Stability Of The Generalized 2-Variable Quadratic Functional Equation

A. Nejadali Abolfazl, Ghoochani Rezvan And Kheirabadi Hamid

Nikshar Branch, Islamic Azad University, Nikshar, Iran
E-mails : nejadali.2000@yahoo.com; re_ghoochani@yahoo.com

Abstract: IN this paper, we derive the stability of the 2-variable quadratic functional equation (0.01)$(x+y,z+t)+f\big(x+\sigma(y),\ z+\sigma(t)\big)=2f(x,z)+2f(y,t)$. and the stability of the 2-variable quadratic functional equation (0.02) $f(x+y,z+t)+g\big(x+\sigma(y),\ z+\sigma(t)\big)=h(x,z)+k(y,t)$. forall $x,y,z,t\in G$, where G is an semigroup and σ is a homomorphism of G such that$\sigma o\sigma$ = I.

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1. INTRODUCTION AND PRELIMINARIES

Let X,Y be real vector spaces. For the mapping $f:X\times X\to Y$, consider the 2-variable quadratic functional equation

(1.0.3) $f(x+y,z+t)+f\big(x+\sigma(y),\ z+\sigma(t)\big)=2f(x,z)+2f(y,t)$.

we define

(1.0.4) $Df(x,y,z,t):=f(x+y,\ z+t)+f\big(x+\sigma(y),\ z+\sigma(t)\big)-2f(x,z)-2f(y,t)$.

$D_{\mu_1,\mu_2}f(x,y,z,t):=f(\mu_1(x+y),\mu_2(z+t))+f\big(\mu_1(x+\sigma(y)\big),\mu_2\big(z+\sigma(t)\big)-2\mu_1\mu_2f(x,z)-2\mu_1\mu_2f(y,t)$.

for all $\mu_1,\mu_2\in T^1:=\{\lambda\in C:|\lambda|=1\}$ and all $x,y,z,t\in X$.

Let X be a set. A function $d:X\times X\to[0,\infty)$ is called a generalized metric on X if and only if d satisfies

(μ_1) $d(x,y)=0$ *if and only if* $x=y$

(μ_2) $d(x,y)=d(y,x)$ *for all* $x,y\in X$

(μ_3) $d(x,z)\le d(x,y)+d(y,z)$

Theorem 1.0.1*Let (X,d) be a generalized complet metric space. Assume that Λ : X → X is a strictly contracting operator with the lipstick constant<.1If there exists a nonnegative integlersuch that*$d(\Lambda^{k+1}f,\Lambda^k f)<\infty$*for some* $f\in X$*, then the following are true.*

(a) The sequence $\{\Lambda^n f\}$ *converges to a fixed point F of Λ;*

(b) F is the unique fixed point of Λ in

(1.0.5) $X^*=\{g\in X\mid d\big(\Lambda^k f,g\big)<\infty\}$;

(c) If $h\in X^*$, then

(1.0.6) $d(h,F)\le\frac{1}{1-L}d(\Lambda h,h)$.

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2. MAIN RESULT

Theorem 2.0.2 *Let σ be an homomorphism of the semigroup G such that σ0σ = I and Y is a Banach space. Suppose that* $f:G\times G\to Y$*satisfies the inequality*

(2.0.7) $\|D_{\mu_1,\mu_2}f(x,y,z,t)\|y\le\delta$

for all $\mu_1,\mu_2\in T^1$ *and for all* $x,y,z,t\in G$ *and for some* $\delta\in[0,\infty)$. *Then there exists a unique 2-variable quadratic mapping* $F:G\times G\to Y$ *such that*

(2.0.8) $\|f(x,z)-F(x,z)\|Y\le\delta$

(2.0.9) $D_{\mu_1,\mu_2}F(x,y,z,t) = 0.$

for all $\mu_1, \mu_2 \in T^1$ *and for all* $x, y, z, t \in G$.

