1 \cap IB_2 \cap \ldots \cap IB_n \subseteq A$. Since I is the interior operator, we have $IB_1 \cap IB_2 \cap \ldots \cap IB_n = I(IB_1 \cap IB_2 \cap \ldots \cap IB_n) \subseteq IA$. Thus $x \in^m IA$. But this is a contradiction. This means that $\cap\Gamma \neq \phi$. There exists $y \in^n \cap\Gamma$. For this element n/y, we have $(m/x, n/y) \in \theta$ and $y \in^n A^c$ that is, $y \notin^n A$. Hence $x \notin^m \underline{\theta}(A)$. It is follows that $\underline{\theta}(A) \subseteq IA$ and therefore $\underline{\theta}(A) = IA$.

4. Separation axioms T_i (i = 0,1,2) of GMA-spaces.

Separations T_i (i = 0,1,2) of M-topological spaces are important topological properties and applied or extended into many branches of mathematics. In this section, we first characterize separations T_i (i = 0,1,2) of Alexandrov topological spaces and then extend them to GMA-spaces. We will also describe T_i^a (i = 0,1,2) GMA-spaces by upper mset approximations.

Definition 4.1. An M-topological space M is called a T_o M-space if for any two different points $m/x,\ n/y \in M$, there exists an open mset A such that $m/x \in A$, $n/y \notin A$ or an open mset B such that $n/y \in B$, $m/x \notin B$.

It is well known that an M-topological space M is a T_0 M-space iff for any $m/x,\ n/y \in M$, $m/x \neq n/y$ implies that $Cl(\{x\}) \neq Cl(\{y\})$. By this fact, one can immediately have the following proposition.

Proposition 4.1. Let $(M;R)$ be a topological GMA-space. Then the following statements are equivalent:

(1) $(M; \tau_R)$ is a T_0 M-space;
(2) R is antisymmetric;
(3) R is a partial order;
(4) for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $\overline{R}(\{m/x\}) \neq \overline{R}(\{n/y\})$.

If R is not a preorder, then statements (2)-(4) in Proposition 4.1 need not be equivalent to each other. So we can extend axiom T_o to GMA-spaces in at least two ways.

Definition 4.2. A GMA-space $(M;R)$ is called a T_o^u GMA-space if for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $\overline{R}(\{m/x\}) \neq \overline{R}(\{n/y\})$.

Proposition 4.2. Let $(M;R)$ be a GMA-space. Then $(M;R)$ is a T_o^u GMA-space iff for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $R_p(m/x) \neq R_p(n/y)$.

Proof. It follows from the fact that for all $m/x \in M$; $\overline{R}(m/x) = R_p(m/x)$.

Definition 4.3. A GMA-space $(M;R)$ is called a T_o^a GMA-space if for all m/x, $n/y \in M$, $m/x \neq n/y$ implies (m/x) R^c (n/y) or (n/y) R^c (m/x).

Proposition 4.3. Let $(M;R)$ be a GMA-space. Then $(M;R)$ is a T_o^a GMA-space iff for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $m/x \notin R_s(n/y)$ or $n/y \notin R_s(m/x)$ iff for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $m/x \notin R_p(n/y)$ or $n/y \notin R_p(m/x)$.

Proof. Clear by Definition 4.3 and the meanings of R^c, R_s and R_p.

By Proposition 4.3, it is easy to see that $(M;R)$ is a T_o^a GMA-space iff for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $m/x \notin \overline{R}(n/y)$ or $n/y \notin \overline{R}(m/x)$.

For an M-topological space $(M; \tau)$ and its induced GMA-space (M, R_τ), we have

Theorem 4.1. If (M,τ) is a T_0 space, then the induced GMA-space $(M;R_\tau)$ is both a T_o^u GMA-space and a T_o^a GMA-space.

Proof. Since $(M; \tau)$ is a T_o space, R_τ is a partial order. Then by Proposition 4.1, Definitions 4.2 and 4.3 we have that (M, R_τ) is both a T_o^u GMA-space and a T_o^a GMA-space.

