On the Existence of Solutions for Stochastic Differential Equations under G-Brownian Motion

Faiz Faizullah[1,2], Waseem Asghar Khan[3] , Muhammad Arif[4] and Riaz Ahmad Khan[5]

[1] Department of Basic Sciences and Humanities, College of E and ME, National University of Sciences and Technology, Islamabad, Pakistan.
[2] College of Physical and Environmental Oceanography, Ocean University of China, Qingdao 266100, PR China.
[3]Department of Mathematics, Sukkar-Institute of Business Administration, Sukkur, Pakistan.
[4] Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan
[5] CAMP, National University of Sciences and Technology (NUST), Islamabad, Pakistan.
faiz_math@yahoo.com (F. Faizullah), marifmaths@aukum.edu.pk (M. Arif)
waseemasg@gmail.com (W. A. Khan)

Abstract: In this paper the existence theory for stochastic differential equations under G-Brownian motion (G-SDEs) of the type $X_t = X_0 + \int_0^t a(v,\ X_v)dv + \int_0^t b(v,\ X_v)d\langle B\rangle_v + \int_0^t \gamma(v,\ X_v)dB_v,\quad t\in[0,\ T],$ is studied. It is valuated that G-SDEs have solutions even if the coefficient b is a discontinuous function. The method of upper and lower solutions is used to establish the above mentioned theory. As an example, a scalar G-SDE whose second coefficient is the sawtooth function is considered. [Faizullah F., Khan W.A., Arif M and Khan R.A. **On the Existence of Solutions for Stochastic Differential Equations under G-Brownian Motion** *Life Sci J* 2013;10(5s):255-260] (ISSN:1097-8135). http://www.lifesciencesite.com. 46

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1 Introduction

The importance of stochastic differential equations (SDEs) is apparent from their wide range of applications inside as well as outside mathematics. For example stochastic dynamical systems are used to model physical, economical, technical and biological dynamical systems under uncertainty. In general one can not obtain the explicit solutions of stochastic dynamical systems and needs to study the behavior and properties of solutions such as existence, uniqueness and stability etc. Existence theory for stochastic dynamical systems is the most important property in the sense that if solutions of the stated systems do not exist then any other property is worthless.

Recently, the theory of G-Brownian motion with related stochastic calculus was introduced by Peng (Peng, 2006). The existence and uniqueness of solutions for stochastic differential equation under G-Brownian motion (G-SDEs) with Lipschitz continuous coefficients was developed by Peng (Peng, 2006; 2008) and Gao (Gao, 2009). Later Faizullah and Piao extended this theory of G-SDEs to discontinuous drift coefficients (Faizullah et al., 2012). Now here the existence theory for G-SDEs with a discontinuous coefficient b via the method of upper and lower solutions is developed. It is shown that G-SDEs have more than one solution if the coefficient b is a discontinuous function.

In this article we consider the following stochastic differential equation under G-Brownian motion

$$X_t = X_0 + \int_0^t a(v,\ X_v)dv + \int_0^t b(v,\ X_v)d\langle B\rangle_v + \int_0^t \gamma(v,\ X_v)dB_v,\quad t\in[0,\ T], \tag{1.1}$$

where $X_0 \in R^n$ is a given constant initial condition, $(\langle B\rangle_t)_{t\geq 0}$ is the quadratic variation process of the G-Brownian motion $(B_t)_{t\geq 0}$ and all the coefficients $a(t,\ x)$, $b(t,\ x)$ and $\gamma(t,\ x)$ are in the space $M_G^2(0,\ T;\ R^n)$ (Peng, 2008). A process X_t belongs to the mentioned space satisfying the G-SDE (1.1) is said to be its solution. It is assumed that $b(t,\ x)$ is a discontinuous function where $a(t,\ x)$ and $\gamma(t,\ x)$ are Lipschitz continuous for all $x\in R^n$.

This paper is organized as follows. In section 2 some basic notions and definitions are included. In section 3 the method of upper and lower solutions for G-SDEs is established. In section 4 the comparison theorem is introduced. In section 4 the existence of solutions for the G-SDEs with a discontinuous coefficient b is developed. In section 5 appendix is given.

2. Preliminaries

For the material of this section see the book (Peng, 2010) and papers (Denis et al., 2010; Faizullah, 2012; Gao, 2009; Li et al., 2011; Song, 2011).