Proof. Letting $\mu_1, \mu_2 = 1$, $y = x$ and $z = t$ in (2.0.7) , we have

(2.0.10) $\| f(2x, 2z) + f(x + \sigma(x),\ z + \sigma(z)) - 4f(x, z)\| \leq \delta.$

for all $x, y, z, t \in G$. Then we obtain

(2.0.11) $\| \frac{f(2x,2z)+f(x+\sigma(x),z+\sigma(z))}{4} - f(x,z)\| \leq \delta.$

for all $x, y, z, t \in G$. Now we set $X = \{h \mid h : G \times G \to Y \ is\ a\ fu\ nction\}$ and introduce a generalized metric on X as follows :

(2.0.12) $d(g, h) = \inf\{\delta \in [0, \infty) \mid \|g(x, y) - h(x, y)\| \leq \delta\}.$

First, we will verify that (X,d) is a complete space. Let $\{g_n\}$ be a Cauchy sequence in (X,d). According to definition Cauchy sequence, for any $\epsilon > 0$ there exists a positive integer N_ϵ such that $d(g_m, g_n) \leq \epsilon$ for all $m, n \geq N_\epsilon$. From the definition of the generalized metric d, it follows that

(2.0.13) $\forall \epsilon > 0\ \exists N_\epsilon \in N\ \forall\, m, n \geq N_\epsilon\ \|g_m(x, y) - g_n(x, y)\| \leq \epsilon.$

This implies that $\{g_n(x, y)\}$ is a Cauchy sequence in *Y*. Since *Y* is a complete space, $\{g_n(x, y)\}$ converges in Y for each $x, y \,\epsilon\, G$. Hence we can define a function $g : G \times G \to Y$ by

(2.0.14) $g(x, y) = \lim_{n\to\infty} g_n(x).$

If we let *m* increase to infinity, it follows from (2.0.13) that for any $\epsilon > 0$, there exists a positive integer N_ϵ with $\|g_n(x, y) - g(x, y)\| \leq \epsilon$ for all $n \geq N_\epsilon$, that is, for any $\epsilon > 0$, there exists a positive integer N_ϵ such that $d(g_n, g\ \ ) \leq \epsilon$ for any $n \geq N_\epsilon$. This fact leads us to the conclusion that $\{g_n\}$ converges in (X,d). Hence (X,d) is a complete space. Now we define an operator $\Lambda : X \to X$ such that

(2.0.15) $(\Lambda f)(x, z) := \frac{f(2x,2z)+f(x+\sigma(x),z+\sigma(z))}{4}$

We assert that Λ is strictly contractive on X. Given , $h \in X$, let $\delta \in [0, \infty)$ be an arbitrary constant with $d(g, h) \leq \delta$, that is,

(2.0.16) $\| g(x, z) - h(x, z)\| \leq \delta$

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It then follows from (2.0.15) that

$$\|(\Lambda g)(x, z) - (\Lambda h)(x, z)\| = \|\frac{g(2x, 2z) + g\big(x + \sigma(x),\ z + \sigma(z)\big)}{4} - \frac{h(2x, 2z) + h\big(x + \sigma(x),\ z + \sigma(z)\big)}{4}\|$$
$$\leq \|\frac{g(2x, 2z) - h(2x, 2z)}{4}\| + \|\frac{g\big(x + \sigma(x),\ z + \sigma(z)\big) - h\big(x + \sigma(x),\ z + \sigma(z)\big)}{4}$$
$$\leq \frac{\delta}{4} + \frac{\delta}{4} \leq \frac{\delta}{2} = \frac{1}{2}\|g(x, z) - h(x, z)\|$$

That is, $d(\Lambda g, \Lambda h) \leq \frac{1}{2} d(g, h)$, for any $g, h \in X$. Hence Λ is a strictly contractive function. It easily follows that

(2.0.17) $(\Lambda^2 f)(x, z) = \frac{f(2^2x,2^2z)+3f(2x+2\sigma(x),2z+2\sigma(z))}{2^{2\times 2}}$

And by direct computation, we obtain

(2.0.18)

$$(\Lambda^n f)(x, z) = \frac{f(2^n x, 2^n z) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x),\ 2^{n-1}z + 2^{n-1}\sigma(z))}{2^{2n}}$$

