A Class of Deformed Hyperbolic Secant Distributions Using Two Parametric Functions

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Abstract: This paper presents a novel class of deformed hyperbolic secant distributions. We apply the deformation technique by introducing two parametric functions under some certain appropriate assumptions. We discuss some important properties of this defined class of distributions. Some measures and functions of this new class of distributions are derived. A simple illustrative example is given.
[S. A. El-Shehawy. **A Class of Deformed Hyperbolic Secant Distributions Using Two Parametric Functions]** *Life Sci J* 2013; 10 (2):897-903] (ISSN:1097-8135). http://www.lifesciencesite.com. 126

Key words: Generalized hyperbolic secant distribution, Hyperbolic secant distribution, *q(t)*-Hyperbolic function, *p*-DHS distribution, *p*-DHS distribution, *pq*-DHS distribution.

Mathematics Subject Classification: 60E05, 60E10, 62E17, 62E20

1. Introduction

The concept of deformation technique has been exploited to a great extent in several fields of sciences [1-4, 6, 12]. The deformation technique is applied for the hyperbolic and trigonometric functions. Recently, this technique has been used especially for the continuous hyperbolic secant distribution "HS-Distribution" which is symmetric about zero with unit variance [8, 9]. This distribution has probability density function "pdf" in the form

$$f_{\mathrm{HS}}(x)=\frac{1}{2}\operatorname{sech}(\pi x/2); \qquad x\in\mathrm{R} \tag{1}$$

and it has some closed forms for some corresponding functions (the moments-generating function "mgf", characteristic function "cf", cumulants-generating function "cgf" and score function "sf") and measures [5, 10, 13]. The family of the continuous *pq*-deformed hyperbolic secant distributions "{*pq*-DHS distribution}" has been constructed and studied [7]. Each *pq*-DHS distribution has been obtained by introducing two positive scalar deformation parameters *p* and *q* respectively as two factors of the exponential growth and decay parts of the hyperbolic secant function "HS function" in the HS distribution. The *pq*-DHS distribution is unimodal with unit variance. Its pdf is given by

$$f_{pq-\mathrm{DHS}}(x;p,q)=\frac{\sqrt{pq}}{2}\operatorname{sech}_{pq}(\pi x/2); \qquad x\in\mathrm{R}. \tag{2}$$

The corresponding mgf, cf, cgf and sf of this distribution have closed forms which depend on the introduced scalar parameters p and q. All moments of this distribution exist and the mean, the median and the mode have equal non-zero values as a function of the introduced real valued positive parameters p and q [7].

The main of this paper is to define and study a class of $p(w)q(w)$-deformed hyperbolic secant distributions, which is denoted by "$p(w)q(w)$-DHS distributions", by introducing two real valued positive parametric functions $p(w)$ and $q(w)$ (the deformation parametric functions). We consider a linear function of the mentioned random variable with coefficients as functions of a scalar parameter *w*. In this study, we will consider some appropriate assumptions with respect to the introduced para-metric functions as well as the used coefficients in the mentioned linear function of the random variable.

2. The *p(w)q(w)*-deformed hyperbolic secant distribution

Firstly, we consider the deformation technique for which two real valued positive parametric functions $p(w)$ and $q(w)$ are introduced respectively as two factors of the exponential growth and decay parts of the HS function in the HS distribution. The $p(w)q(w)$-DHS distribution is defined by means of the $p(w)q(w)$-deformation for the hyperbolic functions. Now, we define the deformed hyperbolic functions by introducing two arbitrary deformation parametric functions $p(w)$ and $q(w)$ and we explain their properties.

Definition 1. *Let* $p(w)$ and $q(w)$ *are two arbitrary real positive deformation parametric differentiable functions of* w, $w\in R$. *We define the* $p(w)q(w)$*-deformed hyperbolic functions to be a family of functions* $\sinh_{p(w)q(w)}\varphi$,

