Some aspect of Analytic Functions Based on Salagean and Ruscheweyh Differential Operators

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Abstract: The purpose of the present paper is to introduce a new subclass of analytic univalent functions with negative coefficients involving the Salagean and Ruscheweyh differential operators. The various results investigated in this paper include coefficient bounds, extreme points, radii of streakiness, convexity and close to convexity. [Shahram B, Nader R, Karim f. **Some aspect of Analytic Functions Based on Salagean and Ruscheweyh Differential Operators**. *Life Sci J* 2013;10(5s):298-301] (ISSN:1097-8135). http://www.lifesciencesite.com. 54

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1 Introduction and preliminaries

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

which are holomorphic in $\Delta = \{z \in C : |z| < 1\}$. We denote by N the subclass of A consisting of functions $f(z) \in A$ which are holomorphic univalent in

Δ and are of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \ , \quad a_k \geq 0 \ . \qquad (2)$$

For more information about univalent analytic functions see [1] and [6].

Definition 1. Let $n \in N \cup \{0\}$ and $\lambda \geq 0$. Let $\Omega_\lambda^n f$ denote the operator defined by $\Omega_\lambda^n : N \to N$, such that

$$\Omega_\lambda^n f(z) = (1-\lambda) S^n f(z) + \lambda R^n f(z), \quad z \in \Delta \ , \qquad (3)$$

where $S^n f$ is the Salagean differential operator [5] and $R^n f$ is the Ruscheweyh differential operator [4].

For $f(z) \in N$ given by (1.2) we get

$$S^n f(z) = z - \sum_{k=n}^{\infty} k^n a_k z^k \qquad (4)$$

and

$$R^n f(z) = z - \sum_{k=2}^{\infty} B_k(n) a_k z^k \ , \qquad (5)$$

Where

$$B_k(n) = \binom{k+n-1}{n} = \frac{(n+1)(n+2)...(n+k-1)}{(k-1)!} . \qquad (6)$$

Further by replacing (1.4) and (1.5) in (1.3) we conclude

$$\Omega_\lambda^n f(z) = z - \sum_{k=2}^{\infty} [K^n (1-\lambda) + \lambda B_k(n)] a_k z^k \ . \qquad (7)$$

It is observed that for $n = 0$,

$\Omega_\lambda^0 f(z) = (1-\lambda) S^0 f(z) + \lambda R^0 f(z) = f(z) = S^0 f(z) = R^0 f(z)$. **Definition 2.** A function $f(z) \in N$ is said to belong to the class $\Psi_\lambda^n(\alpha, \beta)$ if and only if

$$\mathrm{Re}\left\{\frac{\alpha z[\Omega_\lambda^n f(z)]' + z^2 [\Omega_\lambda^n f(z)]''}{\alpha z[\Omega_\lambda^n f(z)]' + (1-\alpha)\Omega_\lambda^n f(z)}\right\} > \beta \ , \qquad (8)$$

Where $0 \leq \beta < 1$, $0 \leq \alpha \leq 1$, $\alpha > \beta$.

In recent years, many authors (e.g. [2,3]) have investigated certain subclasses of N .

2 On Main Results

We begin by proving a necessary and sufficient condition for a function belonging to the class $\Psi_{\lambda}^{n}(\alpha,\beta)$.

Theorem 1 : A function $f(z)$ given by (1.2) is in the class $\Psi_{\lambda}^{n}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^{n}(1-\lambda)+\lambda B_{k}(n)]a_{k}\leq\alpha\beta, \qquad (9)$$

where $B_{k}(n)$ defined by (1.6). The result is best possible for the function

$$H(Z)=z-\frac{\alpha-\beta}{[2(2-\alpha\beta)-(1-\alpha)(2+\beta)][2^{n}(1-\lambda)+\lambda B_{2}(n)]}z^{2}. \qquad (10)$$

Proof : By making use of (1.7) in (1.8) we have

$$\operatorname{Re}\{\frac{\alpha z-\sum_{k=2}^{\infty}k(\alpha+k-1)[k^{n}(1-\lambda)+\lambda B_{n}(n)]a_{k}z^{k}}{z-\sum_{k=2}^{\infty}(\alpha k+1-\alpha)[k^{n}(1-\lambda)+\lambda B_{n}(n)]a_{k}z^{k}}\}>\beta.$$

