Inextensible flows of non-null curves on an pseudo-Euclidean hypersuface in pseudo-Euclidean space R_1^n

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Abstract: We study inextensible flows of non-null curves on pseudo-Euclidean surface in R_1^n and derive equations for inextensible evolution of non-null curves on pseudo-Euclidean hypersurface. Necessary and sufficient conditions for non-null inextensible curve flow on pseudo-Euclidean hypersurface in R_1^n are expressed as a partial differential equation involving the i.geodecic curvature and $a_i = II(E_1, E_i)$ functions.

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1.Introduction

Inextensible curve flows have many applications in computer vision, computer animation and structural mechanics. (Chirikjian and Burdick,1990),(Hamilton and Gage, 1986).

Recently Kwon and Park examine inextensible flow of curves and developable surfaces for plane and space curves (Kwon, 1999),(Kwon and Park, 2005). Later author studied inextensible flow of curves and developable surfaces and space curves in Minkowski 3-space (Gürbüz, 2009).

In this section we give fundamental definitions relation Pseudo-Euclidean space (O'Neill, 1983)..

R^n with the metric

$$-\sum_{i=1}^{\varsigma} x_i y_i + \sum_{\varsigma+1}^{n} x_i y_i, \quad x, y \in R^n,\ 0 \le \varsigma \le n$$

is called pseudo-Euclidean space and is denoted by R_1^n, where ς is called the index of the metric.

Let R_1^n be a pseudo-Euclidean space furnished with a metric tensor $\langle , \rangle$. A vector x to R_1^n is called spacelike if $\langle x, x\rangle > 0$ or, null if $\langle x, x\rangle = 0$ and $x \neq 0$, timelike if $\langle x, x\rangle < 0$.

Let M be a pseudo-Euclidean hypersurface in n-dimensional Lorentz-Minkowski space R_1^n and non-null curve β which lies on M. Apart from the Frenet vector field system $\{E_1, E_2, E_3, ..., E_{n-1}, N\}$, there is also exist a second orthonormal vector field system $\{E_1, E_2, E_3, ..., E_{n-1}, N\}$ at every point of non-null curve β. Let $E_1 = \beta'$ denote the unit tangent vector to β, let N denote the unit hypersurface normal to M. $\{E_1, E_2, E_3, ..., E_{n-1}, N\}$ gives a basis for all vectors at β and $\{E_1, E_2, E_3, ..., E_{n-1}, N\}$ gives a basis for the vectors tangent to M at β. Let II denote the second fundamental form of M. The orthonormal system $\{E_1, E_2, E_3, ..., E_{n-1}, N\}$ is called natural frame field for hypersurface strip β.

Let M be a pseudo-Euclidean hypersurface in R_1^n and β be non-null curve on M. Then, for each i, 1≤i≤n-1, the function,

$$k_{ig} = \langle E_i', E_{i+1}\rangle$$

is called the i. geodesic curvature function of the curve β and $(')$ denote the derivative with respect to the arc length parameter of a non-null curve.

Let M be a pseudo-Euclidean hypersurface in R_1^n. The derivative formulas of orthonormal vector field system $\{E_1, E_2, E_3, ..., E_{n-1}, N\}$ is given by as following:

$$E_1' = \varepsilon_2 k_{1g} E_2 + \varepsilon_n a_1 N$$
$$E_2' = -\varepsilon_1 k_{1g} E_1 + \varepsilon_3 k_{2g} E_3 + \varepsilon_n a_2 N$$
$$\vdots$$
$$E_i' = -\varepsilon_{i-1} k_{(i-1)g} E_{i-1} + \varepsilon_{i+1} k_{ig} E_{i+1} + \varepsilon_n a_i N$$
$$N' = -\varepsilon_1 a_1 E_1 - \varepsilon_2 a_2 E_2 ... + \varepsilon_{(n-1)} a_{(n-1)} E_{n-1}$$

where

$$a_i = II(E_1, E_i) \quad , \quad 1 \le i \le n-1 \text{ and}$$

$$\langle E_1, E_1 \rangle = \varepsilon_1, \quad \langle E_2, E_2 \rangle = \varepsilon_2, ..., \quad \langle E_{n-1}, E_{n-1} \rangle = \varepsilon_{n-1}, \langle N, N \rangle = \varepsilon_n .$$

2. Method.

The arc length of the non-null curve not be subject to elongation can be given by the the following condition

$$s = \int_0^\sigma \frac{\partial v}{\partial t} d\sigma = 0$$

for all $\sigma \in [0, l]$. Here, $v = \left| \left\langle \frac{\partial \beta}{\partial \sigma}, \frac{\partial \beta}{\partial \sigma} \right\rangle \right|^{1/2}$ is non-null curve speed, s is arc length of non-null curve on pseudo-Euclidean space.

