On sufficient condition for Sakaguchi type spiral-like functions of order β

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ABSTRACT: In our present investigation, motivated from Goyal and Goswami work, we obtain a sufficient condition for Sakaguchi type spiral-like function of order β. Some interesting consequences of our main result are also given. [Arif M., Khan W.A, Ayaz M, Islam S. **On sufficient condition for Sakaguchi type spiral-like functions of order β.** *Life Sci J* 2013;10(5s):253-253] (ISSN:1097-8135). http://www.lifesciencesite.com. 45

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1. INTRODUCTION

Let the class of all functions $f(z) = z + a_{n+1}z^{n+1} + \cdots$ which are analytic in $E = \{z: |z| < 1\}$ be denoted by A_n and let $A_1 = A$. A function $f(z) \in A_n$ is said to be in the class $S^*_\lambda(n, \beta, t)$, if

$$\operatorname{Re} e^{i\lambda} \frac{(1-t)zf'(z)}{f(z) - f(tz)} > \beta \cos\lambda, \quad t \in [-1,1),$$

for all $z \in E, 0 \le \beta < 1$ and λ is real with $|\lambda| < \frac{\pi}{2}$. For $\lambda = 0$, this class reduces to $S_n(\beta, t)$ (see, [6]) and for $n = 1$, $\lambda = 0$, we obtain the class $S(\beta, t)$ studied by Owa et.al [9] and Goyal et.al [5]. The class $S^*_0(1,0,-1)$ was introduced by Sakaguchi [11]. Therefore, a function $f(z) \in S^*_0(1, \beta, -1)$ is called Sakaguchi functions of order β (see, [4]). Also we note that for $n = 1, t = 0, \beta = 0$, the class $S^*_\lambda(n, \beta, t)$ reduces to the class of spiral-like functions introduced by Spacek [12] in 1933.

Sufficient conditions for different classes were studied by various authors, see [1,2,3].

In this paper, we obtain a sufficient condition for a function $f(z) \in S^*_\lambda(n, \beta, t)$. To prove our main result, we need the following Lemma proved in [8].

Lemma 1.1. Let Ω be a set in the complex plane C and suppose that ϕ is a mapping from $\mathbb{C}^2 \times E$ to $\mathbb{C}$ which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in E$, and for all real x, y such that $y \le -n(1 + x^2)/2$. If $p(z) = 1 + c_n z^n + \cdots$ is analytic in E and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in E$, then $\operatorname{Re} p(z) > 0$.

2. MAIN RESULTS

Theorem 2.1. If $f(z) \in A_n$, satisfies

$$\operatorname{Re}\left(1e^{i\lambda}\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\right)\left(\frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha tzf'(tz)}{f(z)-f(tz)} + 1\right) > \frac{M^2}{4L} + N,$$

where $0 \le \alpha \le 1$, $0 \le \beta < 1$, $t \in [-1,1)$, λ is real with $|\lambda| < \frac{\pi}{2}$ and

$$L = \alpha(1-\beta)\left[\frac{n}{2}(1-t) + (1-\beta)\cos^2\lambda\right]\cos\lambda$$
$$M = -(1-\beta)^2 \sin 2\lambda \cos\lambda \qquad (2.1)$$
$$N = \alpha\beta\left[(\beta + 2\alpha\sin^2\lambda)\cos^3\lambda + \left(\frac{n}{2} - \cos\lambda\right)(1-t)\right] - \frac{\alpha\sin 2\lambda}{2} + \left(\beta\cos\lambda - \frac{n\alpha}{2}\right)(1-t),$$

then $f(z) \in S^*_\lambda(n, \beta, t)$.

Proof. Set

$$e^{i\lambda}\frac{(1-t)zf'(z)}{f(z)-f(tz)} = q(z) = \cos\lambda\,[(1-\beta)p(z) + \beta] + i\sin\lambda. \qquad (2.2)$$

Then $p(z)$ and $q(z)$ are analytic in E with $p(0) = 1$ and $q(0) = 1$.

