Space Adaptive Technique to Simulate Butterfly Option Using Black-Scholes Equation.

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Abstract: A grid adaptive finite difference technique is instigated to valuate butterfly spread call option for one asset using Black-Scholes equation. The grid is refined near three exercise prices and a coarse grid is generated otherwise. The non unoform finite difference scheme is used in this computation. The numerical experiments show that the adaptive finite difference method is much more efficient than the method with uniform spacing. The grid adaptation technique reduces the points drastically which in turn decreases the computational cost and makes the algorithm highly efficient. A fully implicit and explicit scheme is also compared in this computation.
[N. A. Mir, S. Ahmad and M. Ashraf. **Space Adaptive Technique to Simulate Butterfly Option Using Black-Scholes Equation.** *Life Sci J* 2013;10(5s):76-79] (ISSN:1097-8135). http://www.lifesciencesite.com. 13

Keywords: Non-uniform grid, Strike price, Butterfly spread, Black-Scholes, Finite Difference method.

1. Introduction

The finite difference scheme that was developed in (Ashraf et al. 2012) is extended to work with one-asset butterfly spread option using Black-Scholes equation.There are many types of financial instruments (Duffy, 2006) which go by the name of Options. Options are traded on all of the world's major exchanges. Butterfly options (Khaliq, et al, 2007) are not only very popular in the over-the-counter markets but also important tools for designing more complex financial derivatives (Wilmott and Howison, 1996). In butterfly option, the payoff has a discontinuity at strike prices. In this work, we will focus on butterfly spread call options for one asset.

Fisher-Black and Myron-Schole (Black and Scholes, 1973) derived a celebrated partial differential equation. The Black-Scholes model is the convenient way to calculate the price of an option (Cox et al, 1979). In this article, numerical methods (Smith, 1985) will be used to solve the finite difference equation (Courtadon, 1982) of Black-Scholes. The solution to the Black-Scholes equation is smooth but the final condition has discontinuity which produces oscillation in the numerical solution. Numerical methods have been studied (Dura and Mosneagu, 2010. Zhu et al, 1988) in many application areas in order to cure this oscillation from the initial discontinuities. Finite difference methods (Khaliq et al, 2008. Wade et al, 2007) with variable space-steps are proposed in order to valuate butterfly options.

The purpose of this paper is to develop efficient and accurate numerical technique to price options (Zhongdi and Anbo, 2009) with payoff containing discontinuities. For butterfly options, the discontinuity lies only in the initial condition, therefore we need to use small space-steps initially then use bigger space-steps to keep the efficiency. In proposed study, we focus on adaptivity (Hongjoong, 2011) for space-steps in order to see effects of variable space-steps. In this study, several numerical tests show that the adaptive finite difference methods approximate the solution more efficiently than uniform finite difference methods.

2. Explicit Finite Difference Scheme

Let S(t) be the price of the underlying asset at time t $(0 \le t \le T)$ with a given expiry date T, constant interest rate r >0 and a constant volatility $\sigma > 0$. The value, $V(S,t)$ of butterfly options under classical Black-Scholes model can be computed by solving the following one asset partial differential equation,

$$\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rV = 0. \qquad (2.1)$$

The interval $[0,T]$ is divided into M equally sized subintervals of length Δt. The price of underlying asset will take the values in the unbounded interval $[0,\infty)$. However, an artificial limit $S_{\max}$ is introduced. The size of $S_{\max}$ requires experimentations but normally $S_{\max}$ is taken around three to four times the exercise prices. The interval $\left[0,S_{\max}\right]$ is divided into N subintervals of length ΔS_i. The asset price at an arbitrary point n will be $\sum_{i=0}^{n} \Delta S_i = \Delta S_0 + \Delta S_1 + \Delta S_2 + \cdots \Delta S_{N-1} + \Delta S_n$ Let us assign a variable α_n to this summation, then $\alpha_n = \sum_{i=0}^{n} \Delta S_i$.