$\cosh_{p(w)q(w)}\varphi$, $\tanh_{p(w)q(w)}\varphi$, $\operatorname{sech}_{p(w)q(w)}\varphi$, $\coth_{p(w)q(w)}\varphi$ *and* $\operatorname{csch}_{p(w)q(w)}\varphi$ *as*

$$\sinh_{p(w)q(w)}\varphi=\frac{p(w)e^{\varphi}-q(w)e^{-\varphi}}{2},\quad \cosh_{p(w)q(w)}\varphi=\frac{p(w)e^{\varphi}+q(w)e^{-\varphi}}{2},$$
$$\tanh_{p(w)q(w)}\varphi=\frac{\sinh_{p(w)q(w)}\varphi}{\cosh_{p(w)q(w)}\varphi},\quad \coth_{q(w)}\varphi=\frac{\cosh_{p(w)q(w)}\varphi}{\sinh_{p(w)q(w)}\varphi},$$
$$\operatorname{sech}_{p(w)q(w)}\varphi=\frac{1}{\cosh_{p(w)q(w)}\varphi},\quad \operatorname{csch}_{p(w)q(w)}\varphi=\frac{1}{\sinh_{p(w)q(w)}\varphi};\tag{3}$$

where $\varphi=\varphi(x,w)$ *is a real differentiable function of* x *and* w, *and it is a linear function in* x *with positive partial derivative with respect to* x, *i.e.* $\varphi=C(w)x+D(w)$, $C(w)\in(0,\infty)$ *as a derivative of* φ *with respect to* x, *and* $D(w)\in R$. □

Lemma 1. *A family of the* $p(w)q(w)$*-deformed hyperbolic functions satisfies the following relations of the first derivatives of* $\sinh_{p(w)q(w)}\varphi$, $\tanh_{p(w)q(w)}\varphi$, $\cosh_{p(w)q(w)}\varphi$, $\operatorname{sech}_{p(w)q(w)}\varphi$ *with respect to* x:

$$(\sinh_{p(w)q(w)}\varphi)'=C(w)\cosh_{p(w)q(w)}\varphi,$$
$$(\tanh_{p(w)q(w)}\varphi)'=C(w)p(w)q(w)\operatorname{sech}^2_{p(w)q(w)}\varphi,\tag{4}$$
$$(\cosh_{p(w)q(w)}\varphi)'=C(w)\sinh_{p(w)q(w)}\varphi,$$
$$(\operatorname{sech}_{p(w)q(w)}\varphi)'=-C(w)\operatorname{sech}_{p(w)q(w)}\varphi\tanh_{p(w)q(w)}\varphi.$$

Furthermore, if $p(w)\neq 1$ *or* $q(w)\neq 1$ *then* $\sinh_{p(w)q(w)}\varphi$ *is not odd function with respect to* φ *and* $\cosh_{p(w)q(w)}\varphi$ *is not even function with respect to* φ, *i.e.*

$$\sinh_{p(w)q(w)}(-\varphi)=-p(w)q(w)\sinh_{\frac{1}{p(w)}\frac{1}{q(w)}}\varphi,$$
$$\cosh_{p(w)q(w)}(-\varphi)=p(w)q(w)\cosh_{\frac{1}{p(w)}\frac{1}{q(w)}}\varphi.$$

Moreover, the following relations are satisfied:

$$\cosh^2_{p(w)q(w)}\varphi-\sinh^2_{p(w)q(w)}\varphi=p(w)q(w),$$
$$\tanh^2_{p(w)q(w)}\varphi=1-p(w)q(w)\operatorname{sech}^2_{p(w)q(w)}\varphi,$$
$$\coth^2_{p(w)q(w)}\varphi=p(w)q(w)\operatorname{csch}^2_{p(w)q(w)}\varphi+1.$$

Proof: Based on [1, 7, 11] and Definition 1, we can directly prove this lemma. □

The main idea of the suggested deformation technique is to generalize the HS-distribution in an alternative formula which depends on two real positive parametric functions and also to study its important corresponding characteristics. Here, we extend the random variable X by $\varphi=\varphi(X,w)$, where $w\in R$.

As an immediate consequence of previous definition and lemma, we can define the pdf of the constructed $p(w)q(w)$-DHS distribution as the following.