Now we obtain

$$\|(\Lambda^{n+1}f)(x,z)-(\Lambda^{n}f)(x,z)\| \le \frac{1}{2^{2(n+1)}}\ \|f(2^{n+1}x,2^{n+1}z)$$
$$+f\big(2^nx+2^n\sigma(x),2^nz+2^n\sigma(z)\big)-4f(2^nx,2^nz)$$
$$+\frac{1}{2^{2(n+1)}}\|2(2^n-1)f\big(2^nx+2^n\sigma(x),2^nz+2^n\sigma(z)\big)$$
$$-4(2^n-1)f\big(2^{n-1}x+2^{n-1}\sigma(x),2^{n-1}z+2^{n-1}\sigma(z)\big)\|$$
$$\le \frac{\delta}{2^{2(n+1)}}+\frac{(2^n-1)\delta}{2^{2(n+1)}}=\frac{2^n\delta}{2^{2(n+1)}}=\frac{\delta}{2^{n+2}}<\infty.$$

Hence, if we set $n=0$ then $d(\Lambda f,f)\le\frac{\delta}{4}<\infty$. Thus Theorem (1.0.1)(a) implies that there exists a function $F\in X$, which is a fixed point of Λ, such that $F(x,z):=\lim_{n\to\infty}(\Lambda^nf)(x,z)$for any $x,z\in G$. One can verify F satisfies of (1.0.3). Indeed,

$$\big\|(\Lambda^nf)(x+y,z+t)+(\Lambda^nf)\big(x+\sigma(y),z+\sigma(t)\big)-2(\Lambda^nf)(x,z)-2(\Lambda^nf)(y,t)\big\|$$
$$=\frac{1}{2^{2(n)}}\|f\big(2^n(x+y),2^n(z+t)\big)$$
$$+(2^n-1)f\big(2^{n-1}(x+y)+2^{n-1}\sigma(x+y),2^{n-1}(z+t)+2^{n-1}\sigma(z+t)\big)$$
$$+f\Big(2^n\big(x+\sigma(y)\big),2^n\big(z+\sigma(t)\big)\Big)$$
$$+(2^n-1)f\Big(2^{n-1}\big(x+\sigma(y)\big)+2^{n-1}\big(\sigma(x)+y\big),2^{n-1}\big(z+\sigma(t)\big)+2^{n-1}\big(\sigma(z)+t\big)\Big)$$
$$-2f(2^nx,2^nz)-2(2^n-1)f\big(2^{n-1}x+2^{n-1}\sigma(x),2^{n-1}z+2^{n-1}\sigma(z)\big)$$
$$-2f(2^ny,2^nt)-2(2^n-1)f\big(2^{n-1}y+2^{n-1}\sigma(y),2^{n-1}t+2^{n-1}\sigma(t)\big)\|$$
$$\le\frac{\delta}{2^{2n}}+\frac{(2^n-1)\delta}{2^{2n}}=\frac{2^n\delta}{2^{2n}}=\frac{\delta}{2^n}$$

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Letting $n\to\infty$, we see that F satisfies (1.0.3).

In Theorem (1.0.1) let $k=0$. Since $f\in X^*=\{g\in X\,|d(f,g)<\infty\}$ In Theorem (1.0.1), by Theorem (1.0.1)(c) and (3.0.31), we have

(2.0.19) $$d(f,F)\le\frac{1}{1-L}d(\Lambda f,f)\le 2\frac{\delta}{4}=\frac{\delta}{2}\le\delta.$$