Taking logarithmic differentiation of (2.2), we have

$$\frac{zf''(z)}{f'(z)} + \frac{tzf'(z)}{f(z)-f(tz)} = \frac{(1-t)zq'(z) + e^{-i\lambda}q^2(z) - (1-t)q(z)}{(1-t)q(z)},$$

and hence

$$\left(e^{i\lambda}\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\right)\left(\frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha tzf'(tz)}{f(z)-f(tz)} + 1\right)$$
$$= Azp'(z) + Bp^2(z) + Cp(z) + D$$
$$= \phi(p(z),\ zp'(z);\ z),$$

with

$$A = \alpha(1-t)(1-\beta)\cos\lambda,$$
$$B = \alpha e^{-i\lambda}(1-\beta)^2\cos^2\lambda,$$
$$C = (1-\beta)\left(2\alpha\beta e^{-i\lambda}\cos^2\lambda + i\alpha e^{-i\lambda}\sin 2\lambda + (1-\alpha)(1-t)\cos\lambda\right),$$
$$D = \alpha e^{-i\lambda}(\beta^2\cos^2\lambda - \sin^2\lambda + i\beta\sin 2\lambda) + (1-t)(1-\alpha)(\beta\cos\lambda + i\sin\lambda).$$

Now

$$\phi(r,s;t) = As + Br^2 + Cr + D.$$

For all real x and y satisfying $y \leq -n(1+x^2)/2$, we have

$$\phi(ix,y;z) = Ay - Bx^2 + iCx + D.$$

Taking real part on both sides and then by simple computation, we obtain

$$\text{Re}\,\phi(ix,y;z) \leq -Lx^2 + Mx + N$$
$$= -\left[\sqrt{L}x + \frac{M}{2\sqrt{L}}\right]^2 + \frac{M^2}{4L} + N$$
$$< \frac{M^2}{4L} + N,$$

where L, M and N are given by (2.1).

Let $\Omega = \left\{\omega;\ \text{Re}\omega > \frac{M^2}{4L} + N\right\}$. Then $\phi(p(z), zp'(z); z) \in \Omega$ and $\phi(ix,y;z) \notin \Omega$, for all real x and $y \leq -n(1+x^2)/2, z \in E$. Now by using Lemma 1.1, we obtain the required result.

On taking $\lambda = 0$, in Theorem 2.1, we have the following result proved in [7].

Corollary 2.2. If $f(z) \in A_n$, satisfies

$$\text{Re}\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\left(\frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha tzf'(tz)}{f(z)-f(tz)} + 1\right) > \xi_1,$$

where $0 \leq \alpha \leq 1, 0 \leq \beta < 1, t \in [-1,1)$ and

$$\xi_1 = \alpha\beta\left[\frac{n}{2}(1-t) - (1-t) + \beta\right] + (1-t)\left[\left(\beta - \frac{n\alpha}{2}\right)\right],$$

Then $f(z) \in S_0^*(n,\beta,t)$.

If we put $t = 0, n = 1$ and $\alpha = 0$ in Theorem 2.1, we obtain

Corollary 2.3. If $f(z) \in A$, satisfies

$$\text{Re}\left(e^{i\lambda}\frac{zf'(z)}{f(z)}\right) > \beta\cos\lambda,$$

then $f(z) \in S_\lambda^*(1,\beta,0)$.

If we take $\lambda = 0$, $t = -1$ and $\beta = 0$ in Theorem 2.1, we get

Corollary 2.4. If $f(z) \in A_n$, satisfies

$$\text{Re}\frac{zf'(z)}{f(z)-f(-z)}\left(\frac{\alpha zf''(z)}{f'(z)} - \frac{\alpha zf'(-z)}{f(z)-f(-z)} + 1\right) > \frac{-n\alpha}{4},$$

then $f(z) \in S_0^*(n,0,-1)$.

For $t = 0 = \lambda$, in Theorem 2.1, we have the following result proved in [10].

Corollary2.5. If $f(z) \in A_n$, satisfies

$$\text{Re}\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)} + 1\right) > \alpha\beta\left\{\beta + \frac{n}{2} - 1\right\} + \left\{\beta - \frac{n\alpha}{2}\right\},$$

Then $f(z) \in S_0^*(n,\beta,0)$.

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If we take $\beta = 0$, $n = 1$, $t = 0$ and $\lambda = 0$, in Theorem 2.1, we have the result proved in [7] as:

Corollary 2.6. If $f(z) \in A$, satisfies

$$\text{Re}\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)} + 1\right) > -\frac{\alpha}{2}(z \in E),$$

for some $\alpha(\alpha \geq 0)$, then $f(z) \in S_0^*(1,0,0) = S^*$.

Also for $\beta = \frac{\alpha}{2}, n = 1, t = 0$ and $\lambda = 0$, Theorem 2.1 reduces to the result proved in [7] as follows:

Corollary 2.7. If $f(z) \in A$, satisfies

$$\text{Re}\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)} + 1\right) > \frac{-\alpha^2}{4}(1-\alpha)(z \in E),$$

for some $\alpha(0 \leq \alpha < 2)$, then $f(z) \in S^*\left(\frac{\alpha}{2}\right)$.

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