# A new modification of Newton's method by Gauss integration formula

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**Abstract:** In this paper, we construct a new modification of Newton's method for solving nonlinear equations, which is based on the method of Gauss quadrature integration. It is shown by way of some illustrative examples that the proposed method is a powerful tool for approximation simple root of nonlinear equations. Numerical examples are given to compare the convergent results of this method compared with other existed methods.
[Baghmisheh M, Mahmoudi Y, Jahangirirad M. **A new modification of Newton's method by Gauss integration formula**. *Life Sci J* 2013;10(5s):288-291] (ISSN:1097-8135). http://www.lifesciencesite.com. 52

**keywords:** Nonlinear equations, Iterative methods, Newton's method, Cubic order of convergence

## 1. Introduction and Preliminaries

Solving nonlinear equations is one of the most important problems of numerical computations in mathematics and engineering. There is a wide variety of iterative methods, such as Newton's method and it's modifications to solve nonlinear equations.

In recent years there has been considerable interest in developing new algorithms with high order convergence. Normally, these high order convergence algorithms contain higher derivatives of the function or multi-step. In the former case, various techniques can be used to eliminate the derivatives. However, the resulting interaction function may be more complex than the Original.

Chun et al. [3] developed a new family of fourth-order methods for simple roots free from second derivative. Amat et al. [2] discussed the dynamics of the family of third-order methods do not require second derivatives. In another paper [1] they discussed the dynamics of King and Jarratt's schemes. Neta et al. [6] discussed the best choice of the parameter in King's method.

In this paper we consider a family of third-order algorithms to find a simple root of a nonlinear equation $f(x)=0$ . We use known Gauss integrations formulas to produce new methods. We also prove the convergence rate of the methods.

## 2 . General Method

Newton's method for the calculation of $x^*$, the simple solution of $f(x)=0$, is probably the most widely used iterative method defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}. \qquad (1)$$

It is well known that this method has second order convergence order.

Several third-order methods based on quadratures are given in the literature. A third-order variant of Newton's method appeared in Weerakoon and Fernando [7] where rectangular and trapezoidal approximations to the integral in Newton's theorem

$$f(x)=f(x_n)+\int_{x_n}^{x} f'(t)dt, \qquad (2)$$

were considered to rederive Newton's method and to obtain the cubically convergent method

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \qquad (3)$$

respectively, where from here on

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}. \qquad (4)$$

Frontini and Sormani [5] considered the midpoint rule for the integral (1) to obtain the third-order method

$$y_n=x_n-\frac{f(x_n)}{f'\left(\frac{x_n+y_n}{2}\right)}. \qquad (5)$$

Let us approximate the integral of (1) with a weighted combination of two quadrature formulas of order one as follows

$$\int_{x_n}^{x} f'(t)dt=\lambda(x-x_n)f'(x_n)+(1-\lambda)(x-x_n)f'(x), \qquad (6)$$

where $\lambda$ is an arbitrary real number. After substituting (6) in (1) and putting $x=x_{n+1}$, we get the following formula

$$x_{n+1} = x_n - \frac{f(x_n)}{\lambda f'(x_n)+(1-\lambda)f'(y_n)}, \quad (7)$$

where $y_n$ is defined in (4).
For the method defined in (7) we have the following analysis of convergence.

**Theorem 1:** *Let $x^* \in I$ be a simple zero of a sufficiently differentiable function $f: I \to R$ for an open interval $I$. Let $e_n = x_n - x^*$, then the method defined in (7) is of order two.*
**Proof:** Let $c_k = 1/k!\, f^{(k)}(x^*)/f'(x^*), k = 2,3,.$ Using the Taylor expansion and taking into account $f(x^*) = 0$ and by simple calculations, we easily obtain

$$f(x_n) = f'(x^*)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)], \quad (8)$$

$$f'(x_n) = f'(x^*)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)], \quad (9)$$

$$y_n = x_n - \frac{f(x_n)}{f'(x^n)} = x^* + c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \quad (10)$$

$$f'(y_n) = f'(x^*)[1 + 2c_2^2 e_n^2 - 4c_2(c_2^2 - c_3)e_n^3 + O(e_n^4)], \quad (11)$$

$$\lambda f'(x_n) + (1-\lambda)f'(y_n) = f'(x^*)[1 + 2\lambda c_2 e_n + (3\lambda c_3 + 2(1-\lambda)c_2^2)e_n^2 + (4\lambda c_4 - 4(1-\lambda)c_2(c_2^2 - c_3))e_n^3 + O(e_n^4)], \quad (12)$$

dividing (8) by (12), we get

$$\frac{f(x_n)}{\lambda f'(x_n)+(1-\lambda)f'(y_n)} = e_n + (c_2 - 2c_2\lambda)e_n^2 + (c_3 - 3c_3\lambda - 2c_2^2 + 4c_2^2\lambda^2)e_n^3 + O(e_n^4). \quad (13)$$