Therefor (2.0.8) is true. One can verify F satisfies of (2.0.10). Indeed,

$$\big\|D_{\mu_1,\mu_2}F(x,y,z,t)\big\|$$
$$=\lim_{n\to\infty}\left\|\frac{D_{\mu_1,\mu_2}f(2^nx,2^nz)}{2^{2n}}\right\|$$
$$+\lim_{n\to\infty}\left\|\frac{(2^n-1)D_{\mu_1,\mu_2}f(2^{n-1}x+2^{n-1}\sigma(x),2^{n-1}z+2^{n-1}\sigma(z))}{2^{2n}}\right\|\le\lim_{n\to\infty}\frac{2^n\delta}{2^{2n}}=\lim_{n\to\infty}\frac{\delta}{2^n}=0$$

for all $x,y,z,t\in G$. So $D_{\mu_1,\mu_2}F(x,y,z,t)=0$for all $\mu_1,\mu_2\in T^1$ and for all $x,y,z,t\in G$. Assume that there exists another mapping $H:G\times G\to Y$ which satisfies (1.0.3) and (2.0.8). we obtain

$$\|(\Lambda^nf)(x,z)-H(x,z)\|\le\|(\Lambda^nf)(x,z)-F(x,z)\|+\|F(x,z)-f(x,z)\|+\|f(x,z)-H(x,z)\|$$
$$\infty+\delta+\delta<\infty$$

Thus $d(\Lambda^nf,H)<\infty$, and so $H\in X^*$. On the oder hand $\Lambda H(x,z)=H(x,z)$ for all $x,z\in G$. Hence byTheorem(1.0.1)(b),weget $F=H$.This completes the proof of the theorem.□

3. STABILITY OF EQUATION (0.0.2) IN ABELIAN SEMIGROUPS

In this section we investigate the stability of the 2-variable quadratic functional equation

(3.0.20) $$f(x+y,z+t)+g\big(x+\sigma(y),z+\sigma(t)\big)=h(x,z)+k(y,t).$$

for all $x,y,z,t\in G$, where G is an abliane semigroup and σ is a homomorphism of G such that σ0σ = I.

First we will establish some results which will be instrumental in proving our main results.

In the following lemma, we will present a stability result for Jensen's functional equation :

(3.0.21) $f(x+y,z+t)+f\big(x+\sigma(y),z+\sigma(t)\big)=2f(x,z),$

for all $x,y,z,t\in G$.

Lemma 3.0.3. *Let σ be an homomorphism of the abelian semigroup G such that σ0σ = I. Let Y be a Banach space. Suppose that $f: G\times G\to Y$ satisfies the inequality*

(3.0.22) $\left\|f\big(\mu_1(x+y),\mu_2(z+t)\big)+f\Big(\mu_1\big(x+\sigma(y)\big),\mu_2\big(z+\sigma(t)\big)\Big)-2\mu_1\mu_2 f(x,z)\right\|\le\delta.$

for all $\mu_1,\mu_2\in T^1$, for all $x,y,z,t\in G$ and for some $\delta\ge 0$. Then there exists a 2-variable quadratic mapping $F: G\times G\to Y$ such that

(3.0.23) $\|f(x,z)-F(x,z)\|\le\delta.$

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and

(3.0.24) $F\big(\mu_1(x+y),\mu_2(z+t)\big)+F\Big(\mu_1\big(x+\sigma(y)\big),\mu_2\big(z+\sigma(t)\big)\Big)-2\mu_1\mu_2F(x,z)=0.$

for all $\mu_1,\mu_2\in T^1$ and for all $x,y,z,t\in G$.

Proof. Letting $\mu_1,\mu_2=1$, $y=x$ and $z=t$ in (3.0.33), we have

(3.0.25) $\big\|f(2x,2z)+f\big(x+\sigma(x),z+\sigma(z)\big)-2f(x,z)\big\|\le\delta.$

for all $x,z\in G$. Then we obtain

(3.0.26) $\|\frac{f(2x,2z)+f(x+\sigma(x),z+\sigma(z))}{2}-f(x,z)\|\le\delta.$

for all $x,z\in G$. Now we set $X=\{h\,|h: G\times G\to Y\text{ is a linear function}\}$ and introduce a generalized metric on X as follows :

(3.0.27) $d(g,h)=\inf\{\delta\in[0,\infty)\mid\|g(x,y)-h(x,y)\|\le\delta\}.$

By Theorem (2.0.2), follows that (X,d) is a complete space. Now we define an operator $J_{\frac{1}{2}}\Lambda: X\to X$ such that