Using this nomenclature, we can say that $\alpha_N = \sum_{n=0}^{N} \Delta S_n = S_{\max}$. where ΔS_i are the non-uniform space-steps. Hence, the space $\left[0, S_{\max}\right] \times [0, T]$ is approximated by a grid $(\alpha_n, m\Delta t) \varepsilon [0, \alpha_N] \times [0, M\Delta t]$,
where $n = 0, 1, \ldots, N$ and $m = 0, 1, \ldots, M$. For uniform spacing $\alpha_n = n\Delta S_n$. Let V_n^m denote the numerical approximation of $V(\alpha_n, m\Delta t)$.
The explicit scheme for non uniform grid is (Ashraf et al. 2012).

$$\frac{V_n^{m+1} - V_n^m}{\Delta t} - r \sum_{i=0}^{n} \Delta S_i \left(\frac{V_{n+1}^m - V_{n-1}^m}{\Delta S_n + \Delta S_{n-1}}\right)$$
$$-\frac{1}{2}\sigma^2 \left(\sum_{i=0}^{n} \Delta S_i\right)^2 \left\{\frac{\Delta S_{n-1}(V_{n+1}^m - V_n^m) - \Delta S_n (V_n^m - V_{n-1}^m)}{(\Delta S_n)^2 \times \Delta S_{n-1}}\right\}$$
$$+ r V_n^m = 0$$

In butterfly option, the discontinuity appears at three exercise prices (E_1, E_2 and E_3). In the proposed procedure, dense grid is generated in the vicinity of the exercise prices and coarse grid is generated elsewhere as shown in figure 2.The patches I, II and III have dense grids and coarse grid elsewhere. The grid in each patch is uniform, therefore, the order of the error in each patch is the same as for uniform grid, i.e.$o(\Delta S_n)^2$. The condition of stability can be deduced (Ashraf et al. 2012) ,

$$0 < \Delta t < \frac{1}{\sigma^2 \alpha_{N-2}^2 \beta + \frac{r}{2}},$$

where $\beta = \dfrac{\Delta S_{n-3} + \Delta S_{n-2}}{2(\Delta S_{n-2})^2 \times \Delta S_{n-3}}$.

3 Backward-Euler Finite Difference Scheme

In this method, we use forward difference for V first time derivative, central difference for first S derivative and for second S derivative, we first use forward difference and then backward difference:

$$\frac{\partial V}{\partial t}(\alpha_n, (m+1)\Delta t) \approx \frac{V_n^{m+1} - V_n^m}{\Delta t},$$

$$\frac{\partial V}{\partial S}(\alpha_n, (m+1)\Delta t) \approx \frac{V_{n+1}^{m+1} - V_{n-1}^{m+1}}{\Delta S_n + \Delta S_{n-1}},$$

$$\frac{\partial^2 V}{\partial S^2}(\alpha_n, (m+1)\Delta t)$$
$$\approx \frac{\Delta S_{n-1}(V_{n+1}^{m+1} - V_n^{m+1}) - \Delta S_n (V_n^{m+1} - V_{n-1}^{m+1})}{(\Delta S_n)^2 \times \Delta S_{n-1}}.$$

Using the above substitutions, equation (2.1) takes the form :

$$\frac{V_n^{m+1} - V_n^m}{\Delta t} - r \sum_{i=0}^{n} \Delta S_i \left(\frac{V_{n+1}^m - V_{n-1}^m}{\Delta S_n + \Delta S_{n-1}}\right)$$

$$-\frac{1}{2}\sigma^2 \left(\sum_{i=0}^{n} \Delta S_i\right)^2$$
$$\frac{\Delta S_{n-1}(V_{n+1}^{m+1} - V_n^{m+1}) - \Delta S_n (V_n^{m+1} - V_{n-1}^{m+1})}{(\Delta S_n)^2 \times \Delta S_n}$$
$$+ r V_n^{m+1} = 0$$