Definition 2.1 Non-null curve evolution $\beta(\sigma, t)$ on pseudo-Euclidean hypersurface and its flow $\frac{\partial \beta}{\partial t}$ are called inextensible if $\frac{\partial}{\partial t}\left(\left| \left\langle \frac{\partial \beta}{\partial \sigma}, \frac{\partial \beta}{\partial \sigma} \right\rangle \right|^{1/2} \right) = 0$.

Lemma 2.1 $\frac{\partial v}{\partial t} = \varepsilon_1 \frac{\partial h_1}{\partial \sigma} - h_2 v k_{1g} - h_n v a_1$

Proof.

$$v^2 = \left\langle \frac{\partial \beta}{\partial t}, \frac{\partial \beta}{\partial t} \right\rangle$$

$$2v \frac{\partial v}{\partial t} = 2\left\langle \frac{\partial \beta}{\partial t}, \frac{\partial}{\partial \sigma}\left(\left(\sum_{i=1}^{n-1} h_i E_i \right) + h_n N \right) \right\rangle$$

$$2v \frac{\partial v}{\partial t} = \langle v E_1, \frac{\partial h_1}{\partial \sigma} E_1 + h_1(\varepsilon_2 v k_{1g} E_2 + \varepsilon_n v a_1 N) + \frac{\partial h_2}{\partial \sigma} E_2$$
$$+ h_2(-\varepsilon_1 v k_{1g} E_1 + \varepsilon_3 v k_{2g} E_3 + \varepsilon_n v a_1 N) + ... + \frac{\partial h_n}{\partial \sigma} E_n$$
$$+ h_n(-\varepsilon_1 v a_1 E_1 - \varepsilon_2 a_2 v E_2 - ... - \varepsilon_{n-1} a_{n-1} v E_{n-1}) \rangle$$

$$2v \frac{\partial v}{\partial t} = 2v(\varepsilon_1 \frac{\partial h_1}{\partial \sigma} - h_2 v k_{1g} - h_n v a_1) \quad (2.1)$$

Thus we have

$$\frac{\partial v}{\partial t} = \varepsilon_1 \frac{\partial h_1}{\partial \sigma} - h_2 v k_{1g} - h_n v a_1 .$$

Theorem 2.1 Let M be a pseudo-Euclidean hypersurface in n dimensional semi-Euclidean space R_1^n and $\{E_1, E_2, ..., E_{n-1}, N\}$ be natural frame field for non-null hypersurface strip (β, M). A flow of non-null curve on an pseudo-Euclidean hypersurface

$$\frac{\partial \beta}{\partial t} = h_1 E_1 + ... + h_{n-1} E_{n-1} + h_n N \quad (2.2)$$

is inextensible if and only if

$$\frac{\partial h_1}{\partial s} = \varepsilon_1 h_2 k_{1g} + \varepsilon_1 h_n a_1 . \quad (2.3)$$

Proof.

$$\frac{\partial s(\sigma, t)}{\partial t} = \int_0^\sigma \frac{\partial v}{\partial t} d\sigma = \int_0^\sigma (\varepsilon_1 \frac{\partial h_1}{\partial \sigma} - h_2 v k_{1g} - h_n v a_1) d\sigma = 0 \quad (2.4)$$

From (2.4) , we obtain Equation (2.3).

Corallary 2.1 Inextensible non-null curve flow is independent of $h_3, h_4, ..., h_{n-1}$ components.

Lemma 2.2

$$\frac{\partial E_1}{\partial t} = \left(\sum_{i=2}^{n-1} \frac{\partial h_i}{\partial s} + \varepsilon_i h_{i-1} k_{(i-1)g} - \varepsilon_i h_{i+1} k_{ig} - \varepsilon_i a_i h_n \right) E_i + (\frac{\partial h_n}{\partial s} + \sum_{i=1}^{n-1} \varepsilon_n a_i h_i) N \quad (2.5)$$

$$\frac{\partial E_i}{\partial t} = \left(\sum_{i=2}^{n-1} \varepsilon_1 h_{i+1} k_{ig} + \varepsilon_1 a_i h_n - \varepsilon_1 \varepsilon_i \frac{\partial h_i}{\partial s} - \varepsilon_1 h_{i-1} k_{(i-1)g} \right) E_1 + \sum_{j=2}^{n-1} \varepsilon_j \psi_{ij} E_j \quad (2.6)$$

$$\frac{\partial N}{\partial t} = -\varepsilon_1 \varepsilon_n (\frac{\partial h_n}{\partial s} + \sum_{i=1}^{n-1} \varepsilon_n a_i h_i) E_1 - \sum_{j=2}^{n-1} \varepsilon_j \psi_{jn} E_j \quad (2.7)$$