Definition 2. *Let* $X_{p(w)q(w)-\mathrm{DHS}}$ *be a continuous random variable. This variable has a* $p(w)q(w)$*-DHS distribution with two positive real deformation parametric functions* $p(w)$ *and* $q(w)$, *if its pdf given by*

$$f_{p(w)q(w)-\mathrm{DHS}}(\varphi;p(w),q(w))=\frac{C(w)\sqrt{p(w)q(w)}}{2}\times\operatorname{sech}_{p(w)q(w)}\left(\frac{\pi\varphi}{2}\right);\ x,w\in\mathrm{R},\tag{5}$$

where $p(w),q(w)\in(0,\infty)$ *and* $\varphi=\varphi(x,w)\in R$. *In this case,* $X_{p(w)q(w)-\mathrm{DHS}}$ *is said to be a* $p(w)q(w)$*-DHS random variable with two parametric functions* $p(w)$ *and* $q(w)$, *defined over* R. *Furthermore, the corresponding real valued cdf named* $F_{p(w)q(w)-\mathrm{DHS}}(\varphi;p(w),q(w))$ *is defined as*

$$F_{p(w)q(w)-\mathrm{DHS}}(\varphi;p(w),q(w))=\frac{1}{2}+\frac{1}{\pi}\arctan\left[\frac{1}{\sqrt{p(w)q(w)}}\sinh_{p(w)q(w)}\left(\frac{\pi\varphi}{2}\right)\right],\tag{6}$$

with the inverse cdf (critical value)

$$x_{\alpha}^{p(w)q(w)-\mathrm{DHS}}=\frac{2}{\pi.C(w)}\left[\operatorname{arcsinh}\left[\tan\left(\pi\left(\frac{1}{2}-\alpha\right)\right)\right]-\frac{1}{2}\ln\frac{p(w)}{q(w)}\right]-\frac{D(w)}{C(w)},\tag{7}$$

where

$$P[X>x_{\alpha}^{p(w)q(w)-\mathrm{DHS}}]=1-F_{p(w)q(w)-\mathrm{DHS}}(x_{\alpha}^{p(w)q(w)-\mathrm{DHS}})=\alpha,\ \alpha\in(0,1).$$ □

Without loss of generality, let $D(w)=0$ and in this case the values $x_{\alpha}^{p(w)q(w)-\mathrm{DHS}}$ for some different values of w and for each fixed pair of the parametric functions $p(w)$ and $q(w)$ using (7) can be computed.

Now, we will next present some important properties of this constructed $p(w)q(w)$-DHS distribution. Based on [7-9] and the graphical explanation and under some appropriate assumptions the exponential tail behavior of the $p(w)q(w)$-DHS distribution guarantees the existence of the expectation of $X_{p(w)q(w)-\text{DHS}}$ and generally all moments. In particular, the expectation of the variables $X_{p(w)q(w)-\text{DHS}}$ and also $X^2_{p(w)q(w)-\text{DHS}}$ can be derived and given respectively by

$$\mu = E[X_{p(w)q(w)-\text{DHS}}] = \frac{2}{\pi\, C(w)} \ln[\sqrt{q(w)/p(w)}], \tag{8}$$

$$E[X^2_{p(w)q(w)-\text{DHS}}] = \frac{1}{C^2(w)} + \frac{4}{\pi^2\, C^2(w)} (\ln[\sqrt{q(w)/p(w)}])^2.$$

Moreover, the variance of $X_{p(w)q(w)-\text{DHS}}$ is $\sigma^2 = 1/C^2(w)$.

Proposition 1. *The $p(w)q(w)$-DHS distribution with two positive real deformation parametric functions $p(w)$ and $q(w)$ is symmetric about 0 for $p(w)=q(w)$. Moreover, it skewed more to the right for $p(w)>q(w)$ and skewed more to the left for $p(w)<q(w)$. For all positive real values of $p(w)$ and $q(w)$, the kurtosis is always constant.* □

Based on [7-9], different pdf's for the $p(w)q(w)$-DHS distribution with $p(w)<q(w)$ (or $p(w)>q(w)$) for each fixed pair $(p(w),q(w))$ or $p(w)=q(w)$ for some real values of w can be plotted and illustrated in which it obvious graphically that the Proposition 1 is valid. Computationally we can also find the following results:

- for fixed value of the parametric function $p(w)$, it is clear that the mean of $p(w)q(w)$-DHS distribution is inversely proportional with the value of the parametric function $q(w)$.
- for fixed value of the parametric function $q(w)$, it is clear that the mean of $p(w)q(w)$-DHS distribution is inversely proportional with the value of the parametric function $p(w)$.