By choosing the values of z on the real axis and the $z\to 1^{-}$ through real values, we get

$$(k-\beta)-\sum_{k=2}^{\infty}[k(\alpha+k-1)-\beta(\alpha k+1-\alpha)][k^{n}(1-\lambda)+\lambda B_{n}(n)]a_{k}\geq 0,$$

Or

$$\sum_{k=2}^{\infty}[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^{n}(1-\lambda)+\lambda B_{n}(n)]\leq\alpha-\beta.$$

Conversely, suppose that (2.1) holds true. We will show that (1.80 is satisfied and so $f(z)\in\Psi_{\lambda}^{n}(\alpha,\beta)$. Using the fact that $\operatorname{Re}\omega>\beta$ if and only if $|\omega-(1+\beta)|<|\omega+(1-\beta)|$, it is enough to show that

$$L=\left|\frac{\alpha z[\Omega_{\lambda}^{n}f(z)]'+z^{2}[\Omega_{\lambda}^{n}f(z)]''}{\alpha z[\Omega_{\lambda}^{n}f(z)]'+(1-\alpha)\Omega_{\lambda}^{n}f(z)}-1-\beta\right|$$

$$<\left|\frac{\alpha z[\Omega_{\lambda}^{n}f(z)]'+z^{2}[\Omega_{\lambda}^{n}f(z)]''}{\alpha z[\Omega_{\lambda}^{n}f(z)]'+(1-\alpha)\Omega_{\lambda}^{n}f(z)}-1-\beta\right|=R.$$

If $\omega=\alpha z[\Omega_{\lambda}^{n}f(z)]'+(1-\alpha)\Omega_{\lambda}^{n}f(z)$, we have

$$L=\frac{1}{|\omega|}[\alpha z[\Omega_{\lambda}^{n}f(z)]'+z^{2}[\Omega_{\lambda}^{n}f(z)]''-(1+\beta)\omega]. \qquad (11)$$

By using (1.7) and replacing $[\Omega_{\lambda}^{n}f(z)]'$ and $[\Omega_{\lambda}^{n}f(z)]''$ in (2.3) we conclude

$$L<\frac{|z|}{|\omega|}[(\alpha-\beta-1)-\sum_{k=2}^{\infty}[k(\alpha+k-1)-(1+\beta)(\alpha k-\alpha+1)][k^{n}(1-\lambda)+$$

$$\lambda B_{n}(n)]a_{k}|z|^{k-1}].$$

After same calculation on R, when $z\in\partial\Delta$, it is easy to verify that $R-L>0$ if (2.1) holds and so the proof is complete.

We next find the extreme points of $\Psi_{\lambda}^{n}(\alpha,\beta)$.

Theorem 2 : Let $f(z)=z$ and

$$f_{k}(z)=z-\frac{\alpha-\beta}{k(k-\alpha\beta)-(1-\alpha)(k+\beta)[k^{n}(1-\lambda)+\lambda B_{k}(n)]}z^{k} \ .k=2,3,..$$

then $f(z) \in \Psi_\lambda^n(\alpha,\beta)$ if and only if it can be expressed

where $t_k \geq 0$ and $\sum_{k=1}^{\infty} t_k = 1$. $\qquad f(z) = \sum_{k=1}^{\infty} t_k f_k(z)$,

Proof : Let $f(z)$ be expressed as in the above form. This means we can write

$$f(z) = t_1 f_1(z) \sum_{k=2}^{\infty} t_k f_k(z)$$

$$= t_1 z + \sum_{k=2}^{\infty} t_k z - \sum_{k=2}^{\infty} \frac{\alpha-\beta}{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)[k^n(1-\lambda)+\lambda B_k(n)]} z^k$$

$$= z + \sum_{k=2}^{\infty} d_k z \,,$$

Where $\dfrac{\alpha-\beta}{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)[k^n(1-\lambda)+\lambda B_k(n)]}$. Since

$$\sum_{k=2}^{\infty} \frac{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)[k^n(1-\lambda)+\lambda B_k(n)]}{\alpha-\beta} d_k = \sum_{k=2}^{\infty} t_k = 1 - t_1 < 1,$$

so by Theorem 2.1 we conclude that $f(z) \in \Psi_\lambda^n(\alpha,\beta)$. Conversely, suppose that $f(z) \in \Psi_\lambda^n(\alpha,\beta)$ .By letting

$$t_k = \frac{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)[k^n(1-\lambda)+\lambda B_k(n)]}{\alpha-\beta} a_k$$

And $t_1 = 1 - \sum_{k=2}^{\infty} t_k$, we conclude the required result.

