Improving the Modified Gause – Seidel Method for M - Matrices

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Abstract: In1994, M. Usui et al. have reported the modified Gauss-Seidel method with a preconditioner $(I + U)$. The preconditioning effect is not observed on the n-th row. In this paper, to deal with this drawback, we propose a new preconditioner. In addition, the convergence and comparison theorems of the proposed method are established.

[Nasser Mikaeilvand and, Zahra Lorkojori. **Improving The Modified Gause – Seidel Method For M - Matrices.** *Life Sci J* 2012;9(4):4093-4098]. (ISSN: 1097-8135). http://www.lifesciencesite.com. 608

Keywords: M-Matrices, Preconditioned system, Gauss-Seidel method

1- Introduction:

We consider the following linear system:

$$AX = b, \qquad (1)$$

where $A \in R^{n\times n}, b \in R^n$ are given and $x \in R^n$ is unknown. For simplicity, let $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and strictly upper triangular matrices, respectively.

Now, consider a preconditioned system of (1):

$$PAX = pb, \qquad (2)$$

where P is a non-singular matrix. To effectively solve the preconditioned linear system (2), a variety of preconditioners have been proposed by several authors [1 − 8,11] and the references therein. The preconditioning effect is not observed on the last row of matrix A. For example, the preconditioner $P_{U_1} = I + U$ In [11] where U is a strictly upper triangular part of –A.

In 2009, Zheng et al. [4] proposed the following two preconditioners:

$$P_{max} = I + S_{max} + R_{max}$$

and

$$P_R = I + S_{max} + R \qquad (4)$$

where

$$S_{max} = \begin{cases} -a_{i,k_i} & i=1,\dots,n-1, j>i; \\ 0, & Other\ Wise, \end{cases}$$

$$K_i = min\{j|max||a_{i,j}|i < n\}$$

and

$$(R_{max})_{i,j} = \begin{cases} -a_{n,k_n} & i = n, j = K_n \\ 0, & Other\ Wise \end{cases}$$

with $K_n = min\left\{j \left||a_{n,j}|\right| = max\{|a_{n,l}|, l = 1, \dots, n-1\}\right\}$

and

$$(R)_{i,j} = \begin{cases} -a_{i,j} & i = n, 1 \le j \le n-1, \\ 0, & Other\ Wise \end{cases}$$

The comparison result between the preconditioners P_{max} with P_R [4] shows that the preconditioner P_R is better than P_{max} for solving the preconditioned linear system (2).

In this paper, we propose the following a preconditioner: (3)

$$P_U = (I + U + R) = \begin{pmatrix} 1 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ 0 & 1 & -a_{23} & \cdots & -a_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -a_{n-1n} \\ -a_{n1} & -a_{n2} & \ldots & -a_{nn-1} & 1 \end{pmatrix}$$

Then AU can be written as follows:

$AU = (I + U + R)A$

$= I - L - U + U - UL - U^2 + R - RL - RU = M_U - N_U,$

where

$$M_U = \left(I - D - L - E + R - \acute{\acute{D}} - \acute{\acute{E}}\right), N_U = F + U^2$$

and D, E and F are the diagonal, strictly lower triangular and strictly upper triangular parts of UL, while $\acute{\acute{D}}$ and $\acute{\acute{E}}$ are the diagonal, strictly lower triangular parts of $R(L + U)$, respectively. If M_U is nonsingular, the MGS iterative matrix is $T_U = M_U^{-1} N_U$.

2- **Preliminaries**:

In this section, we present some notation, definitions and lemmas.

For $A = (a_{i,j}), B = (b_{i,j}) \in R^{n \times n}$, we write $A \geq B$ if $a_{i,j} \geq b_{i,j}$ holds for all $i, j = 1,2, \ldots, n$. Calling A nonnegative if $A \geq 0 (a_{i,j} \geq 0; i, j = 1,2, \ldots, n)$. $\rho(.)$ denotes the spectral radius of a matrix.

Definition 2.1. A matrix A is a L-matrix if $a_{i,j} > 0; i = j = 1, \ldots, n$ and $a_{i,j} \leq 0$ for all $i, j = 1,2, \ldots, n; i \neq j$. A nonsingular L-matrix A is a nonsingular M-matrix if $A^{-1} \geq 0$.

Definition 2.2. Let A be a real matrix. Then

$$A = M - N$$

is called a splitting of A if M is a nonsingular matrix. The splitting is called

(a) regular if $M^{-1} \geq 0$ and $N \geq 0$;

(b) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;

(c) nonnegative if $M^{-1}N \geq 0$;

(d) M-splitting if M is a nonsingular M-matrix and $N \geq 0$.

Lemma 2.1 ([14)]. Let $A \in R^{n \times n}$ be nonnegative $n \times n$ matrix. Then

(a) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;

(b) for (A) , there corresponds an eigenvector $X > 0$;

(c) $\rho(A)$ is a simple eigenvalue of A ;

(d) $\rho(A)$ increases when any entry of A increases.

Definition 2.3. We call $A = M - N$ the Gauss-Seidel splitting of A, if $M = (I - L)$ is nonsingular and $N =$. In addition , the splitting is called

(a) Gauss-Seidel convergent if $(M^{-1}N) < 1$;

(b) Gauss-Seidel regular if $M^{-1} = (1 - L)^{-1} \geq 0$ and $N = U \geq 0$.

Lemma 2.2 ([17]). $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.

Lemma 2.3 ([15]). Let A and B be $n \times n$ matrices. Then AB and BA have the same eigenvalues, counting multiplicity.

Lemma 2.4([10]). Let A be a nonsingular M-matrix, and let

$A = M_1 - N_1 = M_2 - N_2