According to the form $S(x) = -(pdf)'/(pdf)$ of the score function "sf", we can derive this function for the $p(w)q(w)$-DHS distribution.

Proposition 2. *The score function of the variable $X_{p(w)q(w)-\text{DHS}}$ with $p(w), q(w) \in (0,\infty)$ is given by*

$$S_{p(w)q(w)-\text{DHS}}(\varphi; p(w), q(w)) = \frac{\pi}{2} C(w) \times \tanh_{p(w)q(w)} \frac{\pi\varphi}{2}. \tag{9}$$

Setting $p(w)=q(w)=1$ and $C(w)=1$, *the last equation reduces to* $S_{\text{HS}}(x) = \frac{\pi}{2}\tanh\frac{\pi x}{2}$, *where $S_{\text{HS}}(x)$ is the sf of HS distribution. Moreover, when $p(w)=p$ and $q(w)=q$ (i.e. parameters)* and $C(w)=1$, *equation (9) reduces to* $S_{pq-\text{DHS}}(x) = \frac{\pi}{2}\tanh_{pq}\frac{\pi x}{2}$ *which is the sf of the variable $X_{pq-\text{DHS}}$ with $p,q \in (0,\infty)$.*

Proof: By using (5), the form (9) can be obtained with the reduced cases $S_{\text{HS}}(x)$ and $S_{pq-\text{DHS}}(x)$ for $p(w)=q(w)=1$, $C(w)=1$ and $p(w)=p$, $q(w)=q$, $C(w)=1$ *respectively.* □

Proposition 3. *The $p(w)q(w)$-DHS distribution is unimodal for $p(w), q(w) \in (0,\infty)$.*

Proof: Based on the pdf of $X_{p(w)q(w)-\text{DHS}}$ in (5), we aim to show that this function is unimodal for all pair of $p(w)$ and $q(w)$. Since this pdf is a continuously differentiable function, the only critical points for this function satisfy $f'_{p(w)q(w)-\text{DHS}}(\varphi; p(w), q(w)) = 0$ (the derivative with respect to x). Now, we want to prove that the last equation has exactly one root, and that this yields a relative maximum. Since

$$\lim_{\varphi\to\pm\infty} f_{p(w)q(w)-\text{DHS}}(\varphi; p(w), q(w)) = 0,$$

then if there is one critical point, it must yield the absolute maximum, so we need to prove there is exactly one root to the derivative equation. After simplification, this can be seen to be equivalent to proving

$$(\text{sech}_{p(w)q(w)}\frac{\pi\varphi}{2}).(\tanh_{p(w)q(w)}\frac{\pi\varphi}{2}) = 0$$

has exactly one root.

Set

$\varphi(x;p(w),q(w))=\frac{2}{\pi}(y+\ln[\sqrt{\frac{q(w)}{p(w)}}])$, the last statement is equivalent to showing that the equation $\operatorname{sech}(y)\tanh(y)=0$ has exactly one root $y=0$ in R. This means that the equation $f'_{p(w)q(w)-\text{DHS}}(\varphi;p(w),q(w))=0$ has only the root $\varphi^*=\varphi(x^*,w)=\frac{2}{\pi}\ln[\sqrt{\frac{q(w)}{p(w)}}]$ (i.e. $x^*=\frac{2}{\pi\, C(w)}\ln[\sqrt{\frac{q(w)}{p(w)}}]$) in R. Since the 2nd derivative $f''_{p(w)q(w)-\text{DHS}}(\varphi^*;p(w),q(w))$ with respect to x is less than 0 with $\varphi^*=\varphi(x^*,w)$, then the point x^* is the maximum value of the $p(w)q(w)$-DHS distribution.

Based on [7-9], this yields a relative maximum (and hence absolute maximum) since the 1st derivative $f'_{p(w)q(w)-\text{DHS}}(\varphi;p(w),q(w))$ is positive to the left of the root x^*, and negative to the right.

□

Note that, the mode for the $p(w)q(w)$-DHS distribution has the above value of the root x^*, which equals the obtained mean.

