Optimal control of an epidemic model of leptospirosis with time delay

Syed farasat saddiq[1] , Muhammad Altaf Khan[2], Saeed Islam[2], Gul Zaman[3], Naeam Khalid[1]
Syed Inayat Ali Shah[1], Zahor-ul-Haq[4]

[1]Department of Mathematics, Islamia College University Peshawar, Khyber Pakhtunkhwa, Pakistan.
[2]Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa , Pakistan
[3]Department of Mathematics, University of Malakand, Chakdara, Dir Lower , Khyber Pakhtunkhwa , Pakistan
[4]Department of Management sciences, Abdul Wali khan university, Mardan Pakistan.
altafdir@gmail.com

Abstract: In this paper, we consider an epidemic model of leptospirosis with nonlinear incidences by applying the optimal control techniques and time delay. First, we formulate the model, with optimal control and time delay. After the formulation of the model we find the existence of the control model. We completely charactrises the optimal control problem by using the Pontryagin's maximum principle. Our naim is to to minimize the infection in the host population, to do this we use three control variables. The numerical simulation of both the system solved by using the backward RungeKutta order four schemes for the solution of the problem. Finally, the numerical results are presented for justification purpose.

[Syed farasat saddiq , Muhammad Altaf Khan, Saeed Islam, Gul Zaman, Naeam Khalid,Syed Inayat Ali Shah, Zahor-ul-Haq. **Optimal control of an epidemic model of leptospirosis with time delay.** *Life Sci J* 2013;10(3s):292-298] (ISSN:1097-8135). http://www.lifesciencesite.com. 41

Keywords: Leptospirosis, Pontryagin's Maximum Principle, Time Delay, Optimal control, Numerical simulations , Subject classification: 92D25, 49J15, 93D20

1. Introduction

Mathematical modeling of infectious disease is one of the important research area now-a-days. The basic and important concern for mathematical models in epidemiology is qualitative analysis, the persistence, performance, asymptotic stability and the existence and uniqueness for the models. Many influential results related to this area have been established and can be found in many articles and books. The first epidemic model for the spread of infectious disease was introduced by [12]. They divided the population in three classes the Susceptible, infected and recovered. They assumed that the susceptible population in a fixed population become infected by contact with infected individuals, infected individuals either die or recover at a constant rate. Their models consist of three differential equations of ODE's which represent the rate of change in their respective population.

In recent years, some mathematical models incorporating delayed effects have been studied. Smith in [24] and Thieme [25] in (1990) derived a scalar delay differential equation for the population of immature and mature age classes. The maturation period is regarded as a time delay. Using the same idea, a system of delayed differential equations for mature population in a patchy environment has been proposed in So et al [26]. More recent studies consider an epidemic model with density dependence to describe disease transmission in variable population size, which can be found in Cooke et al [6,8,28]. Zaman et al [21] studied an SIR epidemic model with control strategies and using the delay. Zaman et al. [11,32] Studied the stability and optimal vaccination of a controlled SIR epidemic model without time delays and their dynamical behavior. Many mathematical models have been proposed to study the optimal control and delay such as [3,5,10,29,30,8,9,11]. For more work, [33-38].

In this paper, we consider a leptospirosis epidemic model [31] with time delay to prevent the spread of disease by using optimal treatment strategies. In order to do this, we first introduce a control variable representing the optimal treatment for infectious host and set an optimal control for our model. Moreover, we show the existence of an optimal control problem. For reducing the infection in a community, we use the control variables. We also analyzed the optimal control and optimality system using optimal control techniques. For the numerical simulation we use the real data of Thailand.Our aim is to maximize the population of susceptibles and recovered and minimize the infection in the infected individuals.

The paper is organized as follows. In Section 2 we study the basic model and applying the optimal control and time delay. Find the Jacobian and Hamiltonian to show the existence of the proposed model. Numerical Simulation of the model with the complete description of the parameters is discussed in Section 3. In the last Section 4 we wind up our work with the conclusion and references.

Optimal Control Techniques in Delay Model

To begin the optimal control procedure, it is necessary to have a model which describes the population dynamics. Youshida and Hara [27] considered an SIR model with time delay. We use an epidemic model of leptospirosis disease model to set our optimal control model. We have a population which consists of five differential equations. The system has two categories, Human and Vector. The human population consists of three sub-classes Susceptible S_h Infected I_h and Recovered R_h . The human population is denoted by N_1 with $N_1 = S_h + I_h + R_h$. The vector population is denoted by N_2 consists of two classes that is susceptible S_v and infected I_v , and $N_2 = S_v + I_v$. The model consists of a system of non-linear differential equation is given in the following.