After simplifying and re-arranging, the above equation takes the form :

$$\left(1 + \frac{\Delta t \sigma^2 \alpha_n^2}{(\Delta S_n)^2} + \frac{\Delta t \sigma^2 \alpha_n^2}{2\Delta S_n \times \Delta S_{n-1}} + r\Delta t\right) V_n^{m+1}$$

$$= V_n^m - \left(\frac{r\Delta t \alpha_n}{\Delta S_n + \Delta S_{n-1}} - \frac{\Delta t \sigma^2 \alpha_n^2}{2\Delta S_n \times \Delta S_{n-1}}\right) V_{n-1}^{m+1}$$

$$+ \left(\frac{\Delta t \sigma^2 \alpha_n^2}{2(\Delta S_n)^2} + \frac{r\Delta t \alpha_n}{\Delta S_n + \Delta S_{n-1}}\right) V_{n+1}^{m+1}.$$

This system of equations can be solved by Gauss-Seidel method. The values V_n^0, V_0^m, V_N^m with $n = 0, \ldots, N$ and $m = 0, \ldots, M$ are known from initial and boundary conditions.

4. Numerical Experiments

We demonstrate some numerical experiments for one asset butterfly spread call option. The butterfly can be created by using call or put options.The strategy is termd "Butterfly" due to the shape of the risk characteristics graph we see, the two wings and the larger body.The butterfly spread is constructed through buying 1 long In The Money(ITM) call, shorting two At The Money (ATM) calls and buying 1 long Out of the money (OTM) call. The ratio between the three options is 1:2:1 and the distance between the strike prices of long options should be equidistant from the short call strike. For example, a butterfly spread could be made of 3 call options with strikes of E_1=10, E_2=20 and E_3=30. The butterfly will result in a net debit transaction as the ITM and OTM call options will total a larger value than the two short ATM calls. The payoff function of butterfly spread call option is given by $V(S,T) = max(S - E_1, 0) - 2max(S - E_2, 0) + max(S - E_3, 0)$.

In butterfly spread call option, the payoff is acting as the initial condition and has a piecewise discontinuity at the strike prices. We use the following parameters for the computation of the butterfly spread call option for one asset: T=0.5, r=0.1, σ=0.4, S=40, E_1=10, E_2=20 and E_3=30, with N=27,57, 85,115, 143, 216, 289 grids in space, different schemes are applied for option valuation. Tables 1 and 2 show the option prices for an at-the-money (S=E_1) butterfly option from various schemes. In Tables 1 and 2, N shows the number of points for uniform and variable space-stepping, $C_{u,e}$ shows the option price for uniform spacing and $C_{a,e}$ shows the option price for adaptive spacing for Explicit scheme. $C_{u,i}$ and $C_{a,i}$ are option prices for implicit scheme. It can be observed that same option values are obtained by using less number of points in adaptive space-stepping as compared to uniform space-stepping and adaptive space-stepping converges more rapidly than uniform space-stepping.

Table 1. Comparison between Explicit and adaptive explicit schemes option values

N	$C_{u,e}$	$C_{a,e}$
29	1.4266	1.2860
59	1.1761	1.1904
89	1.1685	1.1574
119	1.1658	1.1406
149	1.1644	1.1305

Figure 1, depicts the grid for initial conditions for one asset. Here, we refined intervals [E_1-ε,E_1+ε], [E_2-ε, E_2+ε] and [E_3-ε, E_3+ε] around the strike prices. We choose ε(epsilon) as 5 and the grid is refined in these intervals to cure oscillations caused by discontinuity. Figure 3, represents the Gamma plot for for one asset by explicit method. Figure 5 Time plot for payoff function.

Similar results can be obtained for space-stepping by Backward-Euler scheme. This shows that adaptive space-stepping is much better than uniform space-stepping. Figure 4, shows time option evolution plot.

4. Conclusions

We have developed an efficient finite difference numerical technique for one asset to cure oscillations in the solution. The computational domain is descretized embedding more points near the singularities and coarse grid otherwise. We have to modify the numerical scheme to deal with the uneven spacing of the points. The stability analysis of explicit scheme is also performed for one asset Black-Scholes equation. The results are presented for an adaptive explicit scheme, and adaptive implicit scheme.

The oscillations at discontinuities are eliminated by using adaptive space-stepping. The adaptive space-stepping speeds up the solution convergence as compared to the uniform space-stepping. The adaptive finite difference scheme needs less points in its computation and hence is very efficient.