3 Radius Properties

In the last section we obtain the radii of starlikeness, convexity and close to convexity.

Theorem 3 : Let $f(z)\Psi_\lambda^n(\alpha,\beta)$. Then $f(z)$ is starlike of order $\sigma(0 \leq \sigma < 1)$ in $|z| < R_1$, where

$$R_k = \inf_k \left\{ \frac{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)[k^n(1-\lambda)+\lambda B_k(n)](1-\sigma)}{(\alpha-\beta)(k-\sigma)} \right\}^{\frac{1}{k-1}}. \qquad (12)$$

Proof : For $0 \leq \sigma < 1$, we need to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \sigma$. In other words, it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{k=2}^{\infty} (k-1) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} a_k z^{k-1}} \right|$$

$$\leq \left| \frac{\sum_{k=2}^{\infty} (k-1) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}} \right| < 1 - \sigma \,,$$

Or $\sum_{k=2}^{\infty} \left(\frac{k-\sigma}{1-\sigma}\right) a_k |z|^{k-1} < 1$. By using (2.1) it is easy to see that above inequality holds if

$$|z|^{k-1} < \frac{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^n(1-\lambda)+\lambda B_k(n)](1-\sigma)}{(\alpha-\beta)(k-\sigma)}.$$

and this complete the proof.

Since $f(z)$ is convex if and only if $zf'(z)$ is starlike, we obtain the following theorem.

Theorem 4 : Let $f(z)\Psi_\lambda^n(\alpha,\beta)$. Then $f(z)$ is close to convex of order $\sigma(0 \le \sigma < 1)$ in $|z| < R_2$, where

$$R_2 = \inf_k \{\frac{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^n(1-\lambda)+\lambda B_k(n)](1-\sigma)}{k(\alpha-\beta)(k-\sigma)}\}^{\frac{1}{k-1}} \quad (13)$$

Theorem 5 : Let $f(z)\Psi_\lambda^n(\alpha,\beta)$. Then $f(z)$ is close to convex of order $\sigma(0 \le \sigma < 1)$ in $|z| < R_3$, where

$$R_3 = \inf_k \{\frac{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^n(1-\lambda)+\lambda B_k(n)](1-\sigma)}{k(\alpha-\beta)}\}^{\frac{1}{k-1}} \quad (14)$$

Proof : We must show that $|f'(z)-1| \le 1-\sigma$ for $|z| < R_3$ we have R_3 is given by (3.3). Now

$$|f'(z)-1| = |\sum_{k=2}^{\infty} ka_k z^{k-1}| \le \sum_{k=2}^{\infty} ka_k |z|^{k-1}.$$

Thus $|f'(z)-1| \le 1-\sigma$ if $\sum_{k=2}^{\infty} \frac{ka_k}{1-\sigma}|z|^{k-1} < 1$. But, by Theorem 2.1, above inequality holds true if

$$|z|^{k-1} < \frac{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^n(1-\lambda)+\lambda B_k(n)](1-\sigma)}{k(\alpha-\beta)},$$

and this gives the required result.

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References

[1] P. L. Duren, Univalent Functions, Grundelheren der Mathematischen Wissenchaften 259, Springer-Verlag, New York, USA, (1983).
[2] Sh . Najafzadeh and S. R. Kulkarni, Application of Ruscheweyh derivative on univalent functions with negative and fixed finitely many coefficients, Rajasthan acad. phy. Sci. 5(1) (2006), 39-51.
[3] Sh. Najafzadeh and S. R. Kulkarni, Study of certain properties of univalent functions based on Ruscheweyh derivative, Southeast Asian Bull. Math . 32 (2008), 478-488.
[4] St. Ruscheweyh, New criteria for univalent functions, Proc. amer. Math. 49 (1975), 109-115.
[5] G. S. Salagean, Subclasses of univalent functions, Lect. Notes Math. 1013 (1983), 362-372.
[6] H. M. Srivastava and S. Owa (Eds.), Current topics in analytic function theory, World Scientific Publication Company, Singapore, (1992).

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