Proof.

$$\frac{\partial E_1}{\partial t} = \frac{\partial}{\partial t}(\frac{\partial \beta}{\partial s}) = \frac{\partial}{\partial s}(\sum_{i=1}^{n} h_i E_i),$$

$$\frac{\partial E_1}{\partial t} = \frac{\partial h_1}{\partial s} E_1 + h_1(\varepsilon_2 k_{1g} E_2 + \varepsilon_n a_1 N) + \frac{\partial h_2}{\partial s} E_2 + h_2(-\varepsilon_1 k_{1g} E_1 + \varepsilon_3 k_{2g} E_3 + \varepsilon_n a_2 N) + ... + \frac{\partial h_{n-1}}{\partial s} E_{n-1}$$

$$+ h_n(-\varepsilon_1 a_1 E_1 - \varepsilon_2 a_2 E_2 - ... - \varepsilon_{n-1} a_{n-1} E_{n-1}) \quad (2.8)$$

From (2.3) and (2.8), we obtain (2.5).

$$\left\langle \frac{\partial E_1}{\partial t}, E_i \right\rangle + \left\langle E_1, \frac{\partial E_i}{\partial t} \right\rangle = 0 \quad \text{i=2,3,…,n-1}$$

$$\left\langle \left(\sum_{i=2}^{n-1} \left(\frac{\partial h_i}{\partial s} + \varepsilon_i h_{i-1} k_{(i-1)g} - \varepsilon_i h_{i+1} k_{ig} - \varepsilon_i a_i h_n \right) \right) E_i + \left(\frac{\partial h_n}{\partial s} + \sum_{i=1}^{n-1} \varepsilon_n a_i h_i \right) N, E_i \right\rangle + \left\langle E_1, \frac{\partial E_i}{\partial t} \right\rangle = 0 \qquad (2.9)$$

$$\left\langle E_i, \frac{\partial E_j}{\partial t} \right\rangle + \left\langle E_j, \frac{\partial E_i}{\partial t} \right\rangle = 0 \quad , \quad i \neq j, \ \text{j=2,3,…,n-1} \qquad (2.10)$$

From (2.10),

$$\psi_{ij} = \left\langle E_j, \frac{\partial E_i}{\partial t} \right\rangle \ \text{i=1,2,…,n-1} \quad \text{and} \quad \psi_{jn} = \left\langle N, \frac{\partial E_j}{\partial t} \right\rangle \qquad (2.11)$$

Using (2.9), (2.10) and (2.11), we have

$$\frac{\partial E_i}{\partial t} = \left(\sum_{i=1}^{n-1} \varepsilon_1 h_{i+1} k_{ig} + \varepsilon_1 a_i h_n - \varepsilon_1 \varepsilon_i \frac{\partial h_i}{\partial s} - \varepsilon_1 h_{i-1} k_{(i-1)g} \right) E_1 + \sum_{j=2}^{n-1} \varepsilon_j \psi_{ij} E_j \qquad (2.12)$$

$$\frac{\partial}{\partial t} \langle E_1, N \rangle = 0 \qquad (2.13)$$

$$\frac{\partial N}{\partial t} = -\varepsilon_1 \varepsilon_n \left(\frac{\partial h_n}{\partial s} + \sum_{i=1}^{n-1} \varepsilon_n a_i h_i \right) E_1 - \sum_{j=2}^{n-1} \varepsilon_j \psi_{jn} E_j \qquad (2.14)$$

Theorem 2.2 Assume that the curve flow $\frac{\partial \beta}{\partial t} = h_1 E_1 + \ldots + h_{n-1} E_{n-1} + h_n N$ is inextensible. Then the following system of partial differential equations holds

$$\frac{\partial k_{1g}}{\partial t} = \varepsilon_2 \frac{\partial^2 h_2}{\partial s^2} + \frac{\partial (h_1 k_{1g})}{\partial s} - 2a_2 \frac{\partial h_n}{\partial s} - h_n \frac{\partial a_2}{\partial s} - \frac{\partial (h_3 k_{2g})}{\partial s} - \varepsilon_n a_2^2 h_2 - \varepsilon_n a_1 \psi_{2n}$$

$$\frac{\partial a_1}{\partial t} = \varepsilon_n \frac{\partial^2 h_n}{\partial s^2} + \frac{\partial}{\partial s}(a_1 h_1)$$

$$\psi_{1n} = \varepsilon_n \frac{\partial^2 h_n}{\partial s^2} + \frac{\partial}{\partial s}(a_1 h_1)$$

$$\sum_{i=2}^{n-1} \frac{\partial a_i}{\partial t} = \sum_{i=2}^{n-1} \frac{\partial (\psi_{in})}{\partial s}$$

$$\frac{\partial k_{ig}}{\partial t} = \varepsilon_{i-1} k_{(i-1)g} \psi_{(i-1)(i+1)} - \varepsilon_{i+1} k_{ig} \psi_{(i+1)(i+1)} - \varepsilon_n a_i \psi_{(i+1)n} + \frac{\partial \psi_{i(i+1)}}{\partial s}$$

