A new computational technique for Sumudu transforms based on Adomian decomposition method

M. A. Rana

Department of Basic Sciences, Riphah International University, Sector I-14, Islamabad 44000, Pakistan
mafzalrana@gmail.com

Abstract: In this work, Adomian decomposition method is proposed to develop a new computational technique for Sumudu transforms. The proposed method, in contrast of usual method which needs integration, requires simple differentiation. The results reveal that the method is very effective and simple.

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1. Introduction

Adomian decomposition method has been applied to a wide class of functional equations (Adomian, 1985, 1994) since the beginning of the 1980s. Adomian decomposition method gives the solution as an infinite series usually converging to an accurate solution. Abbaoui et al.,1994 applied the standard Adomian decomposition method to nonlinear equations and proved the convergence of series solution. El-Tawil et al., 2004 applied the multistage Adomian decomposition method for solving Riccati differential equation and compared the result with standard Adomian decomposition method. Babolian et al. 2004 considered a new numerical implementation of Adomian decomposition method for cases in which evaluation of terms of the series $\sum_{n=0}^{\infty} u_n(x)$ is impossible analytically. Babolian et al., 2004 employed standard Adomian decomposition method and presented a new computational approach for Laplace transforms. Babolian et al., 2004 modified the standard Adomian method which was proposed in (Abbaoui and Cherruaut, 1994).

Sumudu transform was probably first time introduced by Watagula in his work (Watugala, 1993). Its simple formulation and direct applications to ordinary differential equations immediately sparked interest in this new tool. This new transform was further developed and applied to many problems by various workers. Asiru, 2001, 2002 applied to integro-differential equations, Watugala, 1998, 2000 extended the transform to two variables with emphasis on solution to partial differential equations and applications to engineering control problem, and its fundamentals properties were established by (Belgacem et al., 2003, 2006). Rana et al., 2007 proposed homotopy perturbation method to compute Sumudu transform, Siddiqui et al., 2010 applied Sumudu transform to Newtonian fluid problems. The Sumudu transform has very special and useful properties and can help to solve intricate applications in science and engineering. Having units preserving properties, it may be used to solve problems without resorting to the frequency domain. This is one of many strength points for this new transform, especially with respect to applications in problems with physical dimensions. In fact, the Sumudu transform which is itself linear, preserves linear functions, and hence in particular does not change units (Belgacem et al., 2003). Belgacem et al. 2003 have shown it to be the theoretical dual to the Laplace transform, and hence ought to rival it in problem solving.

The sumudu transform is defined (Watugala, 1998) by

$$G\{u\} = S[f(x)] = \int_0^\infty f(ux)e^{-x}dx, \qquad -\tau_1 < u < \tau_2 \tag{1}$$

over the set of functions,

$$A = \left\{ f(x) / \exists M, \tau_1, \tau_2 > 0, |\mathrm{f(x)}| < Me^{\frac{|x|}{\tau_j}}, \text{if } x \in (-1)^j \times [0,\infty) \right\}. \tag{2}$$

The Laplace and Sumudu Transforms exhibit the following duality relation (Belgacem et al. 2003)

$$G(u) = \frac{F(1/u)}{u} \text{ and } F(s) = \frac{G(1/s)}{s}, \tag{3}$$

where $F(s)$ is the Laplace transform and $G(u)$ Sumudu transform of a given function f.

Consider the first order differential equation

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(0) = 0. \tag{4}$$

The analytical solution of (4) is given by

$$y(x)f(x) = \int Q(x)f(x)dx, \tag{5}$$

where $f(x) = e^{\int P(x)dx}$.

Considering a special case in Equation (4) by taking $P(x) = -s$, where s is a positive constant. Then

$$f(x) = e^{\int P(x)dx} = e^{-sx}. \tag{6}$$

If Equation (5) is considered as a definite integral from zero to infinity, then left hand side of this equation defines the Laplace transform of $Q(x)$. That is

$$L[Q(x)] = \int_0^\infty Q(x)e^{-sx}dx = \{e^{-sx}y(x)\}\Big|_{x=0}^{x=\infty}. \tag{7}$$

In this paper, we apply Adomian decomposition method to propose new computational technique for Sumudu transforms. The results reveal that the proposed method is very effective and simple.

2. Adomian decomposition method

The Equation (4) can be written as

$$y(x) = \frac{Q(x)}{P(x)} - \frac{1}{P(x)}Ly, \tag{8}$$

where $L = d/dx$. The Adomian decomposition method gives the solution $y(x)$ by the series given by

$$y(x) = \sum_{n=0}^{\infty} y_n, \tag{9}$$

where the terms $y_0, y_1, y_2, \cdots$ are determined recursively (Haldar and Datta, 1996) by

$$\left.\begin{aligned} y_0 &= \frac{Q(x)}{P(x)}, \\ y_n &= (-1)^n \frac{1}{P^n} L^n\left(\frac{Q(x)}{P(x)}\right), n = 1,2,3,\cdots. \end{aligned}\right\} \tag{10}$$

Having determined the terms $y_n, n = 0,1,2,\cdots,$ the solution $y(x)$ defined by a series form (9) follows immediately. The convergence of the Adomian decomposition method is established in (Cherruault, 1989, 1992).

