The model particle with torsion for 4 dimensional null Cartan curves

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Abstract: We study the model whose Lagrangian depend the torsion of the particle path for null Cartan curves in Lorentzian space L_4.

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1.Introduction

There are a lot paper concerning geometrical models that describe a relativistic particle. The Lagrangians depending on the first and second curvatures have been investigated in the last twenty years (Langer and Singer,1984), (Plyushchay,1991), (Nersessian, 2000), (Capovilla and Güven, 2002), (Barros, Ferrandez and Javoloyes, 2004), (Ferrandez , Gimenez and Lucas , 2007) .

Lagrangians play important role both in the static and in the kinematic description of curves. Also, they play role in the connection between the motion of curves and integrable systems.

The model of relativistic particle with torsion appears in Bose-Fermi transmutation (Plyushchay, 1991). Investigation of such kind of particle systems became popular after work of Polyakov who study evaluation of the effective action of CP model minimally coupled to the Chern-Simons gauge field .A relativistic model of the anyon, describing the states of the particle with torsion with the maximum value of the mass, was constructed in by Plyushchay.

2. Intrinsic Method

In this paper, we investigate the model of a particle with torsion of relativistic particles for 4 dimensional null Cartan curves as following described by action

$$U(\alpha)=\int_0^{l(w)} h(k_2)ds$$

where h is real arbitrary function and differentiable function.

We obtain the Euler-Lagrange equations describing equilibrium for null Cartan curves in 4 dimensional Lorentzian space L_4.

In this section, some definitions will be given. Let M be 4 dimensional Lorentzian space. If (x_1,x_2,x_3,x_4) and (y_1,y_2,y_3,y_4) are the components of X and Y with respect to an allowable coordinat system , then

$$\langle X,Y\rangle_L = x_1y_1+x_2y_2+x_3y_3-x_4y_4$$

which is called a Lorentzian inner product.

Lemma 2.1. Let α (s) be a non-geodesic null curve of a Lorentzian manifold $M^4{}_1$. There exist only a Frenet frame $F=\{\alpha,N,U_1,U_2\}$ for which (α ,F) is a null Cartan curve with Cartan Frenet equations (Duggal and Jin H, 2008)

$$\begin{aligned}
\nabla_\xi \xi &= \xi' = U_1\\
\nabla_\xi N &= -k_1U_1+k_2U_2\\
\nabla_\xi U_1 &= -k_1\xi+N\\
\nabla_\xi U_2 &= -k_2\xi
\end{aligned} \qquad (2.1)$$

and

$$\langle\alpha'',\alpha''\rangle=\langle\xi',\xi'\rangle=1,\quad \langle U_2,U_2\rangle=1,\quad \langle\xi,\xi\rangle=0,\quad \langle N,N\rangle=0,\quad \langle\xi,N\rangle=-1.$$

Let $\alpha(t)$ be a null immersed curve in Lorentzian 4-space. Let us consider a variation of α ,

$$\Psi(t,w):[0,l]\times(-\varepsilon,\varepsilon)\to M$$

with $\alpha(t,0)=\alpha(t)$. Associated with α are two vector fields along α , $W(t,w)=\frac{\partial\alpha}{\partial w}(t,w)$ and $V(t,w)=\frac{\partial\alpha}{\partial t}$ is velocity vector field . $W=\frac{\partial\alpha}{\partial w}(t,0)$ is the variation vector field along α .

Lemma 2.2. The following assertions hold (Ferrandez, Gimenez and Lucas, 2007).

(1) $\langle \nabla_{\xi} W, \xi \rangle = 0,$

(2) $\frac{\partial v}{\partial t}(t,0) = -\frac{1}{2} g v \qquad g = -\langle \nabla^2{}_{\xi} W, U_1 \rangle$

(3) $\frac{\partial k_1}{\partial w}(t,0) = \langle \nabla^2{}_{\xi} W, N \rangle + k_1 \langle \nabla_{\xi} W, N \rangle + k_1 g - \frac{1}{2}\xi(\xi(g))$

(4) $\frac{\partial k_2}{\partial w}(t,0) = \langle \nabla^4{}_{\xi} W, U_2 \rangle + 2k_1 \langle \nabla^2{}_{\xi} W, U_2 \rangle + k'_1 \langle \nabla_{\xi} W, U_2 \rangle + 2k_2 g$

(2.2)

Let M be a complete, simply connected Lorentzian space and α 4 dimensional null immersed curve in M. W is a Killing vector field along α if and only if it satisfies the following conditions

$$\frac{\partial v}{\partial t}(t,0) = \frac{\partial k_1}{\partial w}(t,0) = \frac{\partial k_2}{\partial w}(t,0) = 0.$$