$$\frac{dS_h}{dt} = b_1 - \mu_h S_h - \frac{\beta_1 S_h I(t)}{N_1(t)} - \frac{\beta_2 S_h I_v(t)}{N_1(t)} + \lambda_h R_h,$$

$$\frac{dI_h}{dt} = \frac{\beta_1 S_h I(t)}{N_1(t)} + \frac{\beta_2 S_h I_v(t)}{N_1(t)} - (\mu_h + \delta_h + \gamma_h) I_h(t),$$

$$\frac{dR_h}{dt} = \gamma_h I_h(t) - (\mu_h + \lambda_h) R_h, \qquad (1)$$

$$\frac{dS_v}{dt} = b_2 - \gamma_v S_v - \frac{\beta_3 S_v I_h(t)}{N_2(t)},$$

$$\frac{dI_v}{dt} = \frac{\beta_3 S_v I_h(t)}{N_2(t)} - (\gamma_v + \delta_v) I_v,$$

Here b_1 is the birth rate of human population, $\beta_1, \beta_2, \beta_3$ respectively represent the transmission Coefficient between human, susceptible human and infected vector and susceptible vector and infected human.Natural mortality rate of human population is represented by μ_h. λ_h is constant of proportionality where the infected human becomes susceptible again. The disease death rate of human population is denoted by δ_h. the natural mortality rate of vector population is shown by γ_v. δ_v is the disease death rate of the vector. α_1 The parameter measure inhibitory effect on human vector population and α_2 the parameter measure inhibitory effect of human population. b_2 is the birth rate for vector population.

The total dynamics of human population are represented by N_h given by,

$$N_h = b_1 - \mu_h N_1 - \delta_h I_h, \qquad (2)$$

and the total dynamics of vector population is denoted by N_h, given by,

$$N_v = b_2 - \gamma_v N_v - \delta_v I_v. \qquad (3)$$

Next we apply the optimal control and delay to our proposed model (1), we will derive an optimal control model to fit our control strategy. The theoretical foundation of optimal control models with underlying dynamics given by ordinary differential equations was developed by Pontragin and his co-worker in Moscow in 1950 [9]. So by Pontryagin's Maximum principle, its extensions and appropriate numerical methods we will set an optimal control problem in the time delayed model of leptospirosis disease. Our main goal is to investigate an effective treatment strategy to control infection diseases. We can make an epidemic model which satisfy that the number of infected individuals is not larger than the susceptible population and want to increase the recovered individuals from the infection. The definition of the control variables u_1 and u_2 is given by,

$u_1(t)$ Represents (cover all cuts, water dry, full-cover boots, shoes and long sleeve shirts when handling animals).

$u_2(t)$ Represents (wash hands thoroughly on a regular basis and shower after work).

To do this, we set an optimal control problem, with the control set defined by

$$U = \{(u_1(t), u_2(t)) \in L^2(0,T) : 0 \le u_1(t), u_2(t) \le 1, \; 0 \le t \le T\}. \qquad (4)$$

Where $u_1(t), u_2(t)$ is Lebesgue measurable and called a control variable.

$$J_{\xi}(u) = \int_0^T [(A_0 I_h + A_1 I_v + A_2 S_v) + \frac{1}{2}(\xi_1 u_1^2 + \xi_2 u_2^2)]dt. \qquad (5)$$

Subject to the control system

$$\frac{dS_h}{dt} = b_1 - \mu_h S_h - \frac{\beta_1 S_h I(t-h)}{N_1(t-h)} - \frac{\beta_2 S_h I_v(t-h)}{N_1(t-h)} + \frac{w u_1(t) I_h(t)}{N_1(t)} + \lambda_h R_h,$$

$$\frac{dI_h}{dt} = \frac{\beta_1 S_h I(t-h)}{N_1(t-h)} + \frac{\beta_2 S_h I_v(t-h)}{N_1(t-h)} - (\mu_h + \delta_h + \gamma_h) I_h(t) - \frac{u_1(t) I_h(t)}{N_1(t)},$$

$$\frac{dR_h}{dt} = \gamma_h I_h(t) - (\mu_h + \lambda_h) R_h + \frac{(1-w) u_1(t) I_h(t)}{N_1(t)}, \qquad (6)$$

$$\frac{dS_v}{dt} = b_2 - \gamma_v S_v - \frac{\beta_3 S_v I_h(t-h)}{N_2(t-h)} - \varepsilon_1 u_2(t) S_v(t),$$

$$\frac{dI_v}{dt} = \frac{\beta_3 S_v I_h(t-h)}{N_2(t-h)} - (\gamma_v - \delta_v) I_v - \varepsilon_2 u_2(t) I_v(t),$$