Definition 4.4. An M-topological space M is called a T_1 M-space (resp., T_2 M-space) if for any two different points m/x, $n/y \in M$, there exist open neighborhoods A of m/x and B of n/y such that $n/y \notin A$ and $m/x \notin B$ (resp., $A \cap B = \phi$).

It is well known that an M-topological space M is a T_1 M-space iff for each $m/x \in M$, $Cl(\{m/x\}) = \{m/x\}$. By this fact and that in a topological GMA-space (M, R), $R_s(m/x)$ is the smallest open neighborhood of m/x, one can easily show the following proposition.

Proposition 4.4. For a topological GMA-space (M, R), the following statements are equivalent:
(1) (M, R_τ) is a T_1 M-space;
(2) (M, R_τ) is a T_2 M-space;
(3) R is the discrete order on M;
(4) for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $m/x \notin R_s(n/y)$ and $n/y \notin R_s(m/x)$;
(5) for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $R_s(n/y) \cap R_s(m/x) = \phi$.

In terms of Proposition 4.4 (4) and (5), we can extend separation axioms T_1 and T_2 to general GMA-spaces as follows.

Definition 4.5. Let (M, R) be a GMA-space. If for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $m/x \notin R_s(n/y)$ and $n/y \notin R_s(m/x)$, then (M, R) is called a T_1^a GMA-space.

Definition 4.6. Let (M, R) be a GMA-space. If for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $R_s(n/y) \cap R_s(m/x) = \phi$, then (M, R) is called a T_2^a GMA-space.

By the definition of T_2^a GMA-spaces, it is easy to see that (U, R) is a T_2^a GMA-space iff for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $m/x \notin \overline{R}(R_s(n/y))$. Similarly, (M, R) is a T_1^a GMA-space iff for all m/x, $n/y \in M$, $m/x \neq n/y$ implies $m/x \notin \overline{R}(n/y)$ and $n/y \notin \overline{R}(m/x)$.

For a M-topological space (M, τ) and its induced GMA-space (M, R_τ), we have

Theorem 4.2. If (M, τ) is a T_1 M-space, then the induced GMA-space (M, R_τ) is both a T_1^a GMA-space and a T_2^a GMA-space.

Proof. Since (M, τ) is a T_1 space, R_τ is a discrete order. Then by Proposition 4.4, Definitions 4.5 and 4.6, we see that (M, R_τ) is both a T_1^a GMA-space and a T_2^a GMA-space.

It is well known that for an M-topological space, T_2 implies T_1 and T_1 implies T_o.

However, for a GMA-space which is not M-topological, we have the following counterexample.

Example 4.1. Let $M = \{3/a, 2/b, 4/c\}$ and $R = \{(3/a, 2/b), (2/b, 3/a), (2/b, 4/c)\}$. Then $R_s(3/a) = \{2/b\}$, $R_s(2/b) = \{3/a, 4/c\}$ and $R_s(4/c) = \phi$. Since $R_s(3/a) \cap R_s(2/b) = R_s(2/b) \cap R_s(4/c) = R_s(4/c) \cap R_s(3/a) = \phi$, (U, R) is a T_2^a GMA-space. However, for $3/a \neq 2/b$, we have $3/a \in R_s(2/b)$ and $2/b \in R_s(3/a)$. By Definitions 4.3 and 4.5, (M, R) is neither a T_1^a GMA-space nor a T_o^a GMA-space.

What is to our surprise is that for a GMA-space, T_1^a implies T_o^a and T_2^a.

Theorem 4.3. If (M, R) is a T_1^a GMA-space, then (M, R) is both a T_o^a GMA-space and a T_2^a GMA-space.

Proof. It is trivial that T_1^a implies T_o^a. Now we show that T_1^a implies T_2^a. For all $m/x, n/y \in M$, $m/x \neq n/y$, it follows from axiom T_1^a that $n/y \notin R_s(m/x)$. So $R_s(m/x) \subseteq \{m/x\}$ for each $x \in^m M$. It follows that for all $m/x, n/y \in M$, $m/x \neq n/y$ implies that $R_s(m/x) \cap R_s(n/y) \subseteq \{m/x\} \cap \{n/y\} = \phi$. Thus (M, R) is a T_2^a GMA-space.

