A Note On Proper Teleparallel Homothetic Motions Of Well Known Spacetime Using Non Diagonal Tetrad

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Abstract: The aim of this paper is to find proper homothetic motions for Einstein universe in context of teleparallel theory of gravitation. We have chosen non diagonal tetrads for the above spacetime and applied direct integration technique to obtain teleparallel homothetic motions. It comes out that the above spacetime do not admit proper teleparallel homothetic motions for the choice of non diagonal tetrad field.
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1. Introduction

In the Universe matter interact with each other through gravitational field. This gravitational field has been remained an unsolved mystery for us. An acceptable interpretation of how matter interacts in this gravitational field was first described by Einstein in his famous general theory of relativity. In this theory he geometrized the matter interaction and curvature of the spacetime has declared responsible for matter interaction. This description of matter interaction was soon challenged as this theory was unable to describe the matter interaction at the quantum level. At this level Einstein once again tried to unify the laws of relativity theory and quantum mechanics. This time an alternate description was given to the matter interaction where gravitation is attributed to the torsion, which is based upon Weitzenböck connection. The new theory known as teleparallel theory of gravitation has zero curvature and torsion in the spacetime compel the particles to feel gravitation. Although this description of gravity failed to unify the laws of gravitation and electromagnetism, this theory has provided an opportunity to reform the laws of physics in an alternate way.

Teleparallel theory of gravitation has gained attention because this theory has given a consistent solution to the problem of localization of energy [Mikhail et al, 1993 and Vargas, 2004]. The energy localization problem was also deeply studied in general relativity by many authors and they defined energy momentum complexes [Lindau et al 1962, Papapetrou, 1948, Tolman, 1934, Bergmann et al 1958]. Those energy-momentum complexes had some limitations and thus were unable to provide physical local energy momentum density. Much interest has been shown to study the teleparallel versions of some exact solutions of general relativity [Obukhov et al 1996, Baeklar et al 1988, Vlachynskyet et al 1996, J. K. Ho et al 1997, Kawai and Toma 1992, Sharif and Amir 2007, Sharif and Amir 2006]. The Lie derivative for a covariant tensor of rank two has been obtained in the presence of torsion by using Weitzenböck connections and applied that Lie derivative to the Einstein Universe to obtain teleparallel Killing vectors [Sharif and Amir 2008]. In their paper [Sharif and Bushra 2009] the authors obtained Killing vectors for spherically symmetric static spacetimes. Later on G. Shabbir and his collaborators classified many well known spacetimes according to teleparallel Killing vector fields [Shabbir and Khan 2010, 2011, Shabbir et al 2011]. In [Shabbir and Khan 2010, 2012] the same authors also classified some spacetimes according to teleparallel homothetic vector fields. In the above papers [Shabbir and Khan 2010, 2011, 2012 and Shabbir et al 2011] an approach has been adopted to look deep into the effect of torsion on symmetries of the spacetime and it is argued that the presence of torsion in spacetime effect the symmetries by reducing the number of known symmetries in general relativity. In teleparallel theory the results of an undergoing study depends upon the choice of a tetrad field. In [Daouda et al] it is shown that two different tetrads can lead us to two different results. In this paper the authors consider two different tetrads for the same spacetimes and get two different equations of motion. In [G. L. Nashed 2010] it is shown that the choice of tetrad for a specific work can be decided by the use of symmetries in teleparallel theory. In this paper the author considers a diagonal and a non diagonal tetrad for the same spacetime. Killing vector fields are obtained for the two different tetrads and these symmetry results are compared to the obtained results of momentum, energy, angular momentum and irreducible mass. It is shown that the results of non diagonal tetrad are much realistic than diagonal one. In [Bushra M 2008] Killing vector fields of cylindrically symmetric static spacetime has been obtained by using a non diagonal tetrad. Later on, in [Shabbir and Khan 2010] teleparallel Killing vector fields and teleparallel proper homothetic vector fields have been found for the same spacetime using a diagonal tetrad. The advantage of the choice of non diagonal tetrad attracted us to look for the teleparallel proper homothetic vector fields in Einstein Universe by choosing a non diagonal tetrad.