Let Ω be a (non-empty) basic space and $\mathbf{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant $c \in \mathbf{H}$ and if $X \in \mathbf{H}$ then $|X| \in \mathbf{H}$. We consider that $\mathbf{H}$ is the space of random variables.

2.1 Definition

A functional $E: \mathbf{H} \to R$ is called sub-linear expectation, if for all $X, Y \in \mathbf{H}$, $c \in R$ and $\lambda \geq 0$ it satisfies the following properties

1. (Monotonicity): If $X \geq Y$ then $E[X] \geq E[Y]$.
2. (Constant preserving): $E[c] = c$.
3. (Sub-additivity): $E[X + Y] \leq E[X] + E[Y]$.
4. (Positive homogeneity): $E[\lambda X] = \lambda E[X]$.

The triple $(\Omega, \mathbf{H}, E)$ is called a sublinear expectation space.

Consider the space of random variables $\mathbf{H}$ such that if $X_1, X_2, ..., X_n \in \mathbf{H}$ then

$\varphi(X_1, X_2, ..., X_n) \in \mathbf{H}$ for each $\varphi \in C_{l.Lip}(R^n)$, where $C_{l.Lip}(R^n)$ is the space of linear functions φ defined as the following

$$C_{l.Lip}(R^n) = \{\varphi: R^n \to R \mid \exists C \in R^+, m \in N \text{ s.t.} |\varphi(x) - \varphi(y)| \leq c(1 + |x|^m + |y|^m)|x - y|\}, \text{ for } x, y \in R^n.$$

2.2 Definition

An n-dimensional random vector $Y = (Y_1, Y_2, ..., Y_n)$ is said to be independent from an m-dimensional random vector $X = (X_1, X_2, ..., X_m)$ if

$$E[\varphi(X, Y)] = E[E[\varphi(x, Y)]_{x=X}],$$

$$\forall \varphi \in C_{l.Lip}(R^m \times R^n).$$

2.3 Definition

Two n-dimensional random vectors X and $\hat{X}$ defined respectively on the sublinear expectation spaces $(\Omega, \mathbf{H}, E)$ and $(\hat{\Omega}, \hat{\mathbf{H}}, \hat{E})$ are said to be identically distributed, denoted by $X \sim \hat{X}$ or $X =^d \hat{X}$, if

$$E[\varphi(X)] = \hat{E}[\varphi(X)], \ \forall \varphi \in C_{l.Lip}(R^n).$$

2.4 Definition

Let $(\Omega, \mathbf{H}, E)$ be a sublinear expectation space and $X \in \mathbf{H}$ with

$$\bar{\sigma}^2 = E[X^2], \ \underline{\sigma}^2 = -E[-X^2].$$

Then X is said to be G-distributed or $\mathbf{N}(0; [\bar{\sigma}^2, \underline{\sigma}^2])$ -distributed, if $\forall \ a, b \geq 0$ we have

$$aX + bY \sim \sqrt{a^2 + b^2} X,$$

for each $Y \in \mathbf{H}$ which is independent to X and $Y \sim X$.

G-expectation and G-Brownian Motion. Let $\Omega = C_0(R^+)$, that is, the space of all R -valued continuous paths $(w_t)_{t \in R^+}$ with $w_0 = 0$ equipped with the distance

$$\rho(w^1, w^2) = \sum_{k=1}^{\infty} \frac{1}{2^k} (\max_{t \in [0, k]} |w_t^1 - w_t^2| \wedge 1),$$

and consider the canonical process $B_t(w) = w_t$ for $t \in [0, \infty)$, $w \in \Omega$ then for each fixed $T \in [0, \infty)$ we have

$$L_{ip}(\Omega_T) = \{\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}) : t_1, t_n \in [0, T], \ \varphi \in C_{l.Lip}(R^n), \ n \in N\},$$

where $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$ for $t \leq T$ and $L_{ip}(\Omega) = \cup_{m=1}^{\infty} L_{ip}(\Omega_m)$.