Proposition 4. *The mode* "$\text{Mode}_{p(w)q(w)\text{-DHS}}$" *and the median* "$\text{Median}_{p(w)q(w)\text{-DHS}}$" *for the* $p(w)q(w)$-*DHS distribution with* $p(w),q(w)\in(0,\infty)$ *have the same value of the mean* μ in (8).

Proof: Due to the unimodality of the distribution, the previous obtained results and the fact that the median of the unimodal distribution lies between the mean and the mode of the same distribution, the given statement in the proposition is valid. □

Note that, in the case when $p(w)=p$ and $q(w)=q$ (where $p,q>0$), $C(w)=1$, the *pq*-DHS distribution is recovered and also the case of $p(w)=q(w)=1$, $C(w)=1$, gives the original HS distribution.

Now, we will derive some closed forms for the corresponding mgf, cgf and cf of the $p(w)q(w)$-DHS distribution. Moreover, we will deduce the corresponding moments, skewness and kurtosis coefficients of this constructed distribution.

Proposition 5. *The* mgf *of the variable* $X_{p(w)q(w)-\text{DHS}}$ *with* $p(w),q(w)\in(0,\infty)$ *is given by*

$$\text{M}_{p(w)q(w)-\text{DHS}}(t;p(w),q(w))=\sec t \times e^{\frac{2t}{\pi\, C(w)}\ln[\sqrt{q(w)/p(w)}]}, \qquad (10)$$

where $|t|<\frac{\pi\, C(w)}{2}$.

In particular, all moments of $X_{p(w)q(w)-\text{DHS}}$ *exist.*

Proof: By using the substitutions

$$\varphi(x;p(w),q(w))=\frac{2}{\pi}[y+\ln(\sqrt{q(w)/p(w)})]$$

and $B=\frac{2t}{\pi\, C(w)}$, we find that

$$\int_{-\infty}^{\infty} e^{By}\operatorname{sech} y\, dy=\pi\sec t,\quad \text{where } |B|<1.$$

Then the mgf in (10) of $X_{p(w)q(w)-\text{DHS}}$ can be directly obtained from the following:

$$\text{M}_{p(w)q(w)-\text{DHS}}(t;p(w),q(w))=\frac{1}{\pi}e^{B\ln[\sqrt{q(w)/p(w)}]}\times\int_{-\infty}^{\infty} e^{By}\operatorname{sech} y\, dy,\ |B|<1.$$

□

Proposition 6. *The first four non-central moments of* $X_{p(w)q(w)-\text{DHS}}$ *with* $p(w),q(w)\in(0,\infty)$ *are given by*

$$\mu_1'=\frac{1}{\pi\, C(w)}\ln[q(w)/p(w)],$$

$$\mu_2'=\frac{1}{C^2(w)}+\frac{1}{\pi^2C^2(w)}(\ln[q(w)/p(w)])^2,$$

$$\mu_3'=\frac{3}{\pi C^3(w)}\ln[q(w)/p(w)]+\frac{1}{\pi^3C^3(w)}(\ln[q(w)/p(w)])^3,$$

$$\mu_4'=\frac{5}{C^4(w)}+\frac{6}{\pi^2C^4(w)}(\ln[q(w)/p(w)])^2+\frac{1}{\pi^4C^4(w)}(\ln[q(w)/p(w)])^4.$$

Proof: The previous forms in this proposition can be directly derived by applying the definition of non-central moment, where the obtained integra-tion can be easily worked out with the help of some mathematical packages. □

From the previous results in Proposition 6 we can find that the first four central moments of $X_{p(w)q(w)\text{-DHS}}$ are

$$\mu_1 = 0\,,\ \mu_2 = \frac{1}{C^2(w)}\,,\ \mu_3 = 0\,,\ \mu_4 = \frac{5}{C^4(w)}\,.$$

Consequently, the skewness and the excess kurtosis are $\gamma = 0$ and $\beta = 2$ respectively.

Using the relation between the cf and mgf, we can obtain the cf of the $p(w)q(w)$-DHS distribution in the following closed form:

$$\Psi_{p(w)q(w)\text{-DHS}}(t; p(w), q(w)) = \operatorname{sech} t \times e^{\frac{2it}{\pi C(w)} \ln[\sqrt{q(w)/p(w)}\,]}\,, \qquad (11)$$

where $|t| < \dfrac{\pi\, C(w)}{2}$.