(3.0.28) $\left(J_{\frac{1}{2}}\Lambda f\right)(x,z):=\frac{1}{2}\left[\frac{f(2x,2z)+f(x+\sigma(x),z+\sigma(z))}{2}\right]$

We assert that $J_{\frac{1}{2}}\Lambda$ is strictly contractive on X. Given $g,h\in X$, let $\delta\in[0,\infty)$ be an arbitrary constant with $d(g,h)\le\delta$, that is,

(3.0.29) $\|g(x,z)-h(x,z)\|\le\delta$

It then follows from (3.0.28) that

$$\left\|\left(J_{\frac{1}{2}}\Lambda g\right)(x,z)-\left(J_{\frac{1}{2}}\Lambda h\right)(x,z)\right\|$$
$$=\frac{1}{2}\left[\left\|\frac{g(2x,2z)+g\big(x+\sigma(x),z+\sigma(z)\big)}{2}-\frac{h(2x,2z)+h\big(x+\sigma(x),z+\sigma(z)\big)}{2}\right\|\right]$$
$$\le\frac{1}{2}\left[\left\|\frac{g(2x,2z)-h(2x,2z)}{2}\right\|+\left\|\frac{g\big(x+\sigma(x),z+\sigma(z)\big)-h\big(x+\sigma(x),z+\sigma(z)\big)}{2}\right\|\right]$$
$$\le\frac{1}{2}\left[\frac{\delta}{2}+\frac{\delta}{2}\right]=\frac{\delta}{2}=\frac{1}{2}\|g(x,z)-h(x,z)\|$$

That is, $d(J_{\frac{1}{2}}\Lambda g,J_{\frac{1}{2}}\Lambda h)\le\frac{1}{2}d(g,h)$, for any $g,h\in X$. Hence $J_{\frac{1}{2}}\Lambda$ is a strictly contractive function. It easily follows that

(3.0.30) $\left(J\ {}^{2}_{\frac{1}{2}}\Lambda f\right)(x,z)=\frac{1}{2^2}\left[\frac{f(2^2x,2^2z)+3f(2x+2\sigma(x),2z+2\sigma(z))}{2^2}\right]$

And by direct computation, er obtain

(3.0.31) $\left(J\ {}^{n}_{\frac{1}{2}}\Lambda f\right)(x,z)=\frac{1}{2^n}\left[\frac{f(2^nx,2^nz)+(2^n-1)f\big(2^{n-1}x+2^{n-1}\sigma(x),2^{n-1}z+2^{n-1}\sigma(z)\big)}{2^n}\right]$

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Now we obtain

$$\|\left(J_{\frac{1}{2}}^{n+1}\Lambda f\right)(x,z)-\left(J_{\frac{1}{2}}^{n}\Lambda f\right)(x,z)\| \leq \frac{1}{2^{n+1}}[\frac{1}{2^{n+1}}\|f(2^{n+1}x,2^{n+1}z)$$
$$+f\left(2^n x+2^n\sigma(x),2^n z+2^n\sigma(z)\right)-4f(2^n x,2^n z)\|$$
$$+\frac{1}{2^{n+1}}\|2(2^n-1)f\left(2^n x+2^n\sigma(x),2^n z+2^n\sigma(z)\right)$$
$$-4(2^n-1)f\left(2^{n-1}x+2^{n-1}\sigma(x),2^{n-1}z+2^{n-1}\sigma(z)\right)\|]$$
$$\leq \frac{1}{2^{n+1}}\left[\frac{\delta}{2^{n+1}}+\frac{(2^n-1)\delta}{2^{n+1}}\right]=\frac{1}{2^{n+1}}\frac{2^n\delta}{2^{n+1}}=\frac{\delta}{2^{n+2}}.$$

