Space-Time Fractional Derivatives of Spatially Nonhomogeneous Diffusion Convection Equations

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Abstract: Mathematical analysis of fractional differential equations containing terms responsible for diffusion and convection are provided analytically and numerically. We derive the solution of the evolution equation in term of Mittag-Leffler functions using Laplace transfrom. In particular we introduce a numerical scheme, perform some simulations and highlight the effects of the fractional derivatives.

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1. Introduction

In recent years fractional derivatives have attracted much attention in a variety of field in the area of applied sciences. They accuracy in predicting evolution of complex systems have led several scientists to explore their solvability. The theoretical research on fractional diffusion equation models has received considerable success in physical modeling and experimental result analysis in the last decade [4-14]]. Nevertheless, numerical methods for solving fractional diffusion equations are still unripe for practical processes in which the spatial behavior is not homogeneous. The major problem is that fractional derivatives fail to obey local rules due to the fact that they are global in nature as compare to continuity and differentiability in the usual sense that are local operations [1-3]. A relatively complete set of one-dimensional analytical solutions for fractional diffusion-convection equations was recently published by Atangana and Kilicman [1]. Here we shall consider a case with nonlocal diffusion convection and space-time fractional derivatives.
The evolution equation reads

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\frac{\partial^\beta u(x,t)}{\partial x^\beta} + C\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} - E\,u(x,t) + F(x,t), \quad u(x,0) = \phi(x), \tag{1.1}$$

where D , C and E are real coefficients and furthermore we have $0 \leq \alpha, \gamma < 1$ and $1 < \beta \leq 2$. This paper is structured as follows: section 2 gives a brief overview of fractional derivatives, section 3 discusses the solvability of the evolution equation, section a covers numerical simulations and section 5 is devoted to the conclusion. Since the concepts of fractional (or non-integer) order derivatives is not widely known, we will give a brief description of the concept in the next section and highlight some relevant properties.

2 . Fractional order derivatives [1]

The concept of fractional calculus is believed to have stemmed from a question raised in the year 1695 by Marquis de L Hospital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's currently popular notation $\frac{d^n y}{dx^n}$ for derivative of order $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\ldots\}$ when $n = \frac{1}{2}$. In his reply, dated 30 September 1695, Leibniz wrote to L' Hospital as follows: " *This is an apparent paradox from which, one day, useful consequences will be drawn…..".* It has emerged that the concept of fractional order derivatives for a function, say $f(x)$, is based on a generalization of the Abel integral: (2.1)

$$D^{-n}f(x) = \iiint f(x)dx_n = \frac{1}{\Gamma(n)}\int_0^x (x-t)^{n-1} f(t)dt$$

where n is a non-zero positive integer, $\Gamma(n)$ the Gamma function [2]. This represents an integral of order n for the continuous function $f(x)$, whenever $f(x)$ and all its derivatives vanish at the origin, $x = 0$. This result can be extended to the concept of an integral of arbitrary order c, defined as:

$$D^{-c}f(x) = D^{-j-s}f(x) = \frac{1}{\Gamma(c)}\int_0^x (x-t)^{c-1} f(t)dt$$

where c is a positive real number and j an integer such that $0 < j \leq 1$.
Let p now be the least positive integer larger than α such that $\alpha = m - \rho;\ 0 < \rho \leq 1$. Equation (2.1) can then be used to define the derivative of

(positive) fractional order, say α, of a function $f(x)$ as: (2.2)

$$D^c f(x) = D^{p-\rho} f(x) = \frac{1}{\Gamma(\rho)} \int_0^x (x-t)^{\rho-1} \frac{d^p f(t)}{dt^p} dt$$

Note that these results, like Abel's integral, are only valid subject to the condition that $f^{(k)}(x)|x = 0$ for $k = 0,1,2,\dots,p$.

Properties of the differential operator can be found in [3] [4], we mention only the following:

For $f \epsilon C_\mu$, $\mu \geq -1$, α, $\beta \geq 0$ and $\gamma > -1$:

$$D^{-\alpha} D^{-\beta} f(x) = D^{-\alpha-\beta} f(x),$$

$$D^{-\alpha} D^{-\beta} f(x) = D^{-\beta} D^{-\alpha} f(x) \quad ;$$

$$D^{-\alpha} x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

3. Formulation of space-time fractional derivative of heat convection equation

In order to include the noise due to the metal diffusivity, the heat convection $\partial_x u(x,t)$ is replaced by the Riemann-Liouville fractional derivatives of order γ, $\partial_x^\gamma u(x,t)$ and in the same way, $\partial_x^2 u(x,t)$ is replaced by $\partial_x^\beta u(x,t)$ and $\partial_t u(x,t)$ is replaced by $\partial_x^\alpha u(x,t)$.