Consider a sequence $\{\xi_i\}_{i=1}^{\infty}$ of n-dimensional random vectors on a sublinear expectation space $(\hat{\Omega}, \hat{\mathbf{H}}_p, \hat{E})$ such that ξ_{i+1} is independent of $(\xi_1, \xi_2, ..., \xi_i)$ for each $i = 1, 2, ..., n-1$ and ξ_i is G-normally distributed. Then a sublinear expectation $E[.]$ defined on $L_{ip}(\Omega)$ is introduced as follows.

For $0 = t_0 < t_1 < ... < t_n < \infty$, $\varphi \in C_{l.Lip}(R^n)$ and each

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega),$$

$$E[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] = \hat{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, ..., \sqrt{t_n - t_{n-1}}\xi_n)].$$

2.5 Definition

The sublinear expectation $E: L_{ip}(\Omega) \to E$ defined above is called a G-expectation and the corresponding canonical process $(B_t)_{t \geq 0}$ is called a G-Brownian motion.

The completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p = (E[|X|^p])^{1/p}$ for $p \geq 1$ is denoted by $L_G^p(\Omega)$ and $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$ for $0 \leq t \leq T < \infty$. The filtration generated by the canonical process $(B_t)_{t \geq 0}$ is denoted by $\mathbf{F}_t = \sigma\{B_s, \ 0 \leq s \leq t\}$, $\mathbf{F} = \{\mathbf{F}_t\}_{t \geq 0}$.

G-Itô's Integral. For any $T \in R^+$, a finite ordered subset $\pi_T = \{t_0, t_1, ..., t_N\}$ such that $0 = t_0 < t_1 < ... < t_N = T$ is a partition of $[0, T]$ and $\mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, 1, ..., N-1\}$.

A sequence of partitions of $[0, T]$ is denoted by $\pi_T^N = \{t_0^N, t_1^N, ..., t_N^N\}$ such that $\lim_{N \to \infty} \mu(\pi_T^N) = 0$.

Consider the following simple process: Let $p \geq 1$ be fixed. For a given partition $\pi_T = \{t_0, t_1, ..., t_N\}$ of $[0, T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t),$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, ..., N-1$. The collection containing the above type of processes, that is, containing $\eta_t(w)$ is denoted by $M_G^{p,0}(0, T)$. The completion of $M_G^{p,0}(0, T)$ under the norm $\|\eta\| = \{\int_0^T E[|\eta_v|^p]dv\}^{1/p}$ is denoted by $M_G^p(0, T)$ and for $1 \leq p \leq q$, $M_G^p(0, T) \supset M_G^q(0, T)$.

2.6 Definition

For each $\eta_t \in M_G^{2,0}(0, T)$, the Itô's integral of G-Brownian motion is defined as

$$I(\eta) = \int_0^T \eta_v dB_v = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}).$$

2.7 Definition

An increasing continuous process $(\langle B \rangle_t)_{t \geq 0}$ with $\langle B \rangle_0 = 0$, defined by

$$\langle B \rangle_t = B_t^2 - 2\int_0^t B_v dB_v,$$

is called the quadratic variation process of G-Brownian motion.

Let $\mathbf{B}(\Omega)$ be the Borel σ-algebra of Ω. It was proved in (Denis et al., 2010) that there exists a weakly compact family $\mathbf{P}$ of probability measures P defined on $(\Omega, \mathbf{B}(\Omega))$ such that

$$E[X] = \sup_{P \in \mathbf{P}} E_P[X], \forall X \in \mathbf{Lip}(\Omega).$$

This makes the following definitions reasonable.

2.8 Definition

The capacity $\hat{c}(.)$ associated to the family $\mathbf{P}$ is defined by

$$\hat{c}(A) = \sup_{P \in \mathbf{P}} P(A), \quad A \in \mathbf{B}(\Omega).$$

2.9 Definition

A set A is said to be polar if its capacity is zero, that is, $\hat{c}(A) = 0$ and a property holds quasi-surely ($q.s.$ in short) if it holds outside a polar set.

Through out the paper for $X = (x_1, x_2, ..., x_n)$, $Y = (y_1, y_2, ..., y_n)$, $X \leq Y$ means $x_i \leq y_i$, $i = 1, 2, ..., n$.

3 Upper and Lower Solutions Method for G-SDEs

We recall that the notion of lower and upper solutions for the classical SDEs was established in (Assing et al., 1995; Halidias et al., 2006; 2008; Ladde, 1980) and for G-SDEs in (Faizullah et al., 2012).