We know that there are many practical problems involving information systems and GMA-spaces. Separations T_i^a (i = 0, 1, 2) of GMA-spaces may provide some concrete information by distinguishing different objects when we apply them to solve such problems.

5 Regularity and normality of GMA-spaces

In this section, we extend regularity and normality of M-topological spaces to GMA-spaces.

Definition 5.1. An M-topological space M is called regular if for each closed mset $A \subseteq M$ and any point $x \in^m A$ there are open msets W and V such that $x \in^m W$, $A \subseteq V$ and $W \cap V = \phi$.

It is well known that an M-topological space M is regular iff for any $x \in^m M$ and any open neighborhood V of m/x, there is an open neighborhood W of m/x such that $W \subseteq V$. When a M-space M has an Alexandrov M-topology then M is regular iff for all $m/x, n/y \in M$, the smallest neighborhood of m/x is different of or equal the smallest neighborhood of n/y. By this fact and Theorem 3.2, we have immediately the following corollary.

Corollary 5.1. Let (M, R) be a topological GMA-space. Then (M, τ_R) is a regular M-space iff for all $m/x, n/y, k/z \in M$, $(m/x)\ R\ (k/z)$ and $(n/y)\ R\ (k/z)$ imply $(m/x)\ R\ (n/y)$.

Making use of the condition in Corollary 5.1, we give the following definition.

Definition 5.2. Let (M, R) be a GMA-space. If $(m/x)\ R\ (k/z)$ and $(n/y)\ R\ (k/z)$ imply $(m/x)\ R\ (n/y)$ for any $m/x, n/y, k/z \in M$, then (M, R) is called a regular GMA-space.

It is easy to see that in a regular GMA-space, one can from $(m/x)\ R\ (k/z)$ and $(n/y)\ R\ (k/z)$ deduce both $(m/x)\ R\ (n/y)$ and $(n/y)\ R\ (m/x)$.

The following proposition shows that regularity of a GMA-space may provide some more local information of the involved GMA-space.

Proposition 5.1. Let (M, R) be a regular GMA-space and $x \in^m M$. If there is $y \in^n M$ such that $(m/x)\ R\ (n/y)$, then $(m/x)\ R\ (m/x)$.

Proof. It follows from $(m/x)\ R\ (n/y)$ and $(m/x)\ R\ (n/y)$ that $(m/x)\ R\ (m/x)$ by Definition 5.2.

Regularity of GMA-spaces can also be characterized by complement and inverse relations.

Proposition 5.2. Let (M, R) be a GMA-space. Then the following statements are equivalent:

(1) (M, R) is a regular GMA-space;

(2) $\overline{R}(R_s(m/x)) \subseteq R_s(m/x)$ for all $x \in^m M$;

(3) $\underline{R}(R_s^c(m/x)) \supseteq R_s^c(m/x)$ for all $x \in^m M$;

(4) R^{-1} is an Euclidean relation.

Proof. (1)$\Rightarrow$ (2): Let $x \in^m M$ and $y \in^n \overline{R}(R_s(m/x))$. Then $R_s(n/y) \cap R_s(m/x) \neq \phi$.

Thus there is $z \in^m M$ such that $(m/x)\ R\ (k/z)$ and $(n/y)\ R\ (k/z)$. It follows from the regularity of (M, R) that $(m/x)\ R\ (n/y)$. So $y \in^n R_s(m/x)$ and $\overline{R}(R_s(m/x)) \subseteq R_s(m/x)$.

(2) $\Rightarrow$(1): Let $m/x, n/y, k/z \in M$ with $(m/x)\ R\ (k/z)$ and $(n/y)\ R\ (k/z)$. Then

$z \in^k R_s(n/y) \cap R_s(m/x) \neq \phi$ and $y \in^n \overline{R}(R_s(m/x))$. By (2), we get $y \in^n R_s(m/x)$, i.e., $(m/x)\, R\, (k/y)$. So, (1) holds.