This leads to the problem

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \frac{\partial^\beta u(x,t)}{\partial x^\beta} + C \frac{\partial^\gamma u(x,t)}{\partial x^\gamma} - E\, u(x,t) + F(x,t),$$

subject to the initial condition

$$u(x,0) = \phi(x),$$

where ϕ is a nonnegative continuous function.

Theoretically the above initial value problem is mathematically well-posed (existence of a unique solution that is stable), however the fractional integro-differentiability nature of the problem makes it difficult to obtain an explicit solution. In this work we will first transform the homogeneous problem into a fractional heat transfer problem by using a well known change of variables

$u(x,t) = e^{ax+bt} v(x,t)$,

where a and b are constants depending on D, C and E. This enables us to reduce the corresponding homogeneous problem to the fractional problem

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = D \frac{\partial^\beta v(x,t)}{\partial x^\beta},$$

subject to the initial condition

$v(x,0) = e^{-ax}\phi(x)$. The idea is to get the solution of the homogeneous fractional heat problem without its initial condition, then use the method of undetermined coefficients or the method of variation of parameters to get the solution to the full non-homogeneous problem, then to apply the initial condition to get the unique exact solution of the whole system. In what follows, we proceed as in [1] to deal with the fractional heat problem in order to solve for $v(x,t)$. We will then explore both fractional derivative in the Riemann-Liouville sense and fractional derivatives in the Caputo sense.

4. Analytical solution

4.1 Fractional derivative in Riemann-Liouville sense

Applying Laplace transform to $\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = D \frac{\partial^\beta v(x,t)}{\partial x^\beta}$ subject to its initial condition $v(x,0) = e^{-ax}\phi(x)$, we get

$$D \frac{\partial^\beta u(x,s)}{\partial x^\beta} + su(x,s) = e^{-ax} u(x,0),$$

where we have assumed $\alpha = 1$. Setting $y(x) = u(x,s)$, this equation becomes

$$D \frac{\partial^\beta y(x)}{\partial x^\beta} + sy(x) = e^{-ax}\phi(x).$$

Applying this time around Laplace transform to the spatial variable, the equation takes the form

$$Y(p) = \mathcal{L}(y)(p) = \sum_{i=1}^{l} h_i \frac{p^{i-1}}{sp^\beta - p^{2\beta} + D}$$

where p is the Laplace variable for the space component and $h_i = \partial_x^{\alpha-i} e^{-ax}\phi(0^+)$.

For $p \in \mathbb{C}$ and $\left|\frac{sp^\beta - p^{2\beta}}{D}\right| < 1$, we have the following expression $\frac{1}{sp^\beta - p^{2\beta} + D}$ can be writing in form of series as follows:

$$\frac{p^{i-1}}{sp^{\beta}-p^{2\beta}+D}=p^{i-1}\sum_{n=0}^{\infty}\frac{(sp^{\beta}-p^{2\beta})^{n}}{D^{n}},$$

and hence substituting the following representation yields

$$Y(p)=\mathcal{L}(y)(p)=\sum_{i=1}^{l}h_i\,p^{i-1}\sum_{n=0}^{\infty}\frac{(sp^{\beta}-p^{2\beta})^{n}}{D^{n}}.$$

The above expression can then simplified as

$$y(x)=\sum_{i=1}^{2}h_i y_i(x)$$

where

$$y_i(x)=\sum_{n=0}^{\infty}\frac{D^n}{n!}x^{n-i}\left(\frac{\partial}{\partial x}\right)^{n}E_{2\beta,\ \beta n+1-i}(x^{-\beta})$$

and

$$\left(\frac{\partial}{\partial x}\right)^{n}E_{\beta}(x)=\sum_{j=0}^{\infty}\frac{\Gamma(n+j+1)}{\Gamma(n+\beta+j)}\frac{x^{j}}{j!}.$$