The next proposition gives the closed form of the cgf and the used closed form to calculate the r-*th* cumulant k_r of $p(w)q(w)$-DHS distribution.

Proposition 7. *The corresponding cgf of the variable* $X_{p(w)q(w)\text{-DHS}}$ *with* $p(w), q(w) \in (0,\infty)$ *is given by*

$$K_{p(w)q(w)\text{-DHS}}(t; p(w), q(w)) = \ln[\sec t\,] + \frac{2t}{\pi C(w)} \ln[\sqrt{\frac{q(w)}{p(w)}}\,], \qquad (12)$$

where $|t| < \dfrac{\pi\, C(w)}{2}$. *Moreover, the r-th cumulant* $k_r,\ r = 1,2,3,\ldots,$ *of* $X_{q(w)\text{-DHS}}$ *is determined by*

$$k_r = [K^{(r)}_{p(w)q(w)\text{-DHS}}(t; p(w), q(w))]_{t=0}\,,\quad r = 1,2,3,\ldots\,, \qquad (13)$$

where the differentiation with respect to t and

$$K^{(1)}_{p(w)q(w)\text{-DHS}}(t; p(w), q(w)) = \frac{2}{\pi C(w)} \ln[\sqrt{\frac{q(w)}{p(w)}}\,] + \tan t\,,$$

$$K^{(2)}_{p(w)q(w)\text{-DHS}}(t; p(w), q(w)) = 1 + \tan^2 t\,,$$

$$K^{(3)}_{p(w)q(w)\text{-DHS}}(t; p(w), q(w)) = 2\tan t\,(1+\tan^2 t)\,,$$

$$K^{(4)}_{p(w)q(w)\text{-DHS}}(t; p(w), q(w)) = 2(1+\tan^2 t)^2 + 4\tan^2 t\,(1+\tan^2 t)\,,\ldots.$$

Proof: The form (12) can be derived by applying the definition of cgf where the obtained integration can be worked out with the help of some mathe-matical packages. Similarly, the r-*th* cumulants k_r of $X_{p(w)q(w)\text{-DHS}}$ for each value of r can be directly determined. □

From the previous results, we find that the moments of $X_{p(w)q(w)\text{-DHS}}$ are related with the cumulants, i.e. $\mu_1' = k_1$, $\mu_2 = \dfrac{1}{C^2(w)} k_2$, $\mu_3 = k_3$,

$$\mu_4 = \frac{1}{C^4(w)}[k_4 + 3(k_2)^2],\ \ldots, \text{ and so on.}$$

3. Maximum Likelihood Parameter Estimation

In this section, we will illustrate the ML Method to determine a certain value of the parametric function that maximizes the probability of the sample data from the $p(w)q(w)$-DHS distribution. To obtain the MLE's for the para-metric functions $p(w)$ and $q(w)$ for the $p(w)q(w)$-DHS distribution, we start with the pdf of the $p(w)q(w)$-DHS distribution which is given in (5).

Suppose that $X_1, X_2, \ldots, X_n$ are an iid random sample from the $p(w)q(w)$-DHS distribution, then the likelihood function is given by

$$L(x_1, x_2, \ldots, x_n \,|\, (p(w), q(w))) = C^n(w)(p(w)q(w))^{n/2} \times \prod_{i=1}^{n} [p(w)\exp(\frac{\pi\varphi_i}{2}) + q(w)\exp(-\frac{\pi\varphi_i}{2})]^{-1}, \qquad (14)$$

with $\varphi_i = \varphi(x_i; w)$. The log-likelihood function is

$$\ell(w) = \frac{n}{2}\ln(C^2(w)p(w)q(w)) - \sum_{i=1}^{n}\ln[p(w) \times \exp(\frac{\pi\varphi_i}{2}) + q(w)\exp(-\frac{\pi\varphi_i}{2})]. \qquad (15)$$

Taking the derivative of $\ell(w) = \ell(p(w), q(w))$ with respect to w and setting it equals 0 yields

$$\frac{C(w)p(w)q(w)}{2C'(w)p(w)q(w) + C(w)[p(w)q'(w) + p'(w)q(w)]} \times \{ \sum_{i=1}^{n} [\pi\varphi_i' \tanh_{p(w)q(w)}(\frac{\pi\varphi_i}{2}) + \{p'(w)\exp(\frac{\pi\varphi_i}{2}) + q'(w)\exp(-\frac{\pi\varphi_i}{2})\} \operatorname{sech}_{p(w)q(w)}(\frac{\pi\varphi_i}{2})] \} = n, \qquad (16)$$

with $\phi_i' = C'(w)x_i + D'(w)$. Solving (16) iteratively, then the MLE's $\hat{p}(w) = p(\hat{w})$ and $\hat{q}(w) = q(\hat{w})$ can be obtained.

