Modified Adomian decomposition method for solving a class of hypersingular integral equations of first kind

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Abstract: In this paper, a new algorithm based on modified Adomian decomposition method is employed to obtain analytical solution of a class of hypersingular integral equations of the first kind. This method avoids the complex function-theoretic, long computations of collocation polynomial-based methods and produces the exact solution.
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1 Introduction

Hypersingular integral equation is considered as an important tool in applied Mathematics as it finds application in solving a large class of mixed boundary value problems arising in mathematical physics. Particularly, the crack problems in fracture mechanics or water wave scattering problems involving barriers, diffraction of electromagnetic waves and aerodynamics problems ([2,9,10,13]) could be reduced to hypersingular integral equations in single or disjoint multiple intervals.

A simple approximation method for solving a general hypersingular integral equation of the first kind where the kernel consists a hypersingular part and a regular part is introduced and developed in [12]. A method based on polynomial approximation is used in [11] to produce the approximate solution of a class of singular integral equations of the second kind. Dutta and Banerjea ([6]) have solved a hypersingular integral equation in two intervals by using the solution of Cauchy type singular integral equations in two disjoint intervals. Gori et.al. ([7]) have constructed a quadrature rule based on the use of suitable refinable quasi-interpolatory operators, for the numerical evaluation of Hadamard finite-part integrals. Chen and Zhou ([3]) have developed an efficient method for solving hypersingular integral equation of the fist kind in reproducing kernel space in order to eliminate the singularity of the equation.
In the present paper, we consider the following hypersingular integral equation of the second kind.

$$a(x)\int_{-1}^{1}\frac{u(t)}{(t-x)^2}dt+\int_{-1}^{1}L(t,x)u(t)dt-f(x)-1\le x\le 1 \qquad (1)$$

where the unknown function $u(x)$ has square-root zero at the end-point, that is, $u(x)=\sqrt{1-x^2}\psi(x)$ with $\psi(x)$ smooth. $a(x)$ is bounded and belongs to $L^2[-1,1]$, $L(t,x)$ is a square-integrable function of t and x, and $f(x)$ is smooth. The first integral in equation (1) is referred to as Hadamard finite part ([13])

$$\int_{-1}^{1}\frac{u(t)}{(t-x)^2}dt = \lim_{\varepsilon\to 0^+}\left[\int_{-1}^{x-\varepsilon}\frac{u(t)}{(x-t)^2}dt+\int_{x+\varepsilon}^{1}\frac{u(t)}{(x-t)^2}dt-\frac{u(x+\varepsilon)+u(x-\varepsilon)}{\varepsilon}\right]. \qquad (2)$$

A simple approximate method for solving (1), with its kernel consisting of a hypersingular part and a regular part is developed using second kind Chebyshev polynomials method by Mandal and Bera in [12]. A simple and efficient method for solving (1) in reproducing kernel spaces is developed by Chen and Zhou [3]. In this paper we use weighted modified Adomian decomposition method (WMADM) to obtain an analytical solution for equation (1). We apply the new method for the case that $f(x)$ be a polynomial.

2 Adomian Decomposition Method

The decomposition procedure of Adomian was first proposed by the American mathematician, G. Adomian (1923-1996) and has been applied already to a wide class of stochastic and deterministic problems in science and engineering. It is based on the search for a solution in the form of a series and on decomposing the nonlinear operator into a series in which the terms are calculated recursively using Adomian polynomials [1].

In this method, the nonlinear function in the equation is decomposed into terms of special polynomials called *Adomian's polynomials* and then the terms of the solution which is regarded as a series, are determined recurrently. Consider the differential equation

$$F(u) = g(x), x \in \Omega, \quad (3)$$

where F represents a general nonlinear ordinary differential operator involving both linear and nonlinear parts and $g(x)$ is a given function. Equation (3) can be written as

$$Lu + Ru + Nu = g(x), \quad (4)$$

where L is an easily invertible operator, which is usually taken as the highest-ordered derivative, R is the reminder of the linear operator, and N is the nonlinear term in $F(u)$.
Applying the inverse operator L^{-1} to (4) yields

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu, \quad (5)$$

and therefore

$$u = c + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \quad (6)$$

where c is the integration constant and satisfies $Lc = 0$. Based on the ADM, the solution of Eq. (3) is regarded as

$$u = \sum_{n=0}^{\infty} u_n, \quad (7)$$

and the nonlinear term Nu is decomposed as follows

$$Nu = \sum_{n=0}^{\infty} A_n, \quad (8)$$

where the components A_n are Adomian's polynomials which are calculated by the formula

$$A_n(u_0, \dots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left[\sum_{k=0}^{\infty} \lambda^k u_k\right]\right]_{\lambda=0}, n \geq 0. \quad (9)$$