3.1 Definition

A process $L_t \in M_G^2(0, T)$ is said to be a lower solution of the G-SDE on the interval $[0, T]$ if for any fixed s the inequality (interpreted component wise)

$$L_t \leq L_s + \int_s^t a(v, L_v)dv + \int_s^t b(v, L_v)d\langle B \rangle_v$$
$$+ \int_s^t \gamma(v, L_v)dB_v, 0 \leq s \leq t \leq T, \quad (3.2)$$

holds q.s.

3.2 Definition

A process $U_t \in M_G^2(0, T)$ is said to be an upper solution of the G-SDE on the interval $[0, T]$ if for any fixed s the inequality (interpreted component wise)

$$U_t \geq U_s + \int_s^t a(v, U_v)dv + \int_s^t b(v, U_v)d\langle B \rangle_v$$
$$+ \int_s^t \gamma(v, U_v)dB_v, 0 \leq s \leq t \leq T, \quad (3.1)$$

holds q.s.

3.3 Example

Consider the scalar stochastic differential equation under G-Brownian motion

$$dX_t = dt + \{X_t\}d\langle B \rangle_t + dB_t, \quad t \in [0, T], \quad (3.3)$$

where $\{x\}: R \to [0, 1)$ is the sawtooth or fractional part function. It has discontinuities at the integers and is defined by

$$\{x\} = x - \lfloor x \rfloor, \quad x \in R,$$

where $\lfloor x \rfloor$ is the floor function (Graham et al., 1994).

Then $U_t = U_0 + \int_0^t dv + \int_0^t d\langle B \rangle_v + \int_0^t dB_v$ and $L_t = L_0 + \int_0^t dv + \int_0^t dB_v$ for $t \in [0, T]$ are the upper and lower solutions of the G-SDE (3.3) respectively, which are shown as follows.

$$U_t = U_0 + \int_0^t dv + \int_0^t d\langle B \rangle_v + \int_0^t dB_v$$
$$= U_s + \int_s^t dv + \int_s^t d\langle B \rangle_v + \int_s^t dB_v$$
$$\geq U_s + \int_s^t dv + \int_s^t \{U_v\}d\langle B \rangle_v + \int_s^t dB_v, \quad 0 \leq s \leq t \leq T,$$

where $U_s = U_0 + \int_0^s dv + \int_0^s d\langle B \rangle_v + \int_0^s dB_v$, for each fixed s such that $0 \leq s \leq t \leq T$. On similar arguments as above one can show that $L_t = L_0 + \int_s^t dv + \int_0^t dB_v$ is a lower solution of the scalar G-SDE (3.3). The existence of solutions for the G-SDE (3.3) will be discussed later in section 5.

Suppose that U_t and L_t are the respective upper and lower solutions of the G-SDE

$$dX_t = a(t, X_t)dt + b(t, w)d\langle B \rangle_t + \gamma(t, X_t)dB_t,$$
$$t \in [0, T]. \quad (3.4)$$

Define two functions $p_{L,U}, q_{L,U} : [0, T] \times R^n \times \Omega \to R^n$ by

$p_{_{L,U}}(t,\ x,\ w) = \max\{L_t(w),\ \min\{U_t(w),\ x\}\},$

$$q_{_{L,U}}(t,\ x,\ w) = p_{_{L,U}}(t,\ x,\ w) - x, \qquad (3.5)$$

and consider the stochastic differential equation

$$dX_t = \tilde{a}(t,\ X_t)dt + \tilde{b}(t,\ X_t)d\langle B\rangle_t + \tilde{\gamma}(t,\ X_t)dB_t, \quad t\in[0,\ T], \qquad (3.6)$$

with a given constant initial condition $X_0 \in R^n$, where

$\tilde{a}(t,\ x,\ w) = a(p_{_{L,U}}(t,\ x,\ w)),$

$\tilde{b}(t,\ x,\ w) = b(t,\ w) + q_{_{L,U}}(t,\ x,\ w),$

$\tilde{\gamma}(t,\ x,\ w) = \gamma(p_{_{L,U}}(t,\ x,\ w)),$

are Lipschitz continuous in x. The stochastic differential equation (3.6) has a unique solution $X_t \in M_G^2(0,\ T;\ R^n)$ see (Gao, 2009; Peng, 2006; 2008).