Thus the series solution of equation can be now given by applying the inverse Laplace operator on $u(x,s)$ to have

$$u_i(x,t)=\mathcal{L}^{-1}\left(\sum_{n=0}^{\infty}\frac{s^{-n}}{n!}\frac{D^n}{n!}x^{n-i}\left(\frac{\partial}{\partial x}\right)^{n}E_{2\beta,\ \beta n+1-i}(x^{-\beta})\right).$$

Since the inverse Laplace operator is a linear operator it follows that

$$u_i(x,t)=\sum_{n=i}^{\infty}\frac{\mathcal{L}^{-1}\{s^{-n}\}}{n(n-1)!}x^{n-i}\left(\frac{\partial}{\partial x}\right)^{n}E_{2\beta,\ \beta n+1-i}(x^{-\beta}).$$

Since

$$\mathcal{L}^{-1}\{s^{-n}\}=\frac{t^{n-1}}{(n-1)!},$$

we have that

$$u_1(x,t)=\sum_{n=1}^{\infty}\frac{t^{n-1}}{n(n-1)!}x^{n-1}\left(\frac{\partial}{\partial x}\right)^{n}E_{2\beta,\ \beta n}(x^{-\beta})$$

and

$$u_2(x,t)=\sum_{n=2}^{\infty}\frac{t^{n-1}}{n(n-1)!}x^{n-2}\left(\frac{\partial}{\partial x}\right)^{n}E_{2\beta,\ \beta n-1}(x^{-\beta}).$$

The coefficients $h_i,\ \ i=1,2$, of

$$y(x)=\sum_{i=1}^{2}h_i y_i(x)$$

are obtained by applying the initial conditions prescribed to the problem.

Example 1

We consider the scenario whereby $\alpha=1$ and $\beta=\frac{1}{2}$, and $\phi(x)=x.$ The coefficient $h_1=\csc^{-1}(x+3x^2)$ and $h_2=\tan^{-1}x^3.$ We obtain

$$u_1(x,t)=\sum_{n=1}^{\infty}\frac{t^{n-1}}{n(n-1)!}x^{n-1}\left(\frac{\partial}{\partial x}\right)^{n}E_{1,\ \beta n}\left(x^{-\frac{1}{2}}\right)$$
$$=\sum_{n=1}^{\infty}\frac{t^{n-1}}{n(n-1)!}x^{n-1}\left(\frac{\partial}{\partial x}\right)^{n}M\left(x,\frac{x+1}{x},\sin x\right),$$

and

$$u_2(x,t)=\sum_{n=2}^{\infty}\frac{t^{n-1}}{n(n-1)!}x^{n-2}\left(\frac{\partial}{\partial x}\right)^{n}E_{1,\ \beta n-1}\left(x^{-\frac{1}{2}}\right)=\sum_{n=2}^{\infty}\frac{t^{n-1}}{n(n-1)!}x^{n-2}\left(\frac{\partial}{\partial x}\right)^{n}$$
$$=exp\frac{x(1-x)}{2\sin x}erfc\frac{x-2t}{t}.$$

4.2 Fractional derivative in Caputo sense

The Riemann-Liouville derivative has smooth properties that can be used to solve regular problems, but it shows certain disadvantages when trying to model singularly perturbed phenomena with fractional differential equations [6]. In what follows we will use Caputo fractional derivative ti investigate the same problem and compare the properties of the solutions obtained.

The Laplace transform for the Caputo derivative is based on the formula

$$(\mathcal{L}\ cD^{\alpha}y)(s)=s^{\alpha}(\mathcal{L}y)(s)-\sum_{i=0}^{1}h_i\,s^{\alpha-i-1}$$

with

$h_i=y^{i}(0)\ \ (i=0,1).$

Thus applying the Laplace transform in both side of our main equation on the component of time, and applying again the Laplace transform on the component of space yields :

$$\mathcal{L}(y)(p) = \sum_{i=0}^{2-1} h_i \frac{p^{-i-1}}{D - p^{2\beta}} - D \sum_{i=0}^{1-1} h_i \frac{p^{\beta-i-1}}{D - p^{2\beta}}$$

For $p \in \mathbb{C}$,

$$\left|\frac{p^{2\beta}}{D}\right| < 1.$$

Following the same trend as the discussion presented earlier for the case of Riemann-Liouville, we have the following

$$\mathcal{L}(y)(p) = \sum_{i=0}^{2-1} h_i \sum_{n=0}^{\infty} \tau^n \frac{p^{(-\beta)-(\beta n+i+1)}}{(D - p^{2\beta})^{n+1}} - \mu \sum_{i=0}^{1-1} \tau^n \frac{p^{(-\beta)-(\beta n+i+1-\beta)}}{(D - p^{2\beta})^{n+1}}$$