Substituting (8) and (7) in (6) results

$$\sum_{n=0}^{\infty} u_n = c + L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \quad (10)$$

Now according to the decomposition procedure of Adomian, we can obtain the components u_ns by the following recurrent relation

$$u_0 = c + L^{-1}g,$$
$$u_n = -L^{-1}Ru_{n-1} - L^{-1}A_{n-1}, n \geq 1. \quad (11)$$

The n-term approximation of the solution is defined as $\theta_n = \sum_{k=0}^{n} u_k$ and $u = \lim_{n\to\infty}\theta_n$. As we know, the more terms added to the approximate solution, the more accurate it would be.

Convergence of Adomian decomposition scheme was established by many authors using fixed point theorem, see for example [4,5,8].

For inhomogeneous equations a simple strategy is used to increas the convergent rate. In performing the ADM, we can expand $g(x) = g_1(x) + g_2(x)$ and then we use the following substitution in (10) to get the exact solution

$$u_0 = c + L^{-1}g_1,$$
$$u_1 = L^{-1}g_2 - L^{-1}Ru_0 - L^{-1}A_0, \quad (12)$$
$$u_n = -L^{-1}Ru_{n-1} - L^{-1}A_{n-1}, n \geq 2,$$

where c is computed with the initial conditions of the problem. The rate of convergence depends on choosing the functions g_1 and g_2. Usually they have chosen in such a way that $u_1 = u_2 = \cdots = 0$. This method is refereed as modified Adomian decomposition method (MADM).

3 Method of Solution

In equation (1) it is assumed that

$$u(x) = \sqrt{1-x^2}\psi(x), \quad (13)$$

where $\psi(x)$ is a smooth function (see [13]). After substituting (13) in (1) we have

$$a(t)\frac{d}{dx}\int_{-1}^{1}\frac{\sqrt{1-t^2}\psi(t)}{t-x}dt + \int_{-1}^{1} L(t,x)\sqrt{1-t^2}\psi(t)dt = f(x). \quad (14)$$

Note that with Liebnites formula we have

$$\int_{-1}^{1}\frac{\sqrt{1-t^2}\psi(t)}{(t-x)^2}dt = \frac{d}{dx}\int_{-1}^{1}\frac{\sqrt{1-t^2}\psi(t)}{t-x}dt.$$

Let us start with the following definition

$$I_j = \int_{-1}^{1}\frac{\sqrt{1-t^2}t^j}{t-x}dt, j = 0,1,\dots. \quad (15)$$

It was proven in [12] that

$$I_j = -\pi x^{j+1} + \sum_{i=0}^{j-1} \frac{1+(-1)^i}{4} \frac{\Gamma(1/2)\Gamma((i+1)/2))}{\Gamma((i+4)/2)} x^{j-1-i}, j = 0,1,\dots \quad (16)$$

For example some first I_js are as follows

$$\begin{aligned}
I_0 &= -\pi x, \\
I_1 &= -\pi x^2 + \frac{1}{2}\pi, \\
I_2 &= -\pi x^3 + \frac{1}{2}\pi x, \\
I_3 &= -\pi x^4 + \frac{1}{2}\pi x^2 + \frac{1}{8}\pi, \\
I_4 &= -\pi x^5 + \frac{1}{2}\pi x^3 + \frac{1}{8}\pi x, \\
I_5 &= -\pi x^6 + \frac{1}{2}\pi x^4 + \frac{1}{8}\pi x^2 + \frac{1}{16}\pi, \\
I_6 &= -\pi x^7 + \frac{1}{2}\pi x^5 + \frac{1}{8}\pi x^3 + \frac{1}{16}\pi x, \\
&\vdots
\end{aligned} \quad (17)$$

Let us rewrite (15) to the form

$$\psi(x) = f(x) + \psi(x) - a(t)\frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi(t)}{t-x} dt - \int_{-1}^{1} L(t,x)\sqrt{1-t^2}\psi(t)dt. \quad (18)$$

For applying ADM to (15) suppose

$$\psi(x) = \sum_{n=0}^{\infty} \psi_n(x). \quad (19)$$

Substituting (19) to (15)

$$\sum_{n=0}^{\infty} \psi_n(x) = f(x) \quad + \sum_{n=0}^{\infty} \psi_n(x) - a(x)\sum_{n=0}^{\infty} \frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi_n(t)}{t-x} dt - \sum_{n=0}^{\infty} \int_{-1}^{1} L(t,x)\sqrt{1-t^2}\psi_n(t)dt. \quad (20)$$