In the next section, Adomian decomposition method is employed to derive a new method for Sumudu transforms.

3. The new computational method

The aim of this communication is to derive a new technique for computing Sumudu transforms. In view of Equation (3), the Sumudu transform of $Q(x)$ can be given by the relation

$$\begin{aligned} S[Q(x)] &= \frac{1}{u}\int_0^\infty Q(x)e^{-x/u}dx \\ &= \frac{1}{u}\left(e^{-x/u}y(x)\right)\Big|_{x=0}^{x=\infty}. \end{aligned} \tag{11}$$

Therefore, according to Equations (9) and (11), we have

$$\begin{aligned} S[Q(x)] &= \frac{1}{u}\int_0^\infty Q(x)e^{-x/u}dx \\ &= \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_{x=0}^{x=\infty}. \end{aligned} \tag{12}$$

Several examples are provided to illustrate the simplicity and reliability of this new computational method.

Example 1. Suppose $Q(x) = 1$. Then from Equation (10), we obtain

$$y_0(x) = \frac{Q(x)}{P(x)} = u,$$

$$y_i(x) = 0, i = 1,2,3,\cdots.$$

Therefore, from Equation (12), we obtain

$$S[1] = \frac{1}{u}\left(e^{-x/u}(-u)\right)\Big|_0^\infty = 1.$$

Example 2. Suppose $Q(x) = x$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -xu,$$

$$y_1(x) = -u^2,$$

$$y_i(x) = 0,\ i = 2,3,4,\cdots,$$

and

$$S[x] = \frac{1}{u}\left(e^{-x/u}(-ux - u^2)\right)\Big|_0^\infty = u.$$

Example 3. Suppose $Q(x) = x^n$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -ux^n,$$

$$y_1(x) = -\frac{1}{P(x)} L\left(\frac{Q(x)}{P(x)}\right) = -nu^2 x^{n-1},$$

$$y_2(x) = (-1)^2 \frac{1}{P^2(x)} L^2\left(\frac{Q(x)}{P(x)}\right)$$
$$= -n(n-1)u^3 x^{n-2},$$

$\vdots$

$$y_n(x) = -n!u^{n+1},$$

$$y_i(x) = 0,\ i = n+1, n+2, \cdots,$$

and

$$S[x^n] = \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_0^{\infty} = n!u^n.$$

Example 4. Suppose $Q(x) = e^{ax}$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -ue^{ax},$$

$$y_1(x) = -\frac{1}{P(x)} L\left(\frac{Q(x)}{P(x)}\right) = -au^2 e^{ax},$$

$$y_2(x) = (-1)^2 \frac{1}{P^2(x)} L^2\left(\frac{Q(x)}{P(x)}\right) = -a^2u^3 e^{ax},$$

$\vdots$

$$y_i(x) = -a^i u^{i+1} e^{ax},\ i = 0,1,2,\cdots,$$

and for $|au| < 1$, we have

$$S[e^{ax}] = \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_0^{\infty}$$
$$= 1 + au + a^2u^2 + \cdots = \frac{1}{1-au}.$$

Example 5. Suppose $Q(x) = e^{-ax}$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -ue^{-ax},$$

$$y_1(x) = -\frac{1}{P(x)} L\left(\frac{Q(x)}{P(x)}\right) = au^2 e^{-ax},$$

$$y_2(x) = (-1)^2 \frac{1}{P^2(x)} L^2\left(\frac{Q(x)}{P(x)}\right) = -a^2u^3 e^{-ax},$$

$$y_3(x) = (-1)^3 \frac{1}{P^3(x)} L^3\left(\frac{Q(x)}{P(x)}\right) = a^3u^4 e^{-ax},$$

$\vdots$

$$y_i(x) = (-1)^{i+1} a^i u^{i+1} e^{-ax},\ i = 0,1,2,\cdots,$$

and for $|au| < 1$, we have

$$S[e^{-ax}] = \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_0^{\infty}$$
$$= 1 - au + a^2u^2 - a^3u^3 + \cdots = \frac{1}{1+au}.$$

Example 6. Suppose $Q(x) = \cos(ax)$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -u\cos(ax),$$

$$y_1(x) = -\frac{1}{P(x)} L\left(\frac{Q(x)}{P(x)}\right) = au^2 \sin(ax),$$

$$y_2(x) = (-1)^2 \frac{1}{P^2(x)} L^2\left(\frac{Q(x)}{P(x)}\right) = a^2u^3 \cos(ax),$$

$$y_3(x) = (-1)^3 \frac{1}{P^3(x)} L^3\left(\frac{Q(x)}{P(x)}\right) = -a^3u^4 \sin(ax),$$

$$y_4(x) = (-1)^4 \frac{1}{P^4(x)} L^4\left(\frac{Q(x)}{P(x)}\right) = -a^4u^5 \cos(ax),$$