Table 2. Comparison between Implicit and adaptive implicit schemes option values

N	$C_{u,i}$	$C_{a,i}$
27	1.4522	1.3101
57	1.2697	1.1938
85	1.2313	1.1612
115	1.1956	1.1419
143	1.1299	1.1321

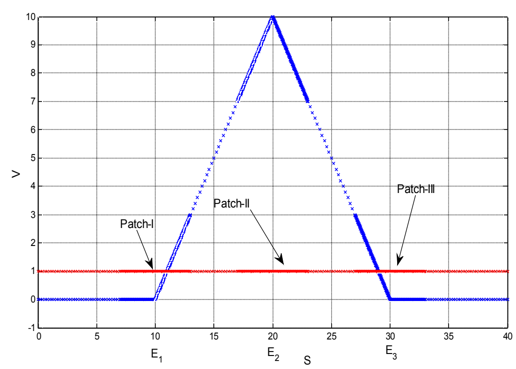

Figure 1Payoff function for one asset butterfly spread

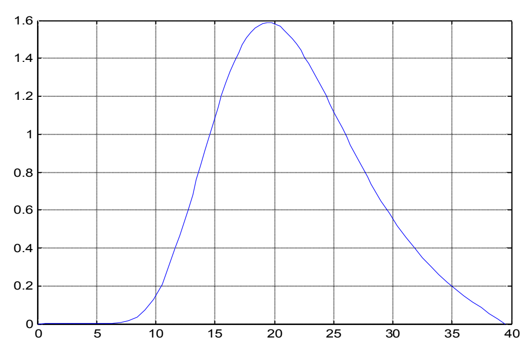

Figure 2Butterfly option simulation using adaptive explicit

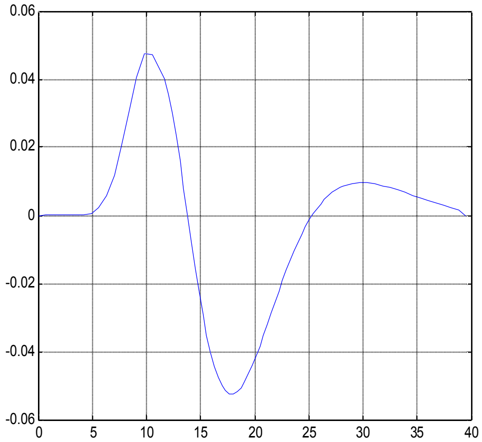

Figure 3 Gamma plot for butterfly option

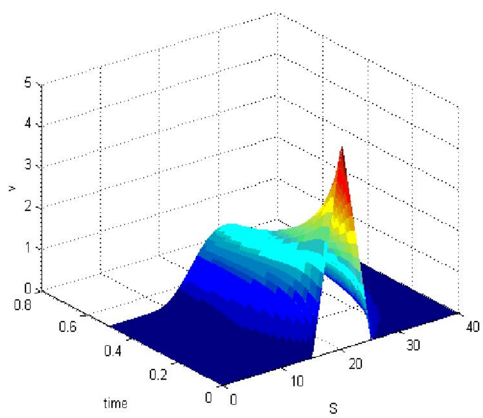

Figure 4Time evolution for butterfly call option

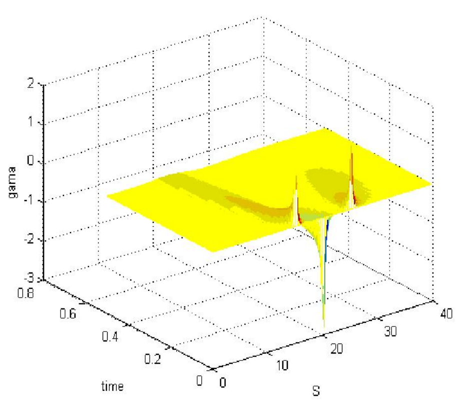

Figure 5Time evolution for gamma

Acknowledgements:

We are thankful to Prof.Dr.A.Q.M.Khaliq, Middle Tennessee State University, USA, for his cooperation in this research.

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2/9/2013