Numerical Solution of Volterra and Fredholm Integral Equations of the Second Kind by Using Variational Iteration Method

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Abstract: In the present article, we apply variational iteration method to obtain the numerical solution of Volterra and Fredholm integral equations of the second kind. The method constructs a convergent sequence of functions, which approximates the exact solution with little iteration. Application of this method in finding the approximate solution of some examples confirms its validity. We use the symbolic algebra program, Maple, 15, to prove our results.

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1. Introduction

Since the integral equations, [2,9,10], appear frequently in modelling of physical phenomena, they have a major role in the fields of science and engineering, and considerable amount of research work has been done in studying them.

The variational iteration method, [1, 3, 4, 6, 11, 12], is one of the useful techniques in solving linear and non- linear problems. In the present study, we aim to employ the variational iteration method (VIM) to obtain the approximate solutions of integral equations. This method gives the exact solution rapidly convergent successive approximations if such a solution exists [7,8].

2. Variational iteration method

Consider the following nonlinear differential equation

$$Ly(x) + Ny(x) = g(x)\,, \qquad (2.1)$$

where L is a linear operator, N is a nonlinear operator and g(x) is a known analytical function. According to the (VIM),a correction functional can be constructed as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\zeta)[Ly_n(\zeta) + N\hat{y}_n(\zeta) - g(\zeta)]d\zeta \qquad (2.2)$$

where λ is a general multiplier and the term $\hat{y}_n$ is considered as a restricted variational i.e. $\delta\hat{y}_n = 0$. Making the above correction function stationary we obtain:

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(\zeta)[Ly_n(\zeta) + N\hat{y}_n(\zeta) - g(\zeta)]d\zeta \qquad (2.3)$$

In order to identify the Lagrange multiplier, from equation (2.3) we have:-

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(\zeta)[Ly_n(\zeta) - g(\zeta)]d\zeta, \qquad (2.4)$$

where y_0 is an initial approximation which possible unknown the successive approximations $y_n(x)$ of the solution $y(x)$ can be readily obtained.

3. Solution of Volterra and Fredholm integral equation of the second kind

Consider the Volterra and Fredholm integral equations of the second kind in the form:

$$y(x) = f(x) + \int_a^x k(x,t)y(t)dt, \qquad (3.1)$$

and

$$y(x) = f(x) + \int_a^b k(x,t)y(t)dt, \qquad (3.2)$$

where $y(x)$ is unknown function, $f(x)$ and $k(x,t)$ are given functions in $a \le x, t \le b$. For Volterra integral equation we take the partial derivatives with respect to x we have**:**

$$y'(x) = f'(x) + \frac{d}{dx}\int_a^x k(x,t)y(t)dt, \qquad (3.3)$$

and for Fredholm integral equation of the second kind, by differentiate both side of that equation by parts we get:

$$y'(x) = f'(x) + \frac{d}{dx}\int_a^b k(x,t)y(t)dt, \qquad (3.4)$$

Consider;

$$\frac{d}{dx}\int_a^x k(x,t)y(t)dt \ and \ \int_a^b k'(x,t)y(t)dt,$$

as a restricted variational , we use (VIM)in direction x .

Then we have the following iteration sequence

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(\zeta)\left[y_n'(\zeta)-f(\zeta)-\frac{d}{d(\zeta)}\int_0^\zeta k(\zeta,t)y_n(t)dt\right]d\zeta. \quad (3.5)$$

Taking with respect to the independent variable $y_n(x)$ and noticing that $\delta y_n(0)=0$, we get:

$$\delta y_{n+1}(x)=\delta y_n(x)+\lambda\delta y_n(x)-\int_0^x \lambda'\delta y_n(\zeta)d\zeta=0.(3.6)$$

Then we apply the following stationary conditions:

$$1+\lambda(\zeta)\big|_{\zeta=x}=0 \ ;\lambda'(\zeta)\big|_{\zeta=x}=0$$

The general Lagrange multiplier can be readily identified:

$$\lambda=-1 \quad (3.7)$$

Then we obtain the following iteration formula:

In case of Volterra integral equations (3.1),

$$y_{n+1}(x)=y_n(x)-\int_0^x\left[y_n'(\zeta)-f(\zeta)-\frac{d}{d(\zeta)}\int_0^\zeta k(\zeta,t)y_n(t)dt\right]d\zeta \quad (3.8)$$

and

$$y_{n+1}(x)=y_n(x)-\int_0^x\left[y_n'(\zeta)-f(\zeta)-\int_0^b\frac{\partial}{\partial\zeta}k(\zeta,t)y_n(t)dt\right]d\zeta. \quad (3.9)$$

in case of Fredholm integral equation (3.2).