Now according to the decomposition procedure of ADM we can obtain the components ψ_n as follows

$$\begin{cases}
\psi_0(x) = f(x), \\
\psi_{n+1}(x) = \psi_n(x) - \frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi_n(t)}{t-x} dt \\
- \int_{-1}^{1} L(t,x)\sqrt{1-t^2}\psi_n(t)dt, n \geq 0.
\end{cases} \quad (21)$$

In performing the MADM, we expand $f(x) = f_1(x) + f_2(x)$ and obtain the components ψ_n as follows

$$\begin{cases}
\psi_0(x) = f_1(x), \\
\psi_1(x) = f_2(x) + \psi_0(x) - \frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi_0(t)}{t-x} dt \\
- \int_{-1}^{1} L(t,x)\sqrt{1-t^2}\psi_0(t)dt \\
\psi_{n+1}(x) = \psi_n(x) - \frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi_n(t)}{t-x} dt \\
- \int_{-1}^{1} L(t,x)\sqrt{1-t^2}\psi_n(t)dt, n \geq 1.
\end{cases} \quad (22)$$

The rate of convergence of MADM depends on how to choose the functions f_1 and f_2. Usually, they have been chosen in such a way that $\psi_1 = \psi_2 = \cdots = 0$. The main problem in performing MADM is to choose f_1 and f_2 such that the rate of convergence to be increased. We apply the new method for the case that $f(x)$ be a polynomial. We introduce a weighted MADM method which helps us in choosing f_1 and f_2 properly.

4 Illustrations

Example 1: We assume the hypersingular integral equation (1) as follows

$$\int_{-1}^{1} \frac{u(t)}{(t-x)^2} dt + \int_{-1}^{1} (t+x)u(t)dt = \frac{\pi}{2}(1-6x^2) - \frac{\pi}{8}x - 1 \leq x \leq 1 \quad (23)$$

Applying $u(x) = \sqrt{1-x^2}\psi(x)$ in (23) we get

$$\frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi(t)}{t-x} dt + \int_{-1}^{1} (t+x)\sqrt{1-t^2}\psi(t)dt = \frac{\pi}{2}(1-6x^2) - \frac{\pi}{8}x.$$

Here $f(x)$ is a polynomial of degree two, then $\psi(x)$ must be a polynomial of two. To start MADM we suppose $f_1(x)$ and $f_2(x)$ as follows

$$f_1(x) = a_2x^2 + a_1x + a_0,$$
$$f_2(x) = (-3\pi - a_2)x^2 + \left(-\frac{\pi}{8} - a_1\right)x + \left(\frac{\pi}{2} - a_0\right). \quad (24)$$

Now applying MADM (22) to (24) we get

$$\psi_0(x) = f_1(x) = a_2x^2 + a_1x + a_0,$$

$$\psi_1(x) = f_2(x) + \psi_0(x) - \frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi_0(t)}{t-x} dt - \int_{-1}^{1} L(t,x)\sqrt{1-t^2}\psi_0(t)dt \quad (25)$$

$$= (-3\pi + 3\pi a_2)x^2 + \left(-\frac{\pi}{8} + \frac{\pi}{8}a_2 + 2\pi a_1 + \frac{\pi}{2}a_0\right)x + \left(\pi a_0 - \frac{\pi}{2}a_2 + \frac{\pi}{2} - \frac{\pi}{8}a_1\right).$$

Note that for the first integral in $\psi_1(x)$ we use (17), where as the second one is straightforward for evaluation. In the MADM, $f_1(x)$ and $f_2(x)$ should be chosen such that $\psi_1(x) = 0$. Putting $\psi_1(x) = 0$ leads us to the following system of equations.

$$\begin{cases} -3\pi + 3\pi a_2 = 0, \\ -\frac{\pi}{8} + \frac{\pi}{8}a_2 + 2\pi a_1 + \frac{\pi}{2}a_0 = 0, \\ \pi a_0 - \frac{\pi}{2}a_2 + \frac{\pi}{2} - \frac{\pi}{8}a_1 = 0. \end{cases} \quad (26)$$

Where $a_0 = 0, a_1 = 0$ and $a_2 = 1$, is the solution, that is

$$\psi(x) = x^2, \quad (27)$$

and finally the solution of the integral equation (23) is given as

$$u(x) = x^2\sqrt{1-x^2}, \quad (28)$$

which is the exact solution of the integral equation.