In [Sharif and Amir 2008] the authors defined Killing equation in teleparallel theory for the vector field X as

$$\underset{X}{L}^{T} g_{\alpha\beta}=g_{\alpha\beta,\rho}X^{\rho}+g_{\rho\beta}X^{\rho}{}_{,\alpha}+g_{\alpha\rho}X^{\rho}{}_{,\beta}+X^{\rho}(g_{\theta\beta}T^{\theta}{}_{\alpha\rho}+g_{\alpha\theta}T^{\theta}{}_{\beta\rho})=0, \quad (1)$$

Here $\underset{X}{L}^{T}$ represents Lie derivative in teleparallel theory, a comma "," denotes partial derivative and $T^{\theta}{}_{\alpha\beta}$ are the components of torsion tensor. Torsion tensor is anti-symmetric in the lower indices. For finding teleparallel proper homothetic motions we shall use the above definition in the extended form as:

$$\underset{X}{L}^{T} g_{\mu\nu}=2\varepsilon g_{\mu\nu}, \quad \varepsilon\in R. \quad (2)$$

2. Main Results

Einstein's Universe in spherical coordinates (t,r,θ,ϕ) is given as [Sharif and Amir 2008]

$$ds^{2}=-dt^{2}+\frac{1}{\lambda^{2}(r)}dr^{2}+r^{2}d\theta^{2}+r^{2}\sin^{2}\theta\, d\phi^{2}, \quad (3)$$

here $\lambda(r)=\sqrt{1-\frac{r^{2}}{\eta^{2}}}$ and η is a constant. We shall follow a well-known procedure given in [Pereira et al 2001] to obtain the tetrad $\mathrm{P}^{a}{}_{\mu}$, its inverse $\mathrm{P}_{a}{}^{\mu}$, non zero Weitzenböck connections $W^{a}{}_{bc}$ and non zero torsion components $T^{\theta}{}_{\alpha\beta}$ for Einstein's Universe as

$$\mathrm{P}^{a}{}_{\mu}=\begin{bmatrix}1&0&0&0\\0&\frac{1}{\lambda}\sin\theta\cos\phi&r\cos\theta\cos\phi&-r\sin\theta\sin\phi\\0&\frac{1}{\lambda}\sin\theta\sin\phi&r\cos\theta\sin\phi&r\sin\theta\cos\phi\\0&\frac{1}{\lambda}\cos\theta&-r\sin\theta&0\end{bmatrix} \quad (4)$$

$$\mathrm{P}_{a}{}^{\mu}=\begin{bmatrix}1&0&0&0\\0&\lambda\sin\theta\cos\phi&\frac{1}{r}\cos\theta\cos\phi&-\frac{\sin\phi}{r\sin\theta}\\0&\lambda\sin\theta\sin\phi&\frac{1}{r}\cos\theta\sin\phi&\frac{\cos\phi}{r\sin\theta}\\0&\lambda\cos\theta&-\frac{1}{r}\sin\theta&0\end{bmatrix} \quad (5)$$

$$W^{1}_{11}=-\frac{\dot{\lambda}}{\lambda},\quad W^{1}_{22}=-r\lambda,\quad W^{1}_{33}=-r\lambda\sin^{2}\theta,$$
$$W^{2}_{12}=W^{3}_{13}=\frac{1}{r\lambda},\quad W^{2}_{21}=W^{3}_{31}=\frac{1}{r},$$
$$W^{2}_{33}=-\sin\theta\cos\theta,\quad W^{2}_{23}=W^{3}_{32}=\cot\theta \quad (6)$$

Here derivative with respect to r is represented by a dot. The non zero torsion components are given as

$$T_{12}^{2}=T_{13}^{3}=\frac{1}{r}\left(1-\frac{1}{\lambda(r)}\right) \quad (7)$$