$$S_h(0) \ge 0, I_h(0) \ge 0, R_h(0) \ge 0,$$

$$S_v(0) \ge 0, I_v(0) \ge 0. \qquad (7)$$

Here ξ_1, ξ_2 are the positive constants to keep balanced of the sized of infected human individuals, infected vector individuals, susceptible vector individuals, I_h, I_v, S_v at time t and $w \in [0,1]$ and ϵ_1 and ϵ_2 are positive constants. In epidemic dynamics, stability, existence and optimal control theory are important research area. At first we will show the existence of solutions for the control system (6). In this control problem , we assume the restriction on the control variables such that $0 \le u_1, u_2 \le 1$, where $(u_1, u_2) \ge 0$ for all $t \in [0, T]$. The total population of host individuals is shown by N_1, and N_2 the populations of vector. Susceptible individuals acquire infection at a per capita $\beta_1 I_h(t-h)N_1(t-h), \beta_2 I_v(t-h)N_1(t-h)$. In our model the incidence rate is $\beta_1 S_h I_h(t-h)N_1(t-h)$ and $\beta_2 I_v(t-h)N_1(t-h)$ and $\frac{\beta_3 S_v I_h(t-h)}{N_2(t-h)}$. This incidence rate seems more reasonable than $\beta_1 I_h(t)N_1(t), \beta_2 I_v(t)N_1(t)$ because the force of infection is proportional to $\frac{I_h(t-h)}{N_1(t-h)}$ with time delay. Note that in some epidemic models, bilinear incidence rate $\beta_1 S_h(t) I_h(t)$ and standard incidence rate $\beta_1 S_h(t) I_h(t)/N$ are frequently used. Actually the infection probability per contact is likely influenced by the number of infected individuals because more infected individuals can increase infection risk. For instance, during SARS outbreak in 2003. The Chinese government did a lot of protection measures and control polices: closing schools, closing restaurants, postponing conferences, isolating infection etc. These actions greatly reduced the contact number per unit time. Then we write the control system (6) in the following form:

$$\frac{dW(t)}{dt} = AW + F(W(t)) \quad (8)$$

Where

$$W(t) = \begin{bmatrix} S_h(t) \\ I_h(t) \\ R_h(t) \\ S_v(t) \\ I_v(t) \end{bmatrix},$$

$$\begin{bmatrix} -\frac{\beta_1 S_h I_h(t-h)}{N_1(t-h)} - \frac{\beta_2 S_v I_v(t-h)}{N_1(t-h)} + b_1 + \frac{w u_1 I_h}{N_1} \\ \frac{\beta_1 S_h I_h(t-h)}{N_1(t-h)} + \frac{\beta_2 S_v I_v(t-h)}{N_1(t-h)} - \frac{u_1 I_h}{N_1} \\ \frac{(1-w)u_1 I_h}{N_1} \\ -\frac{\beta_3 S_v I_h(t-h)}{N_2(t-h)} + b_2 \\ \frac{\beta_3 S_v I_h(t-h)}{N_2(t-h)} \end{bmatrix}$$

$$A = \begin{bmatrix} -\mu_h & 0 & \lambda_h & 0 & 0 \\ 0 & -P_2 & 0 & 0 & 0 \\ 0 & \gamma_h & -P_3 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_v - \epsilon_1 u_2 & 0 \\ 0 & 0 & 0 & 0 & -P_1 - \epsilon_2 u_2 \end{bmatrix}$$

The second term on the right hand side of equation (8) satisfies,

$$|F(W_1) - F(W_2)| \le C_1(|(S_{1h}(t) - S_{2h}(t))| + C_2|(I_{1h}(t) - I_{2h}(t))| + C_3|(R_{1h}(t) - R_{2h}(t))| + C_4|(S_{1v}(t) - S_{2v}(t))| + C_5|(I_{1v}(t) - I_{2v}(t))|),$$

$$\le C(|(S_{1h}(t) - S_{2h}(t))| + |(I_{1h}(t) - I_{2h}(t))| + |(R_{1h}(t) - R_{2h}(t))| + |(S_{1v}(t) - S_{2v}(t))| + |(I_{1v}(t) - I_{2v}(t))|),$$

Where the positive constant $C = \max(C_1, C_2, C_3, C_4, C_5)$ is independent of the state variables . Also we have

$$|G(W_1) - G(W_2)| \le C|W_1 - W_2|, \quad \text{where}$$

$C = C_1 + C_2 + C_3 + C_4 + C_5 + \|M\| < \infty$. So, it follows that the function G is uniformly Lipschitz continuous. From the definition of control variables and non-negative initial conditions we can see that a solution of the system (5) exists see [19]. Now, we consider the control system (6) with the initial conditions (7) to show the existence of the control problem. Note that for bounded Lebesgue measurable controls and non-negative initial conditions, non-negative bounded solutions to the state system exists [19]. In order to find an optimal control pair, we consider the optimal control problem (6-7). First we have to find the Lagrangian and Hamiltonian for the optimal control problem (6-7). The Lagrangian of the control problem is given by,

$$L(I_h, I_v, S_v, u_1, u_2) = (A_0 I_h + A_1 I_v + A_2 S_v) + \frac{1}{2}(\xi_1 u_1^2 + \xi_2 u_2^2).$$