2.1. Equilibrium equations

the following model of relativistic particle with torsion in 4 dimensional Lorentzian space is given by

$$\int_0^{l(w)} h(k_2)\Big|_{w=0}$$

The first derivative restriction of this action can be written

$$\frac{\partial}{\partial w}\int_0^{l(w)} h(k_2)\Big|_{w=0} = \frac{\partial}{\partial w}\int_0^{l(w)} h(k_2) v dt\Big|_{w=0} = \int_0^{l(w)} [h_{k_2}(k_2) w(k_2) v + h(k_2)\frac{\partial v}{\partial w}] dt\Big|_{w=0}$$

where h_{k_2} denotes the partial derivative of h with respect to k_2. In this case, we can give the first variational formulas:

$$\frac{\partial}{\partial w}\int_0^{l(w)} h(k_2)\Big|_{w=0} = \langle \nabla^3{}_{\xi} W, h_{k_2} U_2 \rangle\Big|_0^l - \langle \nabla^2{}_{\xi} W, h_{k_2} k_2 \xi + h'_{k_2} U_2 \rangle\Big|_0^l$$

+ (2.1.1)

$$\left\langle \nabla_{\xi} W, \frac{1}{2} h(k_2) - h_{k_2} k_2) U_1 + (h''_{k_2} + 2h_{k_2} k_1) U_2 \right\rangle\Big|_0^l + \langle W, Z_1 \rangle\Big|_0^l +$$

$$\int_0^l \langle W, p_0 \xi + p_1 U_1 + p_1 U_2 + p_3 N \rangle ds$$

And the boundary term $\Psi(\alpha, W)$ is given by

$\Psi(\alpha, W) =$

$\langle \nabla^3{}_{\xi} W, h_{k_2} U_2 \rangle\Big|_0^l - \langle \nabla^2{}_{\xi} W, h_{k_2} k_2 \xi + h'_{k_2} U_2 \rangle\Big|_0^l +$

$\left\langle \nabla_{\xi} W, \frac{1}{2} h(k_2) - h_{k_2} k_2) U_1 + (h''_{k_2} + 2h_{k_2} k_1) U_2 \right\rangle\Big|_0^l + \langle W, Z_1 \rangle\Big|_0^l$

(2.1.2)

where Z_1 is the vector field given

$Z_1 = (-2$

$h''_{k_2} k_2 - h'_{k_2} k'_2 + k_1 k_2 h_{k_2} (h_{k_2} k_2)'' - 4h_{k_2} k_1 k_2 + \frac{1}{2} h(k_2) k_1) \xi$

(2.1.3)

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$((h'_{k_2} k_2 - \frac{1}{2} h'(k_2)) U_1 + (-h'''_{k_2} - 2(h_{k_2} k_1)' + h_{k_2} k'_1) U_2 + (2h_{k_2} k_2 - \frac{1}{2} h(k_2)) N$

Thus we obtain Euler-Lagrange equations as following:

$$E = p_0(\alpha)\xi + p_1(\alpha) U_1 + p_2(\alpha) U_2 + p_3(\alpha) N = 0$$

Where

$p_0 = 4(h_{k_2} k_2 k_1)' + k_2 (h_{k_2} k_1)' + 3h'''_{k_2} k_2 + 4h''_{k_2} k'_2 + k''_2 h'_{k_2} + 2(h_{k_2} k_2{}^2)' k_2 + h'(k_2) k'_1$

$p_1 = 3h''_{k_2} k'_2 + 2h'_{k_2} k_2' - h_{k_2} k_1 k_2 (h_{k_2} k_2)'' - k_1 k_2 h_{k_2} k_2 + h(k_2) k_1 - h''(k_2)/2$

$p_2 = h'''_{k_2} - h_{k_2} k_2{}^2 + (h_{k_2} k_1)'' - h(k_2) k_2 / 2$

$p_3 = -(h_{k_2} k_2)' - h'(k_2)$

Definition 2.1.1. A regular unit speed null Cartan curve is called h-elastica if it satisfies $E = p_0(\alpha)\xi + p_1(\alpha) U_1 + p_2(\alpha) U_2 + p_3(\alpha) N = 0$. In this case, we have the following differential equations

$$p_0(\alpha) = 0, p_1(\alpha) = 0, p_2(\alpha) = 0, p_3(\alpha) = 0 .$$

For get $k_1 = \kappa$ and $k_2 = \tau$, with aid equation 2.1.2, we obtain

$\Psi(\alpha, W) =$

$\langle \nabla^3{}_{\xi} W, h_{\tau} U_2 \rangle\Big|_0^l - \langle \nabla^2{}_{\xi} W, h'_{\tau} U_2 \rangle\Big|_0^l +$