Hence for $p \in \mathbb{C}$, we have

$$\frac{p^{(-2\beta)-(\beta n+j+1)}}{(D - p^{2\beta})^{n+1}} = \frac{1}{n!}\left(\mathcal{L}\left[x^{n\alpha+i}\left(\frac{\partial}{\partial x}\right)^n E_{\alpha-\beta,\beta n+i+1}\left(\mu x^{-\beta}\right)\right]\right)$$

and

$$\frac{p^{(\alpha-\beta)-(\beta n+j+1+\alpha-\beta)}}{(D - p^{2\beta})^{n+1}} = \frac{1}{n!}\mathcal{L}\left[x^{n\alpha+i-\beta}\left(\frac{\partial}{\partial x}\right)^n E_{-\beta,\beta n+i+1-\beta}\left(\mu x^{-\beta}\right)\right].$$

Thus from the above expression we derive the following solution to the space-time Caputo fractional derivative of our heat convection equation. It follows that

$$u(x,t) = \sum_{i=1}^{2} h_i c_i(x,t)$$

where

$$u_1(x,t) = \sum_{n=0}^{\infty}\left(\frac{R}{D}\right)^n \frac{\exp[-\lambda t]\, t^{-1-n}}{\Gamma(-n)n!} x^{n\alpha+i}\left(\frac{\partial}{\partial x}\right)^n E_{-\beta,\beta n+i+1}\left(\mu x^{-2\beta}\right) - D\sum_{n=0}^{\infty}\left(\frac{R}{D}\right)^n \frac{\exp[-\lambda t]\, t^{-1-n}}{\Gamma(-n)n!} x^{n+i-\beta}\left(\frac{\partial}{\partial x}\right)^n E_{-\beta,\beta n+i+1-2\beta}\left(\mu x^{-\beta}\right)$$

and

$$u_2(x,t) = \sum_{n=0}^{\infty}\left(\frac{R}{D}\right)^n \frac{\exp[-\lambda t]\, t^{-1-n}}{\Gamma(-n)n!} x^{n+i}\left(\frac{\partial}{\partial x}\right)^n E_{-\beta,\beta n+i+1}\left(\mu x^{\alpha-2\beta}\right).$$

As in the previous subsection, the coefficients h_i are found by applying the initial and boundary conditions on $u(x,t)$.

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Example 2

We consider the scenario whereby $\alpha = 1$ and $\beta = \frac{1}{3}$, and $\phi(x) = 1 + x$. The coefficient $h_1 = \sec^{-1} x\,(x+1)$ and $h_2 = (\log_2 x)\, x^3$. This time around instead of using Riemann-Liouville derivative, we make use of the Caputo derivative.

The analytical solution of space-time fractional derivative heat convection equation obtained is given below:

$$u_1(x,t) = \sum_{n=0}^{\infty}\left(\frac{R}{D}\right)^n \frac{\exp[-\lambda t]\, t^{-1-n}}{\Gamma(-n)(n-1)n!} x^{2n}\left(\frac{\partial}{\partial x}\right)^n E_{2-\beta,\beta n+1}\left(\mu x^{2-\beta}\right) - \mu\sum_{n=0}^{\infty}\left(\frac{R}{D}\right)^n \frac{\exp[-\lambda t]\, t^{-1-n}}{\Gamma(-n)(n-1)n!} x^{2n+2-\beta}\left(\frac{\partial}{\partial x}\right)^n E_{3-\beta,\beta n+3-\beta}\left(\mu x^{2-2\beta}\right)$$

and

$$u(x,t) = \sum_{n=0}^{\infty}\left(\frac{R}{D}\right)^n \frac{\exp[-\lambda t]\, t^{-1-n}}{\Gamma(-n)(n-1)n!} x^{2n+1}\left(\frac{\partial}{\partial x}\right)^n E_{4-\beta,\beta n+2}\left(\mu x^{2-2\beta}\right).$$

Note that these solutions are linearly independent and constitute the fundamental system of solution of the evolution equation describing heat convection with lateral loss. Next we present some graphical representation of the solutions obtained the Riemann-Liouville derivative and the Caputo derivative. These solutions are linearly independent and they provide the fundamental system of solutions to space-time Caputo fractional derivative of hydrodynamic advection-dispersion equation. An approximation of this series is given below for possible simulation

5. Numerical simulation

We use the software Mathematica to compute our values and plot the graphs. In entering the codes, the integral parts had to be broken into summations and the derivatives computed as difference equations. The speed of the process was proportional to the order of the fractional derivatives.