Starting with initial approximation y_0 in (3.8) and (3.9), the successive approximations y_n will be easy obtained

4. Numerical examples

In this section, we applied the method in some examples to show the efficiently of the approach.

Example 1.

Consider the following linear Fredholm integral equation

$$y(x)=\sin(x)-\frac{x}{4}+\int_0^{\frac{\pi}{2}} x\,t\,y(t)dt, \quad (3.10)$$

with exact solution

$$y(x)=\sin(x).(3.11)$$

In the view of (VIM), we construct a correction functional in the following form

$$y_{n+1}(x)=y_n(x)-\int_0^x\left[y_n'(\zeta)-\cos(\zeta)+\frac{1}{4}-\frac{1}{4}\int_0^{\frac{\pi}{2}} t y_n(t)dt\right]d\zeta .(3.12)$$

Starting with the initial approximation $y_0(x)=\sin(x)-\frac{x}{4}$ in equation (3.12), successive approximations $y_i(x)$ will be achieved. The absolute error between the exact solution and the 4th order of approximate solution is shown in table 1

Table 1:

x	Approximate solution	Exact solution	Absolute solution
0	0	0	0
$\frac{\pi}{16}$	0.1945561472	0.195090322	5.3417485E-4
$\frac{\pi}{8}$	0.3816150827	0.382683432	1.0683493E-3
$\frac{3\pi}{16}$	0.5539677085	0.555570233	1.0602533E-3
$\frac{\pi}{4}$	0.7049700818	0.7071067812	1.0602533E-3
$\frac{5\pi}{16}$	0.828798738	0.8314696123	2.7670874E-3
$\frac{3\pi}{8}$	0.92067448	0.9238795325	3.2050525E-3
$\frac{7\pi}{16}$	0.9770460564	0.980785258	3.739224E-3
$\frac{\pi}{2}$	0.9957266	1	4.2734E-3

Example2.

Consider the following linear Volterra integral equation[5] :

$$y(x)=1-\int_0^x (x-t)y(t)dt, \quad (3.13)$$

with exact solution:

$$y(x)=\cos(x) \quad (3.14)$$

In the view of (VIM), we construct a correction functional in the following form

$$y_{n+1}(x)=y_n(x)-\int_0^x\left[y_n'(\zeta)+\frac{d}{d\zeta}\int_0^\zeta(\zeta-t)y_n(t)dt\right]d\zeta . (3.15)$$

Starting with the initial approximation $y_0(x)=1$ in equation (3.15), successive approximations $y_i(x)$ will be achieved. The absolute error between the exact solution and the third order of approximate solution is shown in Table2:

x	Approximate solution	Exact solution	Absolute error
0	1	1	0
$\frac{\pi}{8}$	0.923879	0.9238795	5.32511E-7
$\frac{\pi}{4}$	0.7071032	0.7071067812	3.5811865E-6
$\frac{3\pi}{8}$	0.3825928	0.382683432	9.063236E-5
$\frac{\pi}{2}$	-8.94522998E-4	0	-8.94522998E-4

table 2.

Example 3.
Consider the following nonlinear Fredholm integral equation :

$$y(x)=\frac{11}{12}x^2+\frac{1}{2}\int_0^1 x^2 t y^2(t)dt, \qquad (3.16)$$

with exact solution

$$y(x)=x^2 \qquad (3.17)$$

In the view of (VIM), we construct a correction functional in the following form

$$y_{n+1}(x)=y_n(x)-\int_0^x\left[y_n'(\zeta)-\frac{11}{6}\zeta-\frac{1}{2}\int_0^{\frac{\pi}{2}}2\zeta t y^2{}_n(t)dt\right]d\zeta . \quad (3.18)$$