$\left\langle \nabla_{\xi} W, \frac{1}{2} h(\tau) - h_{\tau}\tau) U_1 + (h''_{\tau} - 2h_{\tau}\kappa) U_2 \right\rangle\Big|_0^l + \langle W, Z_1 \rangle\Big|_0^l$

From Equation 2.1.3,

$Z_1 = (-2$

$h''_{\tau}\tau - h'_{\tau}\tau' + \kappa\tau h_{\tau}(h_{\tau}\tau)'' - 4h_{\tau}\kappa\tau + \frac{1}{2} h(\tau)\kappa)\xi$

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$(h'_{\tau}\tau - \frac{1}{2} h'(\tau)) U_1 + (-h'''_{\tau} - 2(h_{\tau}\kappa)' + h_{\tau}\kappa') U_2 - (\frac{1}{2} h(\tau) - h_{\tau}\tau) N$

where Z_1 is constant vector field elastic curve. A vector field in 4- dimensional Lorentzian space produce one parameter family of rotations when its the shape is $\alpha \wedge M_1 \wedge M_2$. M_1 and M_2 are constant vectors. We obtain

$$\Psi(\alpha, \alpha \wedge M_1 \wedge M_2) = \langle T(\alpha, M_1), M_2 \rangle$$

$$T(\alpha, M_1) = -\kappa h_\tau(\xi \wedge U_1 \wedge M_1) + h_\tau(N \wedge U_2 \wedge M_1) - h'_\tau(U_1 \wedge U_2 \wedge M_1)$$
$$+(\frac{1}{2}h(\tau) - h_\tau\tau)(\xi \wedge U_1 \wedge M_1) + (h''_\tau + 2h_\tau\kappa)(\xi \wedge U_2 \wedge M_1) + (\xi \wedge Z_1 \wedge M_1)$$
(2.1.4)

Replacing Z_1 by M_1 in equation 2.1.4, we obtain

$$Q = T(\alpha, Z_1) = v_1\xi + v_2U_1 + v_3U_2 + v_4N,$$

$$v_1 = (h_\tau'\tau - \frac{1}{2}h'(\tau)(h''_\tau + h_\tau\kappa) - 2h'_\tau\tau' + \kappa\tau h_\tau(h_\tau\tau)'' -$$
$$4h_\tau\kappa\tau + \frac{1}{2}h(\tau)\kappa(h''_\tau + h_\tau\kappa) + (\frac{1}{2}h(\tau) - h_\tau h)(-h'''_\tau + h_\tau\kappa' - 2(h_\tau\kappa)',$$

$$v_2 = (\frac{1}{2}h(\tau) - h_\tau\tau)($$
$$-\kappa h_\tau - h''_\tau - 2h_\tau\kappa) - h_\tau(-2h''_\tau - h'_\tau\tau' + \kappa\tau h_\tau(h_\tau\tau)'' - 4h_\tau\kappa\tau + \frac{1}{2}h(\tau)\kappa,$$

$$v_3 = -(\frac{1}{2}h(\tau) - h_\tau\tau)^2,$$

$$v_4 = h'_\tau(\frac{1}{2}h(\tau) - h_\tau\tau) - h_\tau(h'_\tau\tau - \frac{1}{2}h'(\tau).$$

$$v_4 = h'_\tau(\frac{1}{2}h(\tau) - h_\tau\tau) - h_\tau(h'_\tau\tau - \frac{1}{2}h'(\tau).$$

Replacing M_1 to Q in equation 2.1.4, we obtain new constant vector field D. This vector field is the sum of translational and rotational vector field.

$$D = T(\alpha, Q) = ((\kappa h_\tau v_2 - h_\tau v_1 - h'_\tau v_1 - (h''_\tau + 2h_\tau\kappa)v_2)\xi$$
$$+((h_\tau\kappa - (h''_\tau + 2h_\tau\kappa)v_4)U_1$$
$$-h'_\tau v_4U_2 + h_\tau v_2N + \xi \wedge Z_1 \wedge Q$$

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where $J = D - Z_1 \wedge Q \wedge \xi$.

3. Result

D, Q and Z_1 are Killing field along α and α curve is solution of the Euler Lagrange equations if and only if

$\langle Z_1, Z_1 \rangle$ and $\langle Z_1, J \rangle$ vector fields are constant in 4-dimensional Lorentzian space .

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