Example 1

We consider the scenario whereby $\alpha = 1$ and $\beta = \frac{1}{2}$, and $\phi(x) = x$. The coefficient $h_1 = \csc^{-1}(x + 3x^2)$ and $h_2 = \tan^{-1} x^3$.

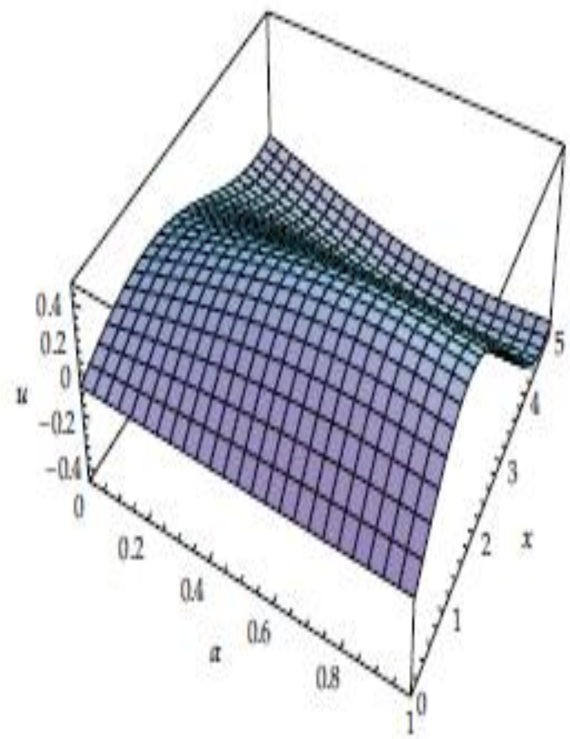

Figure. 1

Example 2

We consider the scenario whereby $\alpha = 1$ and $\beta = \frac{1}{3}$, and $\phi(x) = 1 + x$. The coefficient $h_1 = \sec^{-1} x\,(x+1)$ and $h_2 = (\log_2 x)\,x^3$. ,

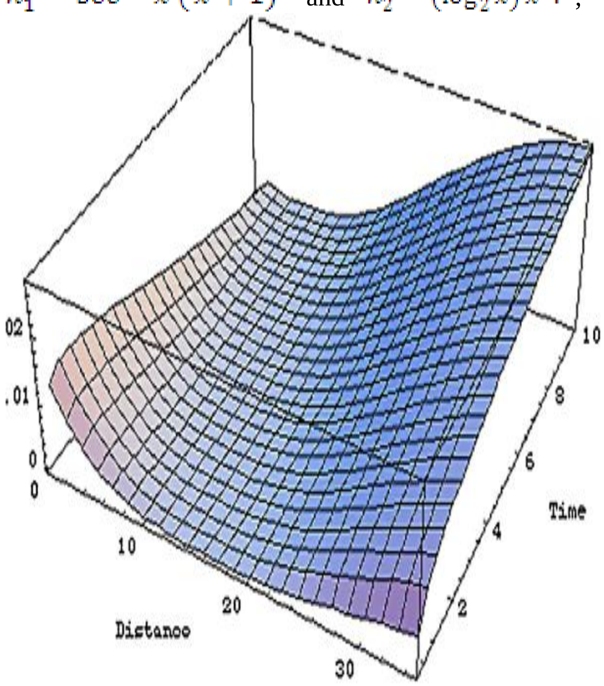

Figure. 2

6. Conclusion

In this paper, the fractional heat convection equation with lateral heat loss was analysed via the Riemann-Liouville derivative and the Caputo derivative. Laplace transform was considered in various instances to enable the solvability of our equations. We showed that the two solutions that satisfied the fractional equation both for space-time Caputo and Riemann-Liouville fractional derivative generated fundamentals solutions of the evolution problem. We highlighted the effects of the order of the fractional derivative by emphasizing on the fact that our solution did not only depend on the space and time variable, but also on the fractional derivatives. The figure 1 and 2 show that the order of the derivative can be used to simulate chaotic world problem and they suggest that these methods could be used to describe with accuracy complex problems in mathematics and engineering.

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