We seek for the minimum value of the Lagrangian and the Hamiltonian for the control system is given by

$$H = L(I_h, S_v, I_v, u_1, u_2) + \lambda_1 \frac{dS_h}{dt} + \lambda_2 \frac{dI_h}{dt}$$

$$+\lambda_3 \frac{dR_h}{dt} + \lambda_4 \frac{dS_v}{dt} + \lambda_5 \frac{dI_v}{dt}. \qquad (9)$$

In order to find an optimal control pair, we consider the optimal control problem (6-7). First we have to find the Lagrangian and Hamiltonian for the optimal control problem (6-7).

The conditions of the methods of optimal control with delay.

The state equations,

$$x_1'(t) = H_\lambda(x, x_h, u, \lambda)(t),$$

The optamility conditions

$$0 = H_u(x, x_h, u, \lambda)(t), \quad (10)$$

and the adjoint equations,

$$-\lambda_1'(t) = H_x(x, x_h, u, \lambda)(t) + \lambda(t+h) H_{x_h}(x, x_h, u, \lambda)(t)$$

Actually, the Lagrangian of the optimal control problem is given by

Theorem: Let $S_h^*(t), I_h^*(t), R_h^*(t), S_v^*(t)$ and $I_v^*(t)$ be the state variables with associated optimal solutions with the corresponding optimal control variables $u_1^*(t), u_2^*(t)$ for the optimal control problem (4-6). Then there exist adjoint variables $\lambda_i, i = 1, 2...5.$ satisfying

$$\frac{d\lambda_1}{dt} = \lambda_1(t)(\mu_h + \frac{\beta_1 I_h^*(t-h)}{N_1^*(t-h)} + \frac{\beta_2 I_v^*(t-h)}{N_1^*(t-h)} - \frac{w u_1^*(t) I_h^*(t)}{(N_1^*(t))^2})$$

$$+\lambda_2(t)(\frac{-\beta_1 I_h^*(t-h)}{N_1^*(t-h)} - \frac{\beta_2 I_v^*(t-h)}{N_1^*(t-h)} + \frac{u_1^*(t) I_h^*(t)}{(N_1^*(t))^2}))$$

$$+\lambda_3(t)((1-w)u_1^*(t)\frac{I_h^*(t)}{(N_1^*(t)^2)})$$

$$+\lambda_1(t+h)(\lambda_2(t)-\lambda_1(t))[\frac{\beta_1 S^*{}_h I^*{}_h(t-h)}{(N_1^*(t-h))^2} + \frac{\beta_1 S^*{}_h I^*{}_v(t-h)}{(N_1^*(t-h))^2}] - A_0, \quad (10)$$

$$\frac{d\lambda_2}{dt} = \lambda_1(t)(\frac{w u_1^*(I)(s_h^*(t) + R_h^*(t))}{(N_1^*(t))^2} + \lambda_h)$$

$$+\lambda_2(t)(\frac{u_1^*(t) I_h^*(t)}{(N_1^*)}^2) + \lambda_3(t)(-\gamma_h + \frac{(1-w)u_1^*(S_h^* + R_h^*)}{(N_1^*(t))^2}) +$$

$$\lambda_2(t+h)(\lambda_1(t)-\lambda_2(t))\left[\frac{\beta_1 S^*{}_h I^*{}_h(t-h)}{(N_1^*(t-h))^2} + \frac{\beta_1 S^*{}_h I^*{}_v(t-h)}{(N_1^*(t-h))^2}\right],$$

$$\frac{d\lambda_3}{dt} = \lambda_1(t)(\frac{w u_1^*(t) I_h^*(t)}{(N_1^*(t))} - \lambda_h) + \lambda_2(t)(\frac{u_1^*(t) I_h^*(t)}{(N_1^*(t))}$$

$$+\lambda_3(t)(\frac{(1-w)u_1^*(t) I_h^*(t)}{(N_1^*(t))^2})$$

$$+\lambda_3(t+h)(\lambda_2(t)-\lambda_1(t))\left[\frac{\beta_1 S^*{}_h I^*{}_h(t-h)}{(N_1^*(t-h))^2}\right]$$

$$\frac{d\lambda_4}{dt} = \lambda_4(t)(\gamma_v + \frac{\beta_3 I_h^*(t-h)}{N_2^*(t-h)} + \epsilon_1 u_2^*(t))$$

$$+\lambda_5(t)(-\frac{\beta_3 I_h^*(t-h)}{N_2^*(t-h)})$$

$$+\lambda_4(t+h)(\lambda_4(t)-\lambda_5(t))\left[\frac{\beta_1 S^*{}_v I^*{}_h(t-h)}{(N_2^*(t-h))^2}\right] - A_1,$$

$$\frac{d\lambda_5}{dt} = \lambda_5(t)((\gamma_v + \delta_v) + \epsilon_2 u_2(t))$$

$$+\lambda_5(t+h)(\lambda_5(t)-\lambda_4(t))\left[\frac{\beta_1 S^*{}_v I^*{}_h(t-h)}{(N_2^*(t-h))^2}\right] - A_2$$