We call X a teleparallel homothetic motion if it satisfies equation (2). Expanding equation (2) with the help of equations (3) and (7) we have

$$X^{0}_{,0}=\varepsilon \quad (8)$$

$$X^{1}_{,0}-\lambda^{2}(r)X^{0}_{,1}=0 \quad (9)$$

$$r^{2}X^{2}_{,0}-X^{0}_{,2}=0 \quad (10)$$

$$r^{2}\sin^{2}\theta X^{3}_{,0}-X^{0}_{,3}=0 \quad (11)$$

$$\lambda(r)X^{1}_{,1}-\dot{\lambda}(r)X^{1}=\varepsilon\lambda(r) \quad (12)$$

$$X^{2}_{,1}+\frac{1}{r^{2}\lambda^{2}(r)}X^{1}_{,2}+\frac{1}{r}\left(1-\frac{1}{\lambda(r)}\right)X^{2}=0 \quad (13)$$

$$X^{3}_{,1}+\frac{1}{r^{2}\sin^{2}\theta\lambda^{2}(r)}X^{1}_{,3}+\frac{1}{r}\left(1-\frac{1}{\lambda(r)}\right)X^{3}=0 \quad (14)$$

$$r\lambda(r)X^{2}_{,2}+X^{1}=\varepsilon r\lambda(r) \quad (15)$$

$$\sin^{2}\theta X^{3}_{,2}+X^{2}_{,3}=0 \quad (16)$$

$$r\cot\theta X^{2}+rX^{3}_{,3}+\frac{1}{\lambda(r)}X^{1}=\varepsilon r \quad (17)$$

Solving equations (8)-(11) we get a system of equations as follows

$$
\begin{aligned}
X^0 &= \varepsilon t + E^1(r,\theta,\phi),\\
X^1 &= t\lambda^2(r)E^1_r(r,\theta,\phi) + E^2(r,\theta,\phi),\\
X^2 &= \frac{t}{r^2}E^1_\theta(r,\theta,\phi) + E^3(r,\theta,\phi),\\
X^3 &= \frac{t}{r^2\sin^2\theta}E^1_\phi(r,\theta,\phi) + E^4(r,\theta,\phi),
\end{aligned} \qquad (18)
$$

where $E^1(r,\theta,\phi)$, $E^2(r,\theta,\phi)$, $E^3(r,\theta,\phi)$, $E^4(r,\theta,\phi)$ are functions of integration. In order to get a complete solution of equations (8)-(17) we will find these unknown functions with the help of equations (12)-(18). To keep our calculations precise we shall avoid the lengthy details involved in the solution of the above non linear partial differential equations. Solving equations (12), (15) and (16) with the help of equation (18) we get the following system of equations:

$$
\begin{aligned}
X^0 &= \varepsilon t + c_1\theta + c_2,\\
X^1 &= \varepsilon\lambda(r)\int\frac{1}{\lambda(r)}dr + \lambda(r)F^1_\theta(\theta,\phi),\\
X^2 &= \frac{c_1 t}{r^2} + \varepsilon\theta - \frac{\varepsilon\theta}{r}\int\frac{1}{\lambda(r)}dr - \frac{1}{r}F^1(\theta,\phi) + F^2(r,\phi),\\
X^3 &= -\frac{\cot\theta}{r}F^1_\phi(\theta,\phi) + \frac{1}{r}\int F^1_{\theta\phi}(\theta,\phi)\cot\theta\, d\theta + \cot\theta F^2_\phi(r,\phi) + F^3(r,\phi),
\end{aligned} \qquad (19)
$$

where $F^1(\theta,\phi)$, $F^2(r,\phi)$, $F^3(r,\phi)$ are functions of integration. Now putting equation (19) in equation (13) and solving the resulting equation by differentiating with respect to t, y and x respectively we get $c_1 = 0$ and $\varepsilon\{\lambda(r) - 1\} = 0 \Rightarrow \varepsilon = 0$. If we take $\lambda(r) = 1$ then the spacetime will become Minkowski. Hence there exist no proper teleparallel homothetic motions for the choice of non diagonal tetrad in Einstein's Universe and teleparallel homothetic motions are just the teleparallel Killing motions given in [Sharif and Amir 2008].

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