$\vdots$

and for $|au| < 1$, we have

$$S[\cos(ax)] = \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_0^{\infty}$$
$$= 1 - a^2u^2 + a^4u^4 - \cdots = \frac{1}{1+a^2u^2}.$$

Example 7. Suppose $Q(x) = \sin(ax)$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -u\sin(ax),$$

$$y_1(x) = -\frac{1}{P(x)} L\left(\frac{Q(x)}{P(x)}\right) = -au^2\cos(ax),$$

$$y_2(x) = (-1)^2 \frac{1}{P^2(x)} L^2\left(\frac{Q(x)}{P(x)}\right) = a^2u^3\sin(ax),$$

$$y_3(x) = (-1)^3 \frac{1}{P^3(x)} L^3\left(\frac{Q(x)}{P(x)}\right) = a^3u^4\cos(ax),$$

$$y_4(x) = (-1)^4 \frac{1}{P^4(x)} L^4\left(\frac{Q(x)}{P(x)}\right) = -a^4u^5\sin(ax),$$

$\vdots$

and for $|au| < 1$, we have

$$S[\sin(ax)] = \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_0^{\infty}$$
$$= au - a^3u^3 + a^5u^5 - \cdots = \frac{au}{1+a^2u^2}.$$

Example 8. Suppose $Q(x) = \cosh(ax)$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -u\cosh(ax),$$

$$y_1(x) = -\frac{1}{P(x)} L\left(\frac{Q(x)}{P(x)}\right) = -au^2\sinh(ax),$$

$$y_2(x) = (-1)^2 \frac{1}{P^2(x)} L^2\left(\frac{Q(x)}{P(x)}\right) = -a^2u^3\cosh(ax),$$

$$y_3(x) = (-1)^3 \frac{1}{P^3(x)} L^3\left(\frac{Q(x)}{P(x)}\right) = -a^3u^4\sinh(ax),$$

$$y_4(x) = (-1)^4 \frac{1}{P^4(x)} L^4\left(\frac{Q(x)}{P(x)}\right) = -a^4u^5\cosh(ax),$$

$\vdots$

and for $|au| < 1$, we have

$$S[\cosh(ax)] = \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_0^{\infty}$$
$$= 1 + a^2u^2 + a^4u^4 + \cdots = \frac{1}{1-a^2u^2}.$$

Example 9. Suppose $Q(x) = \sinh(ax)$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -u\sinh(ax),$$

$$y_1(x) = -\frac{1}{P(x)} L\left(\frac{Q(x)}{P(x)}\right) = -au^2\cosh(ax),$$

$$y_2(x) = (-1)^2 \frac{1}{P^2(x)} L^2\left(\frac{Q(x)}{P(x)}\right) = -a^2u^3\sinh(ax),$$

$$y_3(x) = (-1)^3 \frac{1}{P^3(x)} L^3\left(\frac{Q(x)}{P(x)}\right) = -a^3u^4\cosh(ax),$$

$$y_4(x) = (-1)^4 \frac{1}{P^4(x)} L^4\left(\frac{Q(x)}{P(x)}\right) = -a^4u^5\sinh(ax),$$

$$y_5(x) = (-1)^5 \frac{1}{P^5(x)} L^5\left(\frac{Q(x)}{P(x)}\right) = -a^5u^6\cosh(ax),$$

$\vdots$

and for $|au| < 1$, we have

$$S[\sinh(ax)] = \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_0^{\infty}$$
$$= au + a^3u^3 + a^5u^5 + \cdots = \frac{au}{1-a^2u^2}.$$

Example 10. Suppose $Q(x) = xe^{ax}$, then

$$y_0(x) = \frac{Q(x)}{P(x)} = -ux\,e^{ax},$$

$$y_1(x) = -\frac{1}{P(x)} L\left(\frac{Q(x)}{P(x)}\right) = -u^2(ax+1)e^{ax},$$

$$y_2(x) = (-1)^2 \frac{1}{P^2(x)} L^2\left(\frac{Q(x)}{P(x)}\right) = -au^3(ax+2)e^{ax},$$

$$y_3(x) = (-1)^3 \frac{1}{P^3(x)} L^3\left(\frac{Q(x)}{P(x)}\right) = -a^2u^4(ax+3)e^{ax},$$

$\vdots$

$$y_i(x) = -a^{i-1}u^{i+1}(ax+i)e^{ax}, i = 1,2,3,\cdots$$

therefore,

$$S[xe^{ax}] = \frac{1}{u}\left(e^{-x/u}\sum_{n=0}^{\infty} y_n(x)\right)\Bigg|_0^{\infty}$$

$$= u(1 + 2au + 3a^2u^2 + \cdots) = \frac{u}{(1-au)^2}.$$

4. Conclusion

In this work, we successfully apply Adomian decomposition method to compute Sumudu transform. It gives a simple and a powerful mathematical tool. The proposed method requires simple differentiation in contrast of usual method which needs integration.

Corresponding Author:
Dr. Muhammad Afzal Rana
Department of Basic Sciences,
Riphah International University,
Sector I-14, Islamabad 44000, Pakistan
E-mail: mafzalrana@gmail.com

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