Starting with the initial approximation $y_0(x)=\frac{11}{12}x^2$ in equation (3.18), successive approximations $y_i(x)$ will be achieved. The absolute error between the exact solution and The third order of approximate solution is shown in table 3

Table 3:

x	Approximate solution	Exact solution	Absolute error
0	0	0	0
0.2	0.03999868	0.04	1.32E-6
0.4	0.15999472	0.16	5.28E-6
0.6	0.35998812	0.36	1.188E-5
0.8	0.63997888	0.64	2.112E-5
1	0.999967	1	3.3E-5

Example 4.
Consider the following nonlinear Fredholm integral equation

$$y(x)=\frac{15}{32}x+\frac{1}{2}\int_0^1 x t y^2(t)dt, \qquad (3.19)$$

with exact solution

$$y(x)=\frac{x}{2}. \qquad (3.20)$$

In the view of (VIM), we construct a correction functional in the following form

$$y_{n+1}(x)=y_n(x)-\int_0^x\left[y_n'(\zeta)-\frac{15}{32}\zeta-\frac{1}{2}\int_0^1 t y_n^2(t)dt\right]d\zeta . \quad (3.21)$$

Starting with the initial approximation $y_0(x)=\frac{15}{32}x$ in equation (3.21), successive approximations $y_i(x)$ will be achieved. The absolute error between the exact solution and the 4th order of approximate solution is shown in table 4.

Table 4.

X	Approximate solution	Exact solution	Absolute error
0	0	0	0
0.2	0.0999985	0.1	1.5E-6
0.4	0.1999970477	0.2	2.9523E-6
0.6	0.29999557152	0.3	4.4285E-6
0.8	0.399994095	0.4	5.905E-6
1	0.4999926192	0.5	7.3808E-6

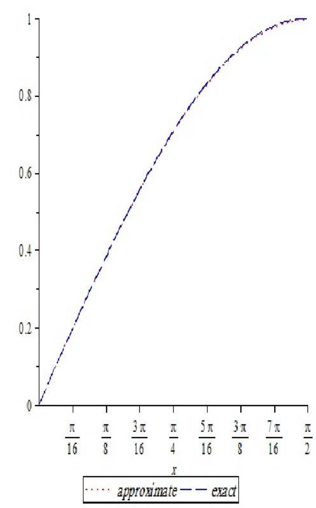

Figure 1: The plots of approximate solution of 4th order and exact solution for Example 1

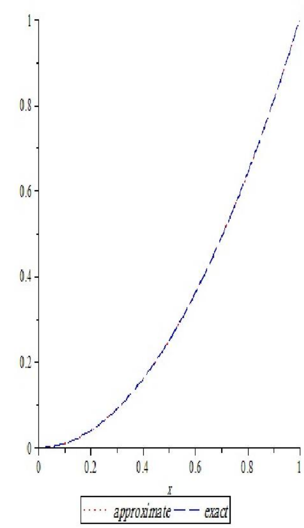

Figure 3: The plots of approximate solution of 5th order and exact solution for Example 3

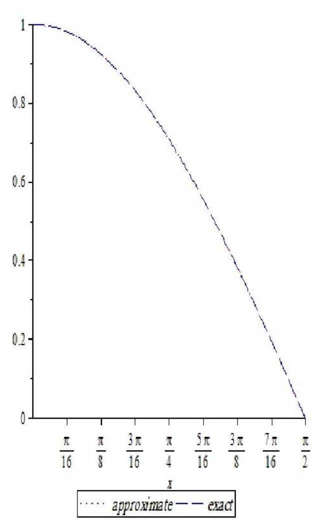

Figure 2: The plots of approximate solution of third order and exact solution for Example 2

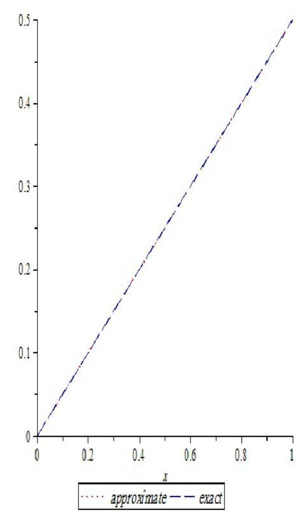

Figure 4: The plots of approximate solution of 3th order and exact solution for Example 4

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