4 Comparison Theorem for G-SDEs

First we give the following two important lemmas. They will be used in the next comparison theorem 4.3.

4.1 Lemma

Suppose that the respective upper and lower solutions U_t and L_t of the G-SDE (3.4) satisfy $L_t \le U_t$ for $t\in[0,\ T]$. Then U_t and L_t are the respective upper and lower solutions of the G-SDE (3.6).

Proof

As $L_t \le U_t$ gives $p_{_{L,U}}(t,\ U_t) = U_t$ and $q_{_{L,U}}(t,\ U_t) = 0$. Hence

$$\begin{aligned} &U_s + \int_s^t \tilde{a}(v,\ U_v)dv + \int_s^t \tilde{b}(v,\ U_v)d\langle B\rangle_v + \int_s^t \tilde{\gamma}(v,\ U_v)dB_v \\ &= U_s + \int_s^t [a(v, p_{_{L,U}}(v,\ U_v))]dv + \int_s^t [b(v,\ w) + q_{_{L,U}}(v,\ U_v)]d\langle B\rangle_v + \int_s^t \gamma(v,\ p_{_{L,U}}(v,\ U_v))dB_v \\ &= U_s + \int_s^t a(v,\ U_v)dv + \int_s^t b(v,\ w)d\langle B\rangle_v + \int_s^t \gamma(v,\ U_v)dB_v \le U_t. \end{aligned}$$

Thus U_t for $0\le s\le t\le T$ is an upper solution of the G-SDE (3.6). Similarly, we can show that L_t is a lower solution of the G-SDE (3.6).

The following lemma can be found in (Faizullah, 2012). The proof is given in appendix.

4.2 Lemma

Let $X_t,\ Y_t \in M_G^{1,0}([0,\ T];\ R^n)$. If $X_t \le Y_t$ for $t\in[0,\ T]$ and any $w\in\Omega$. Then

$$\int_0^T X_t d\langle B\rangle_t \le \int_0^T Y_t d\langle B\rangle_t.$$

4.3 Theorem:

Suppose that

i. The function $b(t,\ x)$ is measurable with $\int_0^t E[|b(v,.)|^2]dv < \infty$ where $a(t,\ x)$ and $\gamma(t,\ x)$ are Lipschitz continuous in x.

ii. The respective upper and lower solutions U_t and L_t of the G-SDE (3.4) with $E[|U_t|^2] < \infty$, $E[|L_t|^2] < \infty$ satisfy $L_t \le U_t$ for $t\in[0,\ T]$.

iii. Also $X_0 \in R^n$ is a given initial value with $E[|X_0|^2] < \infty$ and $L_0 \le X_0 \le U_0$.

Then there exists a unique solution $X_t \in M_G^2(0,\ T;\ R^n)$ of the G-SDE (3.4) such that $L_t \le X_t \le U_t$ for $t\in[0,\ T]$ q.s.

Proof

Consider the G-SDE (3.6) with the functions $p_{_{L,U}},\ q_{_{L,U}} : [0,\ T]\times R^n \times \Omega \to R^n$ defined by (3.5). Now the G-SDE (3.6) has a unique solution and by lemma 4.1 if U_t and L_t are upper and lower solutions of the G-SDE (3.4) respectively then they are the respective upper and lower solutions for the G-SDE (3.6). Also it is obvious to see that any solution X_t of the modified G-SDE (3.6) such that

$$L_t \le X_t \le U_t, \quad t\in[0,\ T], \qquad (4.1)$$

q.s. is also a solution of the G-SDE (3.4). Thus we only have to show that any solution X_t of the problem (3.6) does satisfy the inequality (4.1).