Proof.

$$\frac{\partial}{\partial s}\left(\frac{\partial E_1}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial E_1}{\partial s}\right) \qquad (2.15)$$

$$\frac{\partial}{\partial s}\left(\frac{\partial E_1}{\partial t}\right) = \frac{\partial}{\partial s} \left[\begin{array}{l} \left(\sum_{i=1}^{n-1} \left(\frac{\partial h_i}{\partial s} + \varepsilon_i h_{i-1} k_{(i-1)g} - \varepsilon_i h_{i+1} k_{ig} - \varepsilon_i a_i h_n \right) \right) E_i \\ + \left(\frac{\partial h_n}{\partial s} + \sum_{i=1}^{n-1} \varepsilon_n a_i h_i \right) N \end{array} \right] \qquad (2.16)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial E_1}{\partial s}\right)= \varepsilon_2 \frac{\partial k_{1g}}{\partial t}E_2+\varepsilon_2\frac{\partial E_2}{\partial t}k_{1g}+\varepsilon_n\frac{\partial a_1}{\partial t}N + a_1\left[-\varepsilon_1\left(\frac{\partial h_n}{\partial s}+\sum_{i=1}^{n-1}a_i h_i\right)E_1+\varepsilon_n\sum_{j=2}^{n-1}\varepsilon_j\psi_{jn}E_j\right] \tag{2.17}$$

From (2.15), (2.16) and (2.17) , we obtain

$$\frac{\partial k_{1g}}{\partial t}=\varepsilon_2\frac{\partial^2 h_2}{\partial s^2}+\frac{\partial(h_1k_{1g})}{\partial s}-2a_2\frac{\partial h_n}{\partial s}-h_n\frac{\partial a_2}{\partial s}-\frac{\partial(h_3k_{2g})}{\partial s}-\varepsilon_n a_2^2h_2-\varepsilon_n a_1\psi_{2n}$$

$$\frac{\partial a_1}{\partial t}=\varepsilon_n\frac{\partial^2 h_n}{\partial s^2}+\frac{\partial}{\partial s}(a_1h_1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial E_i}{\partial s}\right)=\frac{\partial}{\partial s}\left(\frac{\partial E_i}{\partial t}\right) \tag{2.18}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial E_i}{\partial s}\right)=-\varepsilon_{i-1}\frac{\partial k_{(i-1)g}}{\partial t}E_{i-1}-\varepsilon_{i-1}k_{(i-1)g}\frac{\partial E_{i-1}}{\partial t}+\varepsilon_{i+1}\frac{\partial k_{ig}}{\partial t}E_{i+1} +\varepsilon_{i+1}k_{ig}\frac{\partial E_{i+1}}{\partial t}+\varepsilon_n\frac{\partial a_i}{\partial t}N+\varepsilon_n a_i\frac{\partial N}{\partial t} \tag{2.19}$$

$$\frac{\partial}{\partial s}\left(\frac{\partial E_i}{\partial t}\right)=\frac{\partial}{\partial s}\left[\left(\sum_{i=2}^{n-1}\varepsilon_1h_{i+1}k_{ig}+\varepsilon_1a_ih_n-\varepsilon_1\varepsilon_i\frac{\partial h_i}{\partial s}-\varepsilon_1h_{i-1}k_{(i-1)g}\right)E_1+\sum_{j=2}^{n-1}\varepsilon_j\psi_{ij}E_j\right] \tag{2.20}$$

From (2.18), (2.19) and (2.20), we obtain

$$\frac{\partial k_{ig}}{\partial t}=\varepsilon_{i-1}k_{(i-1)g}\psi_{(i-1)(i+1)}-\varepsilon_{i+1}k_{ig}\psi_{(i+1)(i+1)}-\varepsilon_n a_i\psi_{(i+1)n}+\frac{\partial\psi_{i(i+1)}}{\partial s}.$$

$$\frac{\partial}{\partial s}\left(\frac{\partial N}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial N}{\partial s}\right) \tag{2.21}$$

From (2.21), we obtain

$$\psi_{1n}=a_1\left(\varepsilon_n\frac{\partial h_n}{\partial s}+a_1h_1\right),$$

$$\sum_{i=2}^{n-1}\frac{\partial a_i}{\partial t}=\sum_{i=2}^{n-1}\frac{\partial(\psi_{in})}{\partial s}.$$

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