Hence, $\{\left(J_{\frac{1}{2}}^{n}\Lambda f\right)(x,z)\}$ is an Cauchy sequence. Since Y is a complete space, implies that there exists a linear function F, such that $F(x,z) := \lim_{n\to\infty}\left(J_{\frac{1}{2}}^{n}\Lambda f\right)(x,z)$for any $x,z \in G$. one can see by Theorem (2.0.2) that F satisfies of (3.0.21), (3.0.23) and (3.0.24).□

By using the proof of preceding lemma, we get the stability of the Jensen's function equation

(3.0.32) $$f(y+x,t+z)+f(\sigma(y)+x,\sigma(t)+z)=2f(x,z).$$

Corollary 3.0.4.*Let σ be an homomorphism of the abelian semigroup G such that $\sigma 0\sigma = I$. Let Y be a Banach space. Suppose that $f : G\times G \to Y$ satisfies the inequality*

(3.0.33) $$\left\|f(\mu_1(y+x),\mu_2 t+z)+f\left(\mu_1(\sigma(y)+x),\mu_2(\sigma(t)+z)\right)-2\mu_1\mu_2 f(x,z)\right\|\leq\delta.$$

for all $\mu_1,\mu_2\in T^1$, for all $x,y,z,t\in G$ and for some $\delta\geq 0$. Then there exists a 2-variable quadratic mapping $F : G\times G\to Y$ such that

(3.0.34) $$\|f(x,z)-F(x,z)\|\leq\delta.$$

and

(3.0.35) $$F\left(\mu_1(y+x),\mu_2(t+z)\right)+F\left(\mu_1(\sigma(y)+x),\mu_2(\sigma(t)+z)\right)-2\mu_1\mu_2 F(x,z)=0.$$

for all $\mu_1,\mu_2 \in T^1$ and for all $x,y,z,t \in G$.

In the following lemma, we obtain a partial stability theorem for the 2-variable quadratic functional equation

(3.0.36) $$f(x+y,z+t)+g\left(x+\sigma(y),z+\sigma(t)\right)=h(x,z)+k(y,t).$$

for all $x,y,z,t \in G$.

Lemma 3.0.5.*Let σ be an homomorphism of the abelian semigroup G such that $\sigma 0\sigma = I$. Let Y be a Banach space. Suppose that $f,g,h,k : G\times G\to Y$ satisfies the inequality*

(3.0.37)

$$\left\|f\left(\mu_1(x+y),\mu_2(z+t)\right)+g\left(\mu_1\left(x+\sigma(y)\right),\mu_2\left(z+\sigma(t)\right)\right)-\mu_1 h(x,z)-\mu_2 k(y,t)\right\|\leq\delta.$$

for all $\mu_1,\mu_2\in T^1$, for all $x,y,z,t\in G$ and for some $\delta\geq 0$. Then there exists a unique2-variable quadratic mapping $Q : G\times G\to Y$ a solution of (1.0.3). Also there exists a solution J_1,J_2 of Jensen's functional equation (3.0.21) and (3.0.32) such that

(3.0.38) $$\|h(x,z)-J_2(x,z)-Q(x,z)-h(e,e)\|\leq 16\delta.$$

(3.0.39) $$\|k(x,z)-J_1(x,z)-Q(x,z)-k(e,e)\|\leq 16\delta.$$

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(3.0.40) $$\left\|f^e(x,z)+g^e(x,z)-Q(x,z)-\tfrac{1}{2}f(e,e)-\tfrac{1}{2}g(e,e)\right\|\leq 12\delta.$$

(3.0.41) $$\|(f^e-g^e)(x+y,z+t)-(f^e-g^e)(x+\sigma(y),z+\sigma(t))\|\leq 12\delta.$$