With the transversality or boundary conditions

$$\lambda_i(T) = 0, i = 1, 2...5. \qquad (11)$$

And the optimal control variables are given as

$$u_1^*(t) = \max(\min$$

$$(\frac{\frac{-w I_h^*(t)\lambda_1}{N_1^*(t)} + \frac{I_h^*(t)\lambda_2}{N_1^*(t)} - \frac{\lambda_3(1-w)I_h^*(t)}{N_1^*(t)}}{\xi_1}, 1, 0) \quad (12)$$

$$u_2^*(t) = \max(\min$$

$$((\frac{\lambda_5 \epsilon_1 S_v^*(t) + \epsilon_2 I_v^*(t)\lambda_4}{\xi_2}, 1), 0). \quad (13)$$

Proof: To prove the above result, i.e the adjoint equation and the transversallity conditions, we use the Hamiltonian (9). The adjoint system was obtained by by using the adjoint equation (10).

$$-\lambda_1'(t) = H_{S_h^*}(t) + \lambda_1(t+h) H_{S_{h_h}^*}(t),$$

$$-\lambda_2'(t) = H_{I_h^*}(t) + \lambda_2(t+h) H_{I_{h_h}^*}(t),$$

$$-\lambda_3'(t) = H_{R_h^*}(t) + \lambda_3(t+h) H_{R_{h_h}^*}(t),$$

$$-\lambda_4^{'}(t) = H_{S_v^*}(t) + \lambda_4(t+h)H_{S_{vv}^*}(t),$$

$$-\lambda_5^{'}(t) = H_{I_v^*}(t) + \lambda_5(t+h)H_{I_{vv}^*}(t),$$

With $\lambda_i(T) = 0.$ To obtain the required characterization of the optimal control given by (12-13), solving the equations,

$\frac{\partial H}{\partial u_1} = 0, and\ \frac{\partial H}{\partial u_2} = 0$ In the interior of the control set and by the control space U, we derive the equation (10-13). Substituting the corresponding derivatives in the above equations and after the arrangement we get the adjoint equations (10-13). In addition, the second derivative of the Lagrangian with respect to u_1^*, u_2^* is positive, which shows that the optimal problem is minimum at control u_1^*, u_2^*. By substituting the value of u_1^*, u_2^* in the control system (6) we get the followingsystem

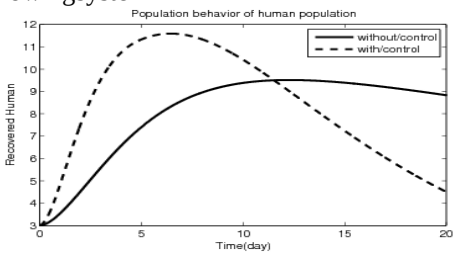

Figure 3. Represents the comparison of recovered human in both the system without control and with control.

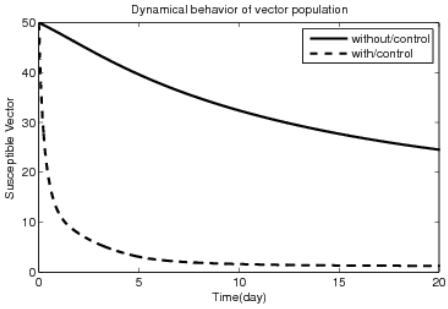

Figure .4. Represents the comparison of susceptible vector in both the system without control and with control.

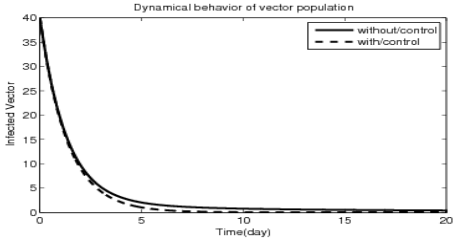

Figure 5. Represents the comparison of infected vector in both the system without control and with control.