Suppose that there exists an arbitrary interval $(t_1,\ t_2)\subset[0,\ T]$ such that $X_{t_1} = L_{t_1}$ and $X_t < L_t$ for $t\in(t_1,\ t_2)$, then we have

$$\begin{aligned} X_t - L_t &= \int_{t_1}^t \tilde{a}(v,\ X_v)dv + \int_{t_1}^t \tilde{b}(v,\ X_v)d\langle B\rangle_v + \int_{t_1}^t \tilde{\gamma}(v,\ X_v)dB_v \\ &\quad - \int_{t_1}^t \tilde{a}(v,\ L_v)dv - \int_{t_1}^t \tilde{b}(v,\ L_v)d\langle B\rangle_v - \int_{t_1}^t \tilde{\gamma}(v,\ L_v)dB_v \\ &= \int_{t_1}^t [a(v, p_{_{L,U}}(v,\ X_v))]dv + \int_{t_1}^t [b(v,\ w) + q_{_{L,U}}(v,\ X_v)]d\langle B\rangle_v \\ &\quad + \int_{t_1}^t \gamma(v,\ p_{_{L,U}}(v,\ X_v))dB_v - \int_{t_1}^t [a(v, p_{_{L,U}}(v,\ L_v))]dv \\ &\quad - \int_{t_1}^t [b(v,\ w) + q_{_{L,U}}(v,\ L_v)]d\langle B\rangle_v - \int_{t_1}^t \gamma(v,\ p_{_{L,U}}(v,\ L_v))dB_v. \end{aligned}$$

As $L_t \le U_t$ yields $p_{_{L,U}}(t,\ L_t) = L_t$ and $X_t < L_t$, $X_t < U_t$ gives $p_{_{L,U}}(t,\ X_t) = L_t$. Also $q_{_{L,U}}(t,\ L_t) = 0$ and $q_{_{L,U}}(t,\ X_t) = L_t - X_t > 0$. Hence

$$X_t - L_t = \int_{t_1}^t q_{_{L,U}}(v,\ X_v)d\langle B\rangle_v > 0,$$

which is a contradiction. Thus $X_t \ge L_t$ for $t\in[0,\ T]$. By the similar arguments as above we can show that $X_t \le U_t$ for $t \in [0, T]$.

5 G-SDEs with Discontinuous Coefficients b

Now take the following G-SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)d\langle B\rangle_t + \gamma(t, X_t)dB_t, \quad t \in [0, T], \quad (5.1)$$

where $b(t, x)$ does not need to be continuous but suppose that it is increasing, that is, if $x \ge y$ then $b(t, x) \ge b(t, y)$ (where the inequalities are interpreted component wise) and $a(t, x)$, $\gamma(t, x)$ are Lipschitz continuous in x.

5.1 Theorem

Assume that

i. The functions $b(t, x)$ is increasing in x where $a(t, x)$, $\gamma(t, x)$ are Lipschitz continuous in x.
ii. U_t and L_t are the respective upper and lower solutions of the G-SDE (5.1) with $\int_0^t E[|b(U_v)|^2]dv < \infty$, $\int_0^t E[|b(L_v)|^2]dv < \infty$ and $L_t \le U_t$ for $t \in [0, T]$.

Then there exists at least one solution $X_t \in M_G^2(0, T; R^n)$ of the G-SDE (5.1) such that $L_t \le X_t \le U_t$ for $t \in [0, T]$ q.s.

Proof

Define the space of all d-dimensional stochastic processes by $\mathbf{H}_2$, that is, $\mathbf{H}_2 = \{X = \{X_t, t \in [0, T]\} : E[|X_t|^2] < \infty\}$ with the norm $\|X\|_2 = \{\int_0^t E[|X_v|^2]dv\}^{1/2}$ for all $t \in [0, T]$, which is a Banach space (Peng, 2006; 2008; 2010). Represent the order interval $[L, U]$ in $\mathbf{H}_2$ by $\mathbf{K}$, that is, $\mathbf{K} = \{X : X \in \hat{\mathbf{H}}_2 \text{ and } L_t \le X_t \le U_t\}$ for $t \in [0, T]$, which is closed and bounded by the above norm. By using the monotone convergence theorem (Denis et al., 2010), one can show the convergence of a monotone sequence that belongs to $\mathbf{K}$ in $\mathbf{H}_2$. Thus $\mathbf{K}$ is a regularly ordered metric space with the above norm. It is clear that for any process $V \in \mathbf{K}$, U_t and L_t are the respective upper and lower solutions for the G-SDE

$$dX_t = a(t, X_t)dt + b(t, V_t)d\langle B\rangle_t + \gamma(t, X_t)dB_t, \quad t \in [0, T]. \quad (5.2)$$

Hence by theorem 4.3, for any $X_0 \in R^n$ with $E[|X_0|^2] < \infty$ and $L_0 \le X_0 \le U_0$, the G-SDE (5.2) has a unique solution $X_t \in M_G^2(0, T; R^n)$ such that $L_t \le X_t \le U_t$ for $t \in [0, T]$ q.s.