Thus putting (13) in (7) we get

$$e_{n+1} = c_2(1-2\lambda)e_n^2 + [c_3(1-3\lambda) - 2c_2^2(1-2\lambda^2)]e_n^3 + O(e_n^4). \quad (14)$$

This means that the method defined by (7) is of order two. This completes the proof. □
From (14) it is obvious that for $\lambda = 1/2$ the method (7) is of order three. In this case the method (7) is the well known Trapezoid Newton method defined in (3).
To produce another method of order three we use two points Gauss quadrature formula which is exact for integration of all polynomials of degree not grater than three. Consider (1) as follows

$$\int_{x_n}^{x} f'(t)dt = \frac{x+x_n}{2}\int_{-1}^{1} f'\left(\frac{x+x_n}{2}s + \frac{x-x_n}{2}\right)ds = \frac{x+x_n}{2}\left[f'\left(-\frac{x+x_n}{2}\frac{\sqrt{3}}{3} + \frac{x-x_n}{2}\right) + f'\left(\frac{x+x_n}{2}\frac{\sqrt{3}}{3} + \frac{x-x_n}{2}\right)\right]. \quad (15)$$

After substituting (15) in (1) and putting $x = x_{n+1}$, we get the following formula

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(\alpha)+f'(\beta)}, \quad (16)$$

where $\alpha$ and $\beta$ are

$$\alpha = -\frac{y_n+x_n}{2}\frac{\sqrt{3}}{3} + \frac{y_n-x_n}{2}, \beta = \frac{y_n+x_n}{2}\frac{\sqrt{3}}{3} + \frac{y_n-x_n}{2},$$

and $y_n$ is defined in (4).
For the method defined in (16) we have the following analysis of convergence.

**Theorem 2:** *Let $x^* \in I$ be a simple zero of a sufficiently differentiable function $f: I \to R$ for an open interval $I$. Let $e_n = x_n - x^*$, then the method defined in (16) is of order three.*

**Proof:** Let $c_k = 1/k!\, f^{(k)}(x^*)/f'(x^*), k = 2,3,$. Using the Taylor expansion and taking into account $f(x^*) = 0$ and by simple calculations, we obtain following

$$\alpha = a y_n - b x_n = x^* - b e_n + a c_2 e_n^2 - 2a(c_2^2 - c_3)e_n^3 + O(e_n^4), \tag{17}$$

$$\beta = b y_n - a x_n = x^* - a e_n + b c_2 e_n^2 - 2b(c_2^2 - c_3)e_n^3 + O(e_n^4), \tag{18}$$

where $a = \frac{1}{2}\left(-\frac{\sqrt{3}}{3} + 1\right)$, $b = \frac{1}{2}\left(\frac{\sqrt{3}}{3} + 1\right)$ and $a + b = 1$.

$$f'(\alpha) = f'(x^*)[1 + 2b c_2 e_n + (2a c_2^2 + 3b^2 c_3)e_n^2 + (4c_2(c_3 - c_2^2) + 6ab c_2 c_3 + 4b^3 c_4)e_n^3 + O(e_n^4)], \tag{19}$$

$$f'(\beta) = f'(x^*)[1 + 2a c_2 e_n + (2b c_2^2 + 3a^2 c_3)e_n^2 + (4c_2(c_3 - c_2^2) + 6ab c_2 c_3 + 4a^3 c_4)e_n^3 + O(e_n^4)], \tag{20}$$

$$\frac{2f(x_n)}{f'(\alpha)+f'(\beta)} = e_n - c_2^2 e_n^3 + O(e_n^4). \tag{21}$$

Thus putting (21) in (16) we get

$$e_{n+1} = -c_2^2 e_n^3 + O(e_n^4). \tag{22}$$

This means that the method defined by (16) is of order three. This completes the proof. □
From (22) it is obvious that if $f''(x^*) = 0$ then the method (16) is of order four.