4. Illustrative Example

We give an illustrative example of the deformed distribution and explain some results. Let $X_{p(w)q(w)-\text{DHS}}$ be a continuous random variable which follows the $p(w)q(w)$-DHS distribution with $p(w)=1$ and $q(w)=\exp(w)$. We consider $\varphi = \cosh(w)\,x + 3$. In this case we can find that pdf of $X_{p(w)q(w)-\text{DHS}}$ can be defined by,

$$f_{p(w)q(w)-\text{DHS}}(\varphi;1,\exp(w)) = \frac{\exp(w/2)}{2\,\text{sech}(w)} \times \text{sech}_{p(w)q(w)}\left(\frac{\pi\varphi}{2}\right); \quad x, w \in \mathbb{R},$$

and the corresponding cdf of $X_{p(w)q(w)-\text{DHS}}$ is

$$F_{p(w)q(w)-\text{DHS}}(\varphi;1,\exp(w)) = \frac{1}{2} + \frac{1}{\pi}\arctan[\exp(-w/2)\,\sinh_{p(w)q(w)}\left(\frac{\pi\varphi}{2}\right)],$$

with the critical value

$$x_\alpha^{p(w)q(w)-\text{DHS}} = \text{sech}(w)\,[\frac{2}{\pi}\,(\text{arcsinh}[\tan(\pi(\frac{1}{2}-\alpha))] + \frac{w}{2}) - 3],$$

where

$$P[X > x_\alpha^{p(w)q(w)-\text{DHS}}] = 1 - F_{p(w)q(w)-\text{DHS}}(x_\alpha^{p(w)q(w)-\text{DHS}}) = \alpha, \quad \alpha \in (0,1).$$

We can find that, the 1st and 2nd non-central moment of $X_{p(w)q(w)-\text{DHS}}$ are given respectively by

$$\mu_1' = \text{sech}(w)\,\frac{w}{\pi} \quad \text{and} \quad \mu_2' = \text{sech}^2(w)\,[1 + \frac{w^2}{\pi^2}].$$

Moreover, the variance is $\sigma^2 = \text{sech}^2(w)$. We can also find that mgf is given by

$$\text{M}_{p(w)q(w)-\text{DHS}}(t;1,\exp(w)) = \sec(t)\exp[\frac{t\,w\,\text{sech}(w)}{\pi}],$$

with $|t| < \dfrac{\pi}{2\,\text{sech}(w)}$.

Moreover, the 3rd and 4th non-central moments of $X_{p(w)q(w)-\text{DHS}}$ can be obtained as

$$\mu_3' = \text{sech}^3(w)\,[\frac{3w}{\pi} + \frac{w^3}{\pi^3}],$$

$$\mu_4' = \text{sech}^4(w)\,[5 + \frac{6w^2}{\pi^2} + \frac{w^4}{\pi^4}].$$

Thus, the first four central moments of the variable $X_{p(w)q(w)-\text{DHS}}$ are $\mu_1 = 0$, $\mu_2 = \text{sech}^2(w)$, $\mu_3 = 0$, $\mu_4 = 5\,\text{sech}^4(w)$ and also $\gamma = 0$ and $\beta = 2$. Directly, we can find that cf and cgf are given respectively by:

$$\Psi_{p(w)q(w)-\text{DHS}}(t;1,\exp(w)) = \text{sech}(t)\exp[\frac{i\,t\,w\,.\text{sech}(w)}{\pi}],$$

$$\text{K}_{p(w)q(w)-\text{DHS}}(t;1,\exp(w)) = \ln[\sec(t)] + \frac{t\,w\,\text{sech}(w)}{\pi},$$

with $|t| < \dfrac{\pi}{2\,\text{sech}(w)}$. From this form of cgf we can find that:

$$\text{K}^{(1)}_{p(w)q(w)-\text{DHS}}(t;1,\exp(w)) = \frac{w\,\text{sech}(w)}{\pi} + \tan(t),$$

and $\text{K}^{(k)}_{p(w)q(w)-\text{DHS}}(t;1,\exp(w)), k = 2,3,4,\ldots,$ are defined in Proposition 7. Moreover, the moments are related with the cumulants,

$$\mu_1' = k_1,\ \mu_2 = \text{sech}^2(w)\,k_2,\ \mu_3 = k_3,$$
$$\mu_4 = \text{sech}^4(w)\,[k_4 + 3(k_2)^2], \ldots,$$

and so on.

Finally, by solving the following nonlinear system in w iteratively,

$$\frac{1}{2\tanh(w)+1}\sum_{i=1}^{n}[\pi\,x_i\,\sinh(w)\,\tanh_{p(w)q(w)}(\frac{\pi\varphi_i}{2}) + \exp(-\frac{\pi\varphi_i}{2} + w)\,\text{sech}_{p(w)q(w)}(\frac{\pi\varphi_i}{2})] = n,$$

with $\varphi_i = \cosh(w)\,x_i + 3$ and $\varphi_i' = \sinh(w)\,x_i$, $i = 1,2,\ldots,n$, one can obtain $\hat{w}$ and thus the MLE of $q(w) = \cosh(w)$ is $\hat{q}(w) = \cosh(\hat{w})$. Different densities for the $p(w)q(w)$-DHS distribution with $q(w) > 1$ (i.e $\exp(w) > 1$) and their corresponding densities with $q(w) < 1$ (i.e $\exp(w) < 1$) for some values of w can be graphically illustrated. Moreover, the derivative of the unimodal pdf of $p(w)q(w)$-DHS distributions is explained in the following figure:

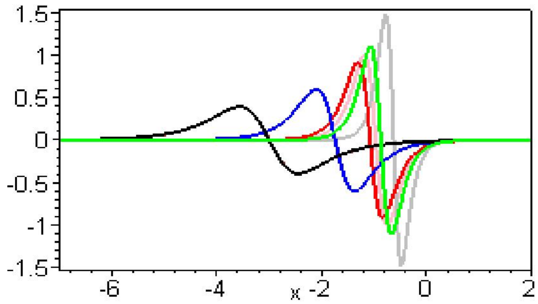

Figure 1: Derivative of the unimodal pdf of $p(w)q(w)$-DHS distributions with $(p(w), q(w)) = (1, \exp(w))$

5. Conclusions

This paper discussed the construction of the class of $p(w)q(w)$-DHS distributions which can be considered as a corresponding extension of the class of pq-DHS distributions. Firstly, we defined the $p(w)q(w)$-deformed hyperbolic functions which have been implemented by introducing two positive real valued parametric functions $p(w)$ and $q(w)$ as two factors of the exponential growth and decay parts of the HS distribution. We studied the effect of these deformation parametric functions in comparing with other previous studies on the HS-distribution. We considered a differentiable real valued function $\varphi = \varphi(X, w)$ instead of X. We assumed that this function is linear function in x with positive partial derivative with respect to x. We found that each $p(w)q(w)$-DHS distribution of the constructed class is unimodal. In general, it has variance with value different than unity. We noted also that the derived closed forms of the corresponding mgf, cf, cgf and sf for the $p(w)q(w)$-DHS distribution depend on $p(w)$, $q(w)$ and the partial derivative of φ with respect to x. Furthermore, some important properties of the constructed class of deformed distributions were discussed. We noted that their moments exist. There is unique value of their mean, median and mode which still also as a function of $p(w)$, $q(w)$ and $C(w)$. The skewness and excess kurtosis of these constructed distributions are still res-pectively equal to 0 and 2. By applying the ML method to determine the MLE for the parameters $p(w)$ and $q(w)$ we obtained a nonlinear system which can be solved iteratively by using high processing systems of computers. An illustrative example of the obtained results has been presented und discussed.

Acknowledgement

I am thankful to Prof. M. F. El-Sayed, Ain Shams University (Cairo), for his encouragement, insightful comments and scientific discussions that contributed to the preparation of this work.

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4/22/2013