Define an operator $S : \mathbf{K} \to \mathbf{K}$ by $S(V) = X$, where X is the unique solution of the G-SDE (5.2). Now suppose that $V_t^{(1)} \le V_t^{(2)}$ for all $t \in [0, T]$ and define $X^{(1)} = S(V^{(1)})$, $X^{(2)} = S(V^{(2)})$ where $V^{(1)}, V^{(2)} \in \mathbf{K}$. As it is given that b is increasing function, thus $X_t^{(1)}$ is a lower solution of the G-SDE

$$X_t = X_0 + \int_0^t a(v, X_v)dv + \int_0^t b(v, V_v^{(2)})d\langle B\rangle_v + \int_0^t \gamma(v, X_v)dB_v, \quad t \in [0, T]. \quad (5.3)$$

But this problem has an upper solution U_t. Hence by theorem 4.3, the G-SDE (5.3) has a solution $X_t^{(2)}$ such that $X_t^{(1)} \le X_t^{(2)} \le U_t$ for $t \in [0, T]$. Thus S is an increasing mapping and by theorem 6.2, it has a fixed point $X^{(*)} = S(X^{(*)}) \in \mathbf{K}$ such that $Y_t \le X_t^{(*)} \le U_t$ q.s. where

$$X_t^{(*)} = X_0 + \int_0^t a(v, X_v^{(*)})dv + \int_0^t b(v, X_v^{(*)})d\langle B\rangle_v + \int_0^t \gamma(v, X_v^{(*)})dB_v, \quad t \in [0, T].$$

Now continuing example 3.3 by the above theorem 5.1, there exists at least one solution $X^{(*)}$ of the G-SDE (3.3) such that $L_0 + B_t \le X_t^{(*)} \le U_0 + t + \langle B\rangle_t + B_t$ for $t \in [0, T]$, where $L_t = L_0 + B_t$ and $U_t = U_0 + t + \langle B\rangle_t + B_t$ are the respective lower and upper solutions of (3.3).

6 Appendix

The following definition and theorem can be found in (Heikkila et al., 1993).

6.1 Definition

An ordered metric space M is called regularly (resp. fully regularly) ordered, if each monotone and order (resp. metrically) bounded ordinary sequence of M converges.

6.2 Theorem

If $[a, b]$ is a nonempty order interval in a regularly ordered metric space, then each increasing mapping $S : [a, b] \to [a, b]$ has the least and the greatest fixed point.

Proof of theorem 4.2

Since $\{\langle B\rangle_t : t \ge 0\}$ is an increasing continuous process with $\langle B\rangle_0 = 0$. Therefore for any fixed $w \in \Omega$ and $t_{i+1} \ge t_i$, $\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i} \ge 0$, $i = 0, 1, \ldots, N-1$. Also for $X_t, Y_t \in M_G^{1,0}([0, T]; R^n)$, $X_t = \sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})}$ and $Y_t = \sum_{i=0}^{N-1} \tilde{\xi}_i I_{[t_i, t_{i+1})}$ where $\xi_i, \tilde{\xi}_i \in L_G^1(\Omega_i)$, $i = 0, 1, \ldots, N-1$. Then $X_t \le Y_t$ implies that

$$\sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})} \le \sum_{i=0}^{N-1} \tilde{\xi}_i I_{[t_i, t_{i+1})}.$$

This gives

$$\sum_{i=0}^{N-1}\xi_i[\langle B\rangle_{t_{i+1}}-\langle B\rangle_{t_i}]\le\sum_{i=0}^{N-1}\tilde{\xi}_i[\langle B\rangle_{t_{i+1}}-\langle B\rangle_{t_i}].$$

Thus

$$\int_0^T X_t d\langle B\rangle_t \le \int_0^T Y_t d\langle B\rangle_t.$$

6.3 Remark

The above lemma shows that G-Ito's integral with respect to the quadratic variation process satisfies the monotonic property. Also if $X_t \le 0$ then $\int_0^T X_t d\langle B\rangle_t \le 0.$

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