$$\frac{dS_h}{dt} = b_1 - \mu_h S_h - \frac{\beta_1 S_h I(t-h)}{N_1(t-h)} - \frac{\beta_2 S_h I_v(t-h)}{N_1(t-h)} +$$

$$\frac{w\max(\min\left(\frac{\frac{-wI_h^*(t)\lambda_1}{N_1^*(t)} + \frac{I_h^*(t)\lambda_2}{N_1^*(t)} - \frac{\lambda_3(1-w)I_h^*(t)}{N_1^*(t)}}{\xi_1}, 1, 0\right)}{N_1(t)}$$

$$I_h(t) + \lambda_h R_h,$$

$$\frac{dI_h}{dt} = \frac{\beta_1 S_h I(t-h)}{N_1(t-h)} + \frac{\beta_2 S_h I_v(t-h)}{N_1(t-h)} - (\mu_h + \delta_h + \gamma_h) I_h(t)$$

$$- \frac{\max(\min\left(\frac{\frac{-wI_h^*(t)\lambda_1}{N_1^*(t)} + \frac{I_h^*(t)\lambda_2}{N_1^*(t)} - \frac{\lambda_3(1-w)I_h^*(t)}{N_1^*(t)}}{\xi_1}, 1, 0\right) I_h(t)}{N_1(t)},$$

$$\frac{dR_h}{dt} = \gamma_h I_h(t) - (\mu_h + \lambda_h) R_h$$

$$+ \frac{(1-w)\max(\min\left(\frac{\frac{-wI_h^*(t)\lambda_1}{N_1^*(t)} + \frac{I_h^*(t)\lambda_2}{N_1^*(t)} - \frac{\lambda_3(1-w)I_h^*(t)}{N_1^*(t)}}{\xi_1}, 1, 0\right) I_h(t)}{N_1(t)},$$

$$\frac{dS_v}{dt} = b_2 - \gamma_v S_v - \frac{\beta_3 S_v I_h(t-h)}{N_2(t-h)}$$

$$-\epsilon_1 \max(\min\left(\left(\frac{\lambda_5 \epsilon_1 S_v^*(t) + \epsilon_2 I_v^*(t)\lambda_4}{\xi_2}, 1\right), 0\right)$$

$$S_v(t), \frac{dI_v}{dt} = \frac{\beta_3 S_v I_h(t-h)}{N_2(t-h)} - (\gamma_v - \delta_v) I_v \quad (14)$$

$$-\epsilon_2 \max(\min\left(\left(\frac{\lambda_5 \epsilon_1 S_v^*(t) + \epsilon_2 I_v^*(t)\lambda_4}{\xi_2}, 1\right), 0\right) I_v(t),$$

With the Hamiltonian H^* at $(I_h^*, S_v^*, I_v^*, u_1^*, u_2^*, \lambda_{1'}, \lambda_{2'}, \lambda_{3'}, \lambda_{4'}, \lambda_{5'})$

$$H^* = A_0 I_h^* + A_1 I_v^* + A_2 S_v^* + \frac{1}{2}$$

$$\left(\xi_1 \left\{\frac{\frac{-wI_h^*(t)\lambda_1}{N_1^*(t)} + \frac{I_h^*(t)\lambda_2}{N_1^*(t)} - \frac{\lambda_3(1-w)I_h^*(t)}{N_1^*(t)}}{\xi_1}\right\}\right)^2$$

$$+\xi_2 \left(\max\left(\min\left(\left(\frac{\lambda_5 \epsilon_1 S_v^*(t) + \epsilon_2 I_v^*(t)\lambda_4}{\xi_2}, 1\right), 0\right)\right)^2\right.$$

$$+\lambda_1\lambda_{1'} + \lambda_2\lambda_{2'} + \lambda_3\lambda_{3'} + \lambda_4\lambda_{4'} + \lambda_5\lambda_{5'}. \quad (15)$$

Numerical Simulation and Summary

In this section, we present the numerical simulations of the proposed model (1) and the delay control model (6) by using Runge-Kutta method. We solve first the model (1) and then solving the proposed model (6).

Then we solve the adjoint equation (10) with the boundary conditions (11) numerically by Runge-kutta order four backward scheme. The constants used in the objective functional with their numerical values we assumed in the numerical simulation is $A_2 = 0.001, A_3 = 0.002, \xi_1 = 0.7, \xi_2 = 0.3$ and $\epsilon_2 = 0.0031$. The values of parameters used in the numerical simulations are presented in **Table 1.**

In this simulation the bold line shows the system with no control and the dashed line shows the system with control throughout Figure 1 to Figure 5. Figure 6 and Figure 7 represents the control variable u_1 and u_2 respectively. The aim of this paper is to control the infection in the host population by using the control variables in the form of treatment or prevention or educational compaign. The control shows in the Figure 1, the population of susceptible human increases and Figure 2 the infection in the host decreases. Figure 3 shows the recovered individuals of human population which increases. Also the population of susceptible vector and infected vector and susceptible vector also decreases in Figure 4 and Figure 5.

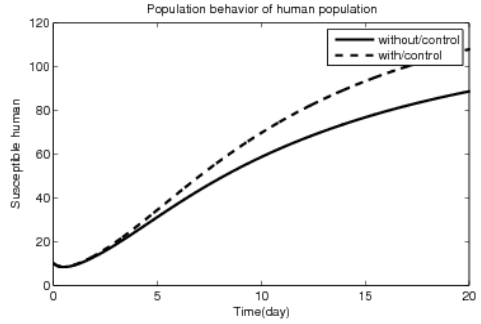

Figure 1. Represents the comparison of susceptible human in both the system without control and with control.