(2) ⇔ (3): Noticing that $R_s^c(m/x) = \{y \in^n M : (m/x) R^c (n/y)\}$ $= \{y \in^n M : (m/x) R (n/y)\}^c = (R_s(m/x))^c$, we have

$\overline{R}(R_s(m/x)) \subseteq R_s(m/x) \Leftrightarrow$

$(\overline{R}(R_s(m/x)))^c \supseteq (R_s(m/x))^c \Leftrightarrow$

$\underline{R}((R_s(m/x))^c \supseteq (R_s(m/x))^c \Leftrightarrow$

$\underline{R}(R_s^c(m/x)) \supseteq R_s^c(m/x)$.

(1) ⇔(4): (M, R) is regular, for all m/x, n/y, $k/z \in M$ with $(m/x)\, R\, (k/z)$ and $(n/y)\, R\, (k/z)$ imply $(m/x)\, R\, (n/y)$, for all m/x, n/y, $k/z \in M$, $(k/z)\, R^{-1}(n/y)$ and $(k/z)\, R^{-1}(m/x)$ imply $(n/y)\, R^{-1}(m/x)$ ⇔ R^{-1} is an Euclidean relation.

For a topological space (M, τ) and its induced GMA-space (M, R_τ), we have

Theorem 5.1. If (M, τ) is a regular M-topological space, then (M, R_τ) is a regular GMA-space.

Proof. Argue by the way of contradiction. Assume that (M, R_τ) is not regular. Then there are m/x, n/y, $k/z \in M$ with $(m/x)\, R_\tau\, (k/z)$ and $(n/y)\, R_\tau\, (k/z)$ but $(m/x) R_\tau^c (n/y)$. Then $x \in^m Cl(\{n/y\})$. Since (M, τ) is regular, there are open neighborhood V_1 of m/x and open neighborhood V_2 of $Cl(\{n/y\})$ such that $V_1 \cap V_2 = \phi$. Since V_1 is an open neighborhood of m/x and $(m/x)\, R_\tau\, (k/z)$, one has $z \in^k V_1$. Noticing that V_2 is also an open neighborhood of n/y and $(n/y)\, R_\tau\, (k/z)$, we have $z \in^k V_2$. Thus $z \in^k V_1 \cap V_2$ and so $V_1 \cap V_2 \neq \phi$, which is a contradiction.

For M-topological spaces, generally, there is no mutual implication between regularity and separation axioms T_o, T_1, T_2. But for GMA-spaces, T_1^a implies regularity.

Theorem 5.2. Every T_1^a GMA-space is a regular GMA-space.

Proof. Let (M, R) be a T_1^a GMA-space and m/x, n/y, $k/z \in M$ with $(m/x)\, R\, (k/z)$ and $(n/y)\, R\, (k/z)$. Then by T_1^a and $(n/y)\, R\, (k/z)$, we get that $n/y = k/z$ and $(m/x)\, R\, (n/y)$. So (M, R) is regular.

For M-topological spaces, regularity and T_o imply T_1. Similarly, for GMA-spaces we have

Theorem 5.3. If (M, R) is a regular T_o^a GMA-space, then (M, R) is a T_2^a GMA-space.

Proof. Let (M, R) be a regular T_o^a GMA-space and m/x, $n/y \in M$ with $m/x \neq n/y$.

We claim that $R_s(m/x) \cap R_s(n/y) = \phi$. If not, that is, there is $z \in^k M$ such that $(m/x)\, R\, (k/z)$ and $(n/y)\, R\, (k/z)$, then by the regularity of (M, R), we have $(m/x)\, R\, (n/y)$ and $(n/y)\, R\, (m/x)$, contradicting that (M, R) is a T_o^a GMA-space. So (M, R) is a T_2^a GMA-space.

Now we start to consider normality of GMA-spaces.

Definition 5.3. A M-topological space M is called normal if for any disjoint closed mets A and B there are open sub msets W, $V \subseteq M$ such that $A \subseteq W$, $B \subseteq V$ and $W \cap V = \phi$.