(3.0.42) $$\left\|f^e(x,z)-\tfrac{1}{2}J_1(x,z)-\tfrac{1}{2}J_2(x,z)\right\|\leq 10\delta.$$

and

(3.0.43) $$\left\|g^e(x,z)-\tfrac{1}{2}J_2(x,z)+\tfrac{1}{2}J_1(x,z)\right\|\leq 10\delta.$$

and

(3.0.44) $$D_{\mu_1,\mu_2}Q(x,y,z,t)=0.$$

(3.0.45)$J_1(\mu_1(x+y),\mu_2(z+t))+J_1\left(\mu_1(x+\sigma(y)),\mu_2(z+\sigma(t))\right)-2\mu_1\mu_2 J_1(x,z)=0.$

and

(3.0.46)$J_2(\mu_1(y+x),\mu_2(t+z))+J_2(\mu_1(\sigma(y)+x),\mu_2(\sigma(t)+z))-2\mu_1\mu_2 J_2(x,z)=0.$

for all $\mu_1,\mu_2 \in T^1$ and for all $x,y,z,t \in G$.

Proof. Letting $\mu_1,\mu_2=1$, Let us denote by $f^e(x,z)=\frac{f(x,z)+f(\sigma(x),\sigma(z))}{2}$ then even part of f and by $f^o(x,z)=\frac{f(x,z)-f(\sigma(x),\sigma(z))}{2}$ the odd part of *f*. For any function $f:G\times G\to Y$, we define $F(x,z)=f(x,z)-f(e,e)$. By putting $x=y=z=t=e$ in (3.0.37), we get

(3.0.47) $\|f(e,e)+g(e,e)-h(e,e)-k(e,e)\|\le\delta.$

Consequently, if we subtract the inequality (3.0.37) from the new inequality (3.0.47), we obtain

(3.0.48) $\left\|F_1(x+y,z+t)+F_2(x+\sigma(y),z+\sigma(t))-F_3(x,z)-F_4(y,t)\right\|\le 2\delta.$

Now by replacing *x* by *σ(x)* and *y* by *σ(y)* and *z* by *σ(z)* and *t* by *σ(t)* in (3.0.48) and we add the inequality obtained in (3.0.48), we deducethat

(3.0.49) $\left\|F_1^e(x+y,z+t)+F_2^e(x+\sigma(y),z+\sigma(t))-F_3^e(x,z)-F_4^e(y,t)\right\|\le 2\delta.$

and

(3.0.50) $\left\|F_1^o(x+y,z+t)+F_2^o(x+\sigma(y),z+\sigma(t))-F_3^o(x,z)-F_4^o(y,t)\right\|\le 2\delta.$

for all. Hence, if we replace *y,t* by *e* and *x,z* by *e* respectively in (3.0.49), we get

(3.0.51) $\|F_1^e(x,z)+F_2^e(x,z)-F_3^e(x,z)\|\le 2\delta.$

and

(3.0.52) $\|F_1^e(y,t)+F_2^e(y,t)-F_3^e(y,t)\|\le 2\delta.$

So, in view of (3.0.49), (3.0.51) and (3.0.52), we obtain

$$\left\|F_1^e(x+y,z+t)+F_2^e(x+\sigma(y),z+\sigma(t))-(F_1^e+F_2^e)(x,z)-(F_1^e+F_2^e)(y,t)\right\|\left\|F_1^e(x+y,z+t)+F_2^e(x+\sigma(y),z+\sigma(t))-F_3^e(x,z)-F_4^e(y,t)\right\|+\|F_1^e(x,z)+F_2^e(x,z)-F_3^e(x,z)\|+\|F_1^e(y,t)+F_2^e(y,t)-F_4^e(y,t)\|\le 6\delta.$$

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By replacing *y,t* by $\sigma(y),\sigma(t)$in above, we get the following

(3.0.53)

$$\left\|F_1^e(x+\sigma(y),z+\sigma(t))+F_2^e(x+y,z+t)-(F_1^e+F_2^e)(x,z)-(F_1^e+F_2^e)(y,t)\right\|\le 6\delta$$

If we add the inequality above to (3.0.53), we get

$$\|(F_1^e+F_2^e)(x+y,z+t)+(F_1^e+F_2^e)(x+\sigma(y),z+\sigma(t))-2(F_1^e+F_2^e)(x,z)-2(F_1^e+F_2^e)(y,t)\|\le 12\delta,$$