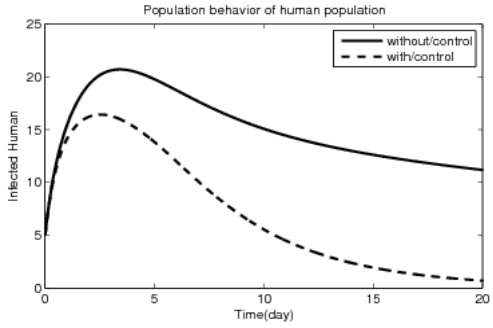

Figure .2. Represents the comparison of infected human in both the system without control and with control.

In this paper, we have presented an epidemic model by applying the optical control and time delay. First we have obtained the formulatation of the model and then we applied the time delay and optimal control with control variables u_1, u_2. Then we have proved the existence of the control system and obtained the numerical solution of the both the system without control and with control. Finally, we conclude our work by references.

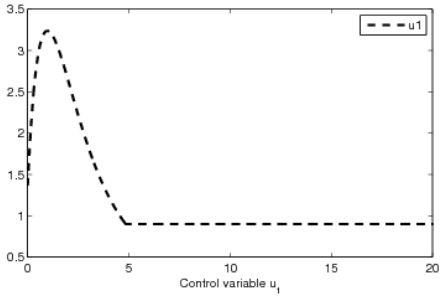

Figure .6. Represents the the contro variable $\mathbf{u_1}$

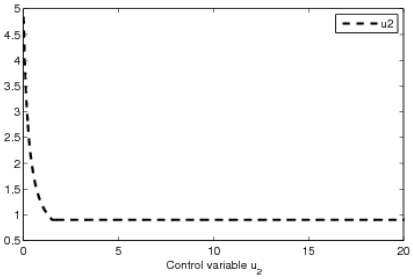

Figure .7. Represents the the control variable $\mathbf{u_2}$.

Table 1. Parameters value used in the numerical simulation of the optimal control problem.

.Notation	Parameters Description	Value
b_1	Recruitment rate for human population	13
β_1	Transmission rate for human population	0.01
β_2	Transmission rate for vector population	0.95 0.09
β_3	Transmission rate for S_v and I_h	
μ_h	Natural mortality rate of human population	0.00001
λ_h	Proportionality constant	0.02
δ_h	Disease death rate for human population	0.051
γ_v	Natural mortality rate of vector population	0.051
δ_v	Disease death rate for vector population	0.051
b_2	Recruitment rate for vector population	3

Corresponding Author: Muhammad Altaf Khan Phd Fellow Department of Mathematics
Abdul Wali Khan University Mardan Pakistan.
E-mail: altafdir@gmail.com

References

1. Anderson, R.M., May, R.M., 1979. Population biology of infectious diseases: Part I. Nature 280 (5721), 361–367.
2. Bachar,M., Dorfmayr, A., 2004. HIV treatment models with time delay C. R. Biol. 327, 983–994.