Recall that a normal M-topological space M is also characterized by that for any closed mset $A \subseteq M$ and any open neighborhood V of A, there is an open neighborhood W of A such that $Cl(W) \subseteq V$. When a M-space M has an Alexandrov M-topology then M is normal iff for any closed mset $A \subseteq M$, the smallest open neighborhood of A is a closed mset. We have immediately the following corollary.

Corollary 5.1. Let (M, R) be a topological GMA-space. Then (M, τ_R) is a normal M-space iff for all m/x, n/y, k/z, $l/u \in M$, if $(n/y)\, R\, (m/x)$, $(n/y)\, R\, (k/z)$ and $(l/u)\, R\, (k/z)$, then there is $v \in^t M$ such that $(t/v)\, R\, (m/x)$ and $(t/v)\, R\, (l/u)$.

In terms of the condition in Corollary 5.1, we can extend normality to GMA-spaces.

Definition 5.4. Let (M,R) be a GMA-space. If for any m/x, n/y, k/z, $l/u \in M$, if $(n/y)\, R\, (m/x)$, $(n/y)\, R\, (k/z)$ and $(l/u)\, R\, (k/z)$, imply that there is $v \in^t M$ such that $(t/v)\, R\, (m/x)$ and $(t/v)\, R\, (l/u)$, then (M, R) is called a normal GMA-space.

From Definition 5.4, we can characterize normality of GMA-spaces by upper mset approximations. It is easy to check that a GMA-space (M, R) is normal iff

$\overline{R}(\overline{R^{-1}}(\overline{R}(\{m/x\}))) \subseteq \overline{R^{-1}}(\overline{R}(\{m/x\}))$ for all $x \in^m M$.

Proposition 5.3. If (M, R) is a normal GMA-space, then for all m/x, $n/y \in M$, $(m/x)\, R\, (n/y)$ implies that there is $v \in^t M$ such that $(t/v)\, R\, (m/x)$ and $(t/v)\, R\, (n/y)$.

Proof. Let m/x, $n/y \in M$, $(m/x)\, R\, (n/y)$. Then by Definition 5.4, it follows from $(m/x)\, R\, (n/y)$, $(m/x)\, R\, (n/y)$ and $(m/x)\, R\, (n/y)$ that there is $v \in^t M$ such that $(t/v)\, R\, (n/y)$ and $(t/v)\, R\, (m/x)$, as desired.

For an M-topological space (M, τ) and its induced GMA-space (M, R_τ), we have

Theorem 5.4. If (M, τ) is a normal M-topological space, then (M, R_τ) is a normal GMA-space.

Proof. Assume that m/x, n/y, k/z, $l/u \in M$, and $(n/y)\, R_\tau\, (m/x)$, $(n/y)\, R_\tau\, (k/z)$ and $(l/u)\, R_\tau\, (k/z)$. If there is no $v \in^t M$ such that $(t/v)\ R_\tau\ (m/x)$ and $(t/v)\ R_\tau\ (l/u)$, then

$Cl(\{m/x\}) \cap Cl(\{l/u\}) = \phi$. Since (M, τ) is normal, there is open neighborhood V_1 of $Cl(\{m/x\})$ and open neighborhood V_2 of $Cl(\{l/u\})$ such that $V_1 \cap V_2 = \phi$. Since $(n/y)\ R_\tau\ (m/x)$, we have $y \in^n Cl(\{m/x\})$ and V_1 is also an open neighborhood of n/y. Then it follows from $(n/y)\ R_\tau (k/z)$ that $z \in^k V_1$. Since V_2 is an open neighborhood of l/u and $(l/u)\ R_\tau\ (k/z)$, we have $z \in^k V_2$. So, $z \in^k V_1 \cap V_2$, contradicting $V_1 \cap V_2 = \phi$. This shows that for all m/x, n/y, k/z, $l/u \in M$ and $(n/y)\ R_\tau\ (m/x)$, $(n/y)\ R_\tau\ (k/z)$ and $(l/u)\ R_\tau\ (k/z)$, there is $v \in^t M$ such that $(t/v)\ R_\tau\ (m/x)$ and $(t/v)\ R_\tau\ (l/u)$. So (M, R_τ) is a normal GMA-space.