(3.0.54) $\left\|(F_1^e-F_2^e)(x+y,z+t)-(F_1^e-F_2^e)(x+\sigma(y),z+\sigma(t))\right\|\le 12\delta,$

for all $x,y,z,t \in G$. Hence, in view of Theorem (2.0.2), there exists a unique function *Q*, a solution of equation (1.0.3) such that

(3.0.55) $\|(F_1^e+F_2^e)(x,z)-Q(x,z)\|\le 12\delta,$

Consequently, from (3.0.54) and (3.0.55), we deduce that

(3.0.56) $\|F_3^e(x,z)-Q(x,z)\|\le 16\delta,$

and

(3.0.57)$\|F_4^e(x,z)-Q(x,z)\|\le 16\delta,$

for all $x,z \in G$. On the other hand, from (3.0.50) we get

(3.0.58) $\|F_3^o(x,z)-F_1^o(x,z)-F_2^o(x,z)\|\le 2\delta.$

and

(3.0.59) $\|F_4^o(x,z)-F_1^o(x,z)+F_2^o(x,z)\|\le 2\delta.$

for all $x,z \in G$. Hence, we obtain

(3.0.60) $\|2F_1^o(x,z) - F_3^o(x,z) - F_4^o(x,z)\| \leq 4\delta.$

and

(3.0.61) $\|2F_2^o(x,z) - F_3^o(x,z) + F_4^o(x,z)\| \leq 4\delta.$

for all $x, z \in G$ and Consequently, we have

$$\|F_3^o(x+y,z+t) + F_3^o\big(x+\sigma(y),z+\sigma(t)\big) - 2F_3^o(x,z)$$
$$\leq \|F_3^o(x+y,z+t) - F_1^o(x+y,z+t) - F_2^o(x+y,z+t)\|$$
$$+\big\|F_3^o\big(x+\sigma(y),z+\sigma(t)\big) - F_1^o\big(x+\sigma(y),z+\sigma(t)\big) - F_2^o\big(x+\sigma(y),z+\sigma(t)\big)\big\|$$
$$+\big\|F_1^o(x+y,z+t) + F_2^o\big(x+\sigma(y),z+\sigma(t)\big) - F_3^o(x,z) - F_4^o(y,t)\big\|$$
$$+\big\|F_1^o\big(x+\sigma(y),z+\sigma(t)\big) + F_2^o(x+y,z+t) - F_3^o(x,z) - F_4^o\big(\sigma(y),\sigma(t)\big)\big\| \leq 8\delta.$$

and

$$\|F_4^o(y+x,t+z) + F_4^o(\sigma(y)+x,\sigma(t)+z) - 2F_4^o(x,z)$$
$$\leq \|F_4^o(y+x,t+z) - F_1^o(y+x,t+z) - F_2^o(y+x,t+z)\|$$
$$+\|F_4^o(\sigma(y)+x,\sigma(t)+z) - F_1^o(\sigma(y)+x,\sigma(t)+z) - F_2^o(\sigma(y)+x,\sigma(t)+z)\|$$
$$+\|F_1^o(y+x,t+z) + F_2^o(\sigma(y)+x,\sigma(t)+z) - F_3^o(y,t) - F_4^o(x,z)\|$$
$$+\|F_1^o(\sigma(y)+x,\sigma(t)+z) + F_2^o\big(\sigma(y)+\sigma(x),\sigma(t)+\sigma(z)\big)$$
$$-F_3^o\big(\sigma(y),\sigma(t)\big) - F_4^o(x,z)\| \leq 8\delta.$$

for all $x, y, z, t \in G$. Now from lemma (3.0.3) and corollary (3.0.4) there exists two solution of Jensen's functional equation $J_1, J_2 : G \times G \to Y$ such that

(3.0.62) $\|F_4^o(x,z) - J_1^o(x,z)\| \leq 8\delta.$

and

(3.0.63) $\|F_3^o(x,z) - J_2^o(x,z)\| \leq 8\delta.$

for all $x, z \in G$. Now, by small computations, we obtain the rest of the proof. □

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