Example 2: Another hypersingular integral equation is given by

$$\int_{-1}^{1} \frac{u(t)}{(t-x)^2} dt + \int_{-1}^{1} txu(t)dt = -8\pi x^3 + \frac{17}{8}\pi x - \pi, -1 \leq x \leq 1 \quad (29)$$

Applying $u(x) = \sqrt{1-x^2}\psi(x)$ in (30) we get

$$\frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi(t)}{t-x} dt + \int_{-1}^{1} tx\sqrt{1-t^2}\psi(t)dt = -8\pi x^3 + \frac{17}{8}\pi x - \pi. \quad (30)$$

Here $f(x)$ is a polynomial of degree three, then $\psi(x)$ must be a polynomial of three. We suppose $f_1(x)$ and $f_2(x)$ as follows

$$f_1(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$
$$f_2(x) = (-8\pi - a_3)x^3 - a_2x^2 + \left(\frac{17}{8}\pi - a_1\right)x + (-\pi - a_0). \quad (31)$$

Now applying MADM (22) to (31) we get

$$\psi_0(x) = f_1(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$
$$\psi_1(x) = (-8\pi + 4\pi a_3)x^3 + 3\pi a_2x^2 + \left(\frac{17}{8}\pi - \frac{17}{16}\pi a_3 + \frac{15}{8}\pi a_1\right)x + \left(-\pi - \frac{1}{2}\pi a_2 + \pi a_0\right). \quad (32)$$

Putting $\psi_1(x) = 0$ leads us to the following system of equations.

$$\begin{cases} -8\pi + 4\pi a_3 = 0, \\ 3\pi a_2 = 0, \\ \frac{17}{8}\pi - \frac{17}{16}\pi a_3 + \frac{15}{8}\pi a_1 = 0, \\ -\pi - \frac{1}{2}\pi a_2 + \pi a_0 = 0. \end{cases} \quad (33)$$

Where $a_0 = 1$, $a_1 = a_2 = 0$, and $a_3 = 2$, is the solution, that is

$$\psi(x) = 1 + 2x^3, \quad (34)$$

and finally the solution of the integral equation ((29)) is given as

$$u(x) = (1 + 2x^3)\sqrt{1 - x^2}, \quad (35)$$

which is the exact solution of the integral equation.

Example 3: We assume the hypersingular integral equation ((1)) as follows

$$\int_{-1}^{1} \frac{u(t)}{(t-x)^2}dt + \int_{-1}^{1} t^2(1 - x^2)u(t)dt = \frac{-\pi^2}{16}(7 + x^2), -1 \le x \le 1 \quad (37)$$

Applying $u(x) = \sqrt{1 - x^2}\psi(x)$ in (**Error! Reference source not found.**) we get

$$\frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}\psi(t)}{t-x}dt + \int_{-1}^{1} t^2(1 - x^2)\sqrt{1 - t^2}\psi(t)dt = \frac{-\pi^2}{16}(7 + x^2). \quad (36)$$

Here $f(x)$ is a polynomial of degree two, then $\psi(x)$ must be a polynomial of two. To start MADM we suppose $f_1(x)$ and $f_2(x)$ as follows

$$f_1(x) = a_2x^2 + a_1x + a_0,$$
$$f_2(x) = \left(\frac{-\pi^2}{16} - a_2\right)x^2 - a_1x + \left(-\frac{7\pi^2}{16} - a_0\right). \quad (37)$$

Now applying MADM (22) to (38) we get

$$\psi_0(x) = a_2x^2 + a_1x + a_0,$$
$$\psi_1(x) = \left(\frac{-\pi^2}{16} + \frac{\pi}{8}a_0 + \frac{49\pi}{16}a_2\right)x^2 + 2\pi a_1x + \left(-\frac{9\pi}{16}a_2 - \frac{7\pi^2}{16} + \frac{7\pi}{8}a_0\right). \quad (38)$$

Putting $\psi_1(x) = 0$ we get $a_0 = \pi/2$ and $a_1 = a_2 = 0$, that is

$$\psi(x) = \frac{\pi}{2}, \quad (39)$$

and finally the solution of the integral equation (**Error! Reference source not found.**) is given as

$$u(x) = \frac{\pi}{2}\sqrt{1 - x^2}, \quad (40)$$

which is the exact solution of the integral equation.

5 Conclusion

In this paper, a new scheme of modified Adomian decomposition method is introduced and used to solve a class of hypersingular integral equations of the second kind analytically. Unlike existing analytical and numerical schemes for solving the hypersingular integral equations, the proposed scheme is easy in use and avoids the complex function-theoretic of analytical method and long computations of numerical methods.

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