3. Baker, C.T.H., Paul, C.A.H., Will, D.R., 1995. Issues in the numerical solution of evolutionary delay differential equations, Adv. Comput. Math. 3, 171–196.
4. Birkhoff, G., Rota, G.C., 1989. Ordinary Differential Equations, Fourth Ed. John Wiley and Sons, New York.
5. Brauer, F., Castillo-Chavez, C., 2001. Mathematical Models in Population Biology and Epidemiology Springer, New York, USA.
6. Chwell, G., Hengartner, N.W., Castillo-Chavez, C., Fenimore, P.W., Hyman, J.M., 2004. The basic reproductive number of Ebola and the effects of public health measures: the case of Congo and Uganda J. Theo. Biol. 229, 119–126.
7. Cooke, K., van den Driessche, P., Zou, X., 1999. Interaction of maturation delay and nonlinear birth in population and epidemic models, J. Math. Biol. 39, 332–352.
8. Halanay, A., 1968. Optimal control for systems with time lag, SIAM J. Control 6, 215–234.
9. Hethcote, H.W., van den Driessche, P., 2000. Two SIS epidemiological models with delays, J. Math. Biol. 40, 3–26.
10. Kamien, M.I., Schwartz, N.L., 2000. Dynamics Optimization: The calculus of Variations and Optimal Control in Economics and Management, Elsevier Science, The Netherland.
11. Zaman, G., Kang, Y.H., Jung, I.H., 2008. Stability analysis and optimal vaccination of an SIR epidemic model, Biosystems 93, 240–249.
12. Katri, P., Ruan, S., 2004. Dynamics of human T-cell lymphotropic virus I (HTLV-I) infection of CD4+ T-cells, C. R. Biol. 327, 1009–1016.
13. Kermack,W.O., Mckendrick, A.G., 1927. Contribution to the mathematical theory of epidemics, Proc. R. Soc. Lond. A 115, 700–721.
14. Lekone, P.E., Finkenstädt, B.F., 2006. Statistical inference in a stochastic epidemic SEIR model with control intervention: Ebola as a case study,Biometrics 62, 1170–1177.
15. Lenhart, S., John, T.W., 2007. Optimal Control Applied to Biological Models. Chapman and Hall/CRC, Mathematical and Computational Biology Series, London, UK.
16. Linda, J.S.A., Amy, M.B., 2000. Comparison of deterministic and stochastic SIS and SIR models in discrete time, Math. Biosci. 163, 1–33.
17. Linda, J.S.A., 2007. An Introduction to Mathematical Biology, Pearson Education Ltd., USA, pp. 123–127.
18. Lukes, D.L., 1982. Differential Equations: Classical to Controlled. Mathematics in Science and Engineering}, vol. 162. Academic Press, New York, p. 182.
19. Ma, Z., Liu, J., Li, J., 2003. Stability analysis for differential infectivity epidemic models, Nonlinear Anal. Real World Appl. 4, 841–856.
20. Ma,W., Takeuchi, Y., Hara, T., Beretta, E., 2002. Permanence of an SIR epidemic model with distributed time delays}, Tohoku Math. J. 54, 581-591.
21. May, R.M., 1983. Parasitic infections as regulators of animal populations, Am. Sci. 71, 36–45.
22. G. Zaman, Y. H. Kang, Il Hyo Jung, Optimal treatment of an SIR epidemic model with time delay}, BioSystems 98 (2009) 43–50.
23. Mills, C.E., Robins, J.M., Lipsitch, M., 2004. Transmissibility of 1918 pandemic influenza, Nature 432 (7019), 904–906.
24. Milner, F.A., Pugliese, A., 1999. Periodic solutions: a robust numerical method for an S–I–R model of epidemics, J. Math. Biol. 39, 471–492.
25. Smith, H.L., 1994. A structured population model and related function–differential equation: global attractors and uniform persistence, J. Dyn. Diff. Eqns. 6, 71–99.
26. Smith, H.L., Thieme, H., 1990. Monotone semiflows in scalar non-quasi-monotone functional deferential equations}, J. Math. Anal. Appl. 21, 673–692.
27. So, J.W.-H., Wu, J., Zou, X., 2001. Structured population on two patches: modeling dispersion and delay}, J. Math. Biol. 43, 37–51.
28. G. Zaman, M. A. Khan, S. Islam,et al, mdeling dynamical interaction between leptospirosis infected vector and human population ,Vol 6 2012 no.26, 1287-1302.
29. M. A. Khan, et al, Optimal Campaign in Leptospirosis Epidemic by Multiple Control Variables, *Applied Mathematics*, 2012, 3, 1655-1663.
30. Hamid Sharif Nia, et al, Relationship of some risk factors and symptoms in patients with acute coronary syndrome, Life Science Journal 2012;9(4)
31. M.A. Khan et al., Global Analysis of leptospirosis epidemic model with nonlinear incidences ,Accepted, 2013.
32. G. Zaman,M.A. khan, S. Islam, M.I. Chohan, II Hyo,Jung, Modeling dynamical interaction between leptospirosis infected vector an human pulation, Appl. Math. Sci, Vol. 6, 2012, no. 26, 1287 – 1302.
33. Ullah, R., Zaman, G., & Islam, S. (2013). Stability analysis of a general SIR epidemic model. *VFASTTransactions on Mathematics*, *1*(1).
34. Saddiq, S. F., Khan, M. A., Khan, S. A., Ahmad, F., & Ullah, M. (2013). Analytical solution of an SEIV epidemic model by Homotopy Perturbation method. *VFAST Transactions on Mathematics*, *1*(2).
35. Muhammad Altaf Khan, Saeed Islam, Muhammad Arif, and Zahoor ul Haq, Transmission Model of Hepatitis B Virus with the Migration Effect, BioMed Research International, vol. 2013, Article ID 150681, 10 pages, 2013. doi:10.1155/2013/150681.
36. Muhammad Altaf Khan, Saeed Islam, Sher Afzal Khan, and Gul Zaman, Global Stability of Vector-Host Disease with Variable Population Size, BioMed Research International, vol. 2013, Article ID 710917, 9 pages, 2013. doi:10.1155/2013/710917.
37. M. A. Khan, et al, Application of Homotopy Perturbation Method to Vector Host Epidemic Model with Non-Linear Incidences, Resaerch Journal of Recent Sciences**,** Vol. **2(6),** 90-95.