The following theorem gives us an unexpected result about the relation of normality and regularity for GMA-spaces, which is very different from the situation for

M-topological spaces.

Theorem 5.5. Every regular GMA-space is a normal GMA-space.

Proof. Assume that $(M;R)$ is a regular GMA-space. Let m/x, n/y, k/z, $l/u \in M$, and $(n/y)\ R\ (m/x)$, $(n/y)\ R\ (k/z)$ and $(l/u)\ R\ (k/z)$. By regularity of (M, R), $(n/y)\ R\ (k/z)$ and $(l/u)\ R\ (k/z)$ imply $(n/y)\ R\ (l/u)$. Pick $t/v = n/y \in M$, then we have $(t/v)\ R\ (m/x)$ and $(t/v)\ R\ (l/u)$. So (M, R) is a normal GMA-space.

The converse of Theorem 5.5 is not true in general. The following example provides a GMA-space which is normal but not regular.

Example 5.1. Let $M = \{3/a, 2/b, 4/c\}$ and $R = \{(3/a, 3/a), (2/b, 2/b), (4/c, 4/c), (4/c, 3/a), (4/c, 2/b)\}$. Then R is a partial order mset relation on M and (M, R) is a

topological GMA-space. For the induced M-topological space (M, τ_R), the family of open msets is $\tau_R = \{\phi, \{3/a\}, \{2/b\}, M\}$ and the family of closed msets is $F_R = \{M, \{2/b, 4/c\}, \{3/a, 4/c\}, \phi\}$. Since there is no disjoint non-empty closed sets in F_R, (M, τ_R) and hence (M, R) is normal. But for the point $3/a$ and the closed mset $\{2/b, 4/c\}$ not containing $3/a$, there is no open msets to separate them, showing that (M, τ_R) and hence (M, R) is not regular.

It is known that every normal T_1 M-topological space is regular. Similarly, for GMA-spaces, axiom T_2^a and normality imply regularity.

Theorem 5.6. If (M, R) is a normal T_2^a GMA-space, then (M, R) is regular.

Proof. Let m/x, n/y, k/z, $l/u \in M$ with $(m/x)\ R\ (k/z)$ and $(n/y)\ R\ (k/z)$. It follows from $(m/x)\ R\ (k/z)$, $(m/x)\ R\ (k/z)$, $(n/y)\ R\ (k/z)$ and normality of (M, R) that there is $v \in^t M$ such that $(t/v)\ R\ (k/z)$ and $(t/v)\ R\ (n/y)$. If $t/v \neq m/x$, then by T_2^a, $R_s(t/v) \cap R_s(m/x) = \phi$. Clearly $z \in^k R_s(t/v) \cap R_s(m/x)$, which is a contradiction, So $t/v = m/x$ and $(m/x)\ R\ (n/y)$, showing that (M, R) is a regular GMA-space.

6. Conclusion

In this paper, we studied GMA-spaces in terms of topological methods and gave further connections between M-topology and rough mset theory. And we discuss the following statements.

1- Any reflexive binary mset relation determines an M-topology.

2- If R is a reflexive and symmetric mset relation on a mset M, then $\tau = \{A \subseteq M : \underline{R}(A) = A\}$ is a M-topology such that A is open if and only if it is closed.

3- For every topological space (M, τ) satisfying the condition that A is open if and only if it is closed, there exists a reflexive and symmetric relation θ such that $\tau = \{A \subseteq M : \underline{\theta}(A) = A\}$.

4- Classical separation axioms for M-topological spaces are extended to generalized mset approximation spaces.

5- Relationships among separation axioms for generalized mset approximation spaces and relationships between M-topological spaces and their induced generalized mset approximation spaces are investigated.

In the future work, we will focus on another topological properties saying connectedness and compactness on generalized rough msets.

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