## 3 Numerical examples

We present some numerical test results. The method of Gauss Newton (GN) were compared with the classic Newton's method (NM) and the Trapezoid Newton's method (TN). All computations were done using MAPLE using 200 point arithmetics (Digits:=200). We use the following two stoping criterias for computer programs: (i) $|x_{n+1} - x_n| < \varepsilon$, (ii) $|f(x_{n+1}| < \varepsilon$, and so when the stopping criterion is satisfied, $x_{n+1}$ is taken as the exact root $x^*$ computed. For numerical illustrations in this section we use the fixed stopping criterion $\varepsilon = 10^{-25}$. We used the test functions as the Weerakoon and Fernando [7] and the test functions in Neta **Error! Reference source not found.** as listed in Table 1.

Table 1: Test functions

| Test | Function | $x_0$ | $x^*$ |
|---|---|---|---|
| 1 | $x^3 + 4x^2 - 15$ | 1.0 | 1.631980805566063517522106445 54 |
| 2 | $\sin^2(x) - x^2 + 1$ | 1.0 | 1.404491648215341226035086817 8 |
| 3 | $x^2 - e^x - 3x + 1$ | 2.0 | 0.257530285439860760455367304 99 |
| 4 | $\cos(x) - x$ | 1.5 | 0.739085133215160641655312087 67 |
| 5 | $xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$ | -1.0 | -1.207647827130918927009416758 4 |
| 6 | $x^5 + x - 10000$ | 4.0 | 6.308777129972689094767571771 8 |
| 7 | $e^x + x - 20$ | 2.0 | 2.842439953784447067816585940 2 |
| 8 | $\ln(x) + \sqrt{x} - 5$ | 1.0 | 8.309432694231571795346955682 7 |

As convergence criterion, it is required that the distance of two consecutive approximations $\delta$ be less than $\varepsilon = 10^{-25}$ that is $|x_{n+1}| < 10^{-25}$. Table 2. displays the numerical results for the test functions of Table 1. In this table IT is the number of iterations to approximate the root, NFE denotes the number of function and its derivatives evaluations which counts the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative and CO denotes the convergence order which is approximated as follows

$$CO = \frac{\ln\frac{[|x_{k+1} - x_k|}{|x_k - x_{k-1}|}}{\ln\frac{|x_k - x_{k-1}|}{|x_{k-1} - x_{k-2}|}}, \tag{23}$$

The test results in table 2. shows that for most of the functions, the Gauss Newton method have equal or better performance compared to the Trapezoid Newton method and both of these methods have better performance compared with the classical Newton's method.

Table 2: Numerical results of the proposed examples

| Test | Method | IT | NFE | $\lvert f(x_{n+1})\rvert$ | $\lvert x_{n+1}-x_n\rvert$ | CO |
|---|---|---|---|---|---|---|
| 1 | NM | 7 | 14 | 58E-62 | 78E-62 | 2 |
| | GN | 5 | 15 | 27E-117 | 13E-118 | 3 |
| | TN | 5 | 15 | 44E-110 | 21E-111 | 3 |
| 2 | NM | 7 | 14 | 10E-51 | 31E-51 | 2 |
| | GN | 5 | 15 | 58E-95 | 23E-95 | 3 |
| | TN | 5 | 15 | 89E-90 | 36E-90 | 3 |
| 3 | NM | 6 | 12 | 29E-56 | 91E-56 | 2 |
| | GN | 5 | 15 | 13E-162 | 34E-163 | 3 |
| | TN | 4 | 12 | 81E-68 | 21E-68 | 3 |
| 4 | NM | 6 | 12 | 38E-65 | 22E-65 | 2 |
| | GN | 4 | 12 | 31E-81 | 34E-81 | 3 |
| | TN | 4 | 12 | 31E-81 | 34E-81 | 3 |
| 5 | NM | 7 | 14 | 23E-64 | 11E-65 | 2 |
| | GN | 5 | 15 | 25E-121 | 12E-122 | 3 |
| | TN | 5 | 15 | 25E-121 | 12E-122 | 3 |
| 6 | NM | 10 | 20 | 17E-63 | 22E-67 | 2 |
| | GN | 7 | 21 | 23E-79 | 51E-83 | 4 |
| | TN | 7 | 21 | 40E-79 | 22E-67 | 4 |
| 7 | NM | 8 | 16 | 29E-93 | 16E-94 | 2 |
| | GN | 5 | 15 | 18E-87 | 10E-90 | 3 |
| | TN | 5 | 15 | 38E-69 | 21E-70 | 3 |
| 8 | NM | 8 | 16 | 25E-80 | 84E-80 | 2 |
| | GN | 5 | 15 | 13E-77 | 43E-77 | 3 |
| | TN | 6 | 18 | 14E-163 | 48E-163 | 3 |

**Acknowledgments**
The author would like to thank Tabriz Branch, Islamic Azad University for the financial support of this research, which is based on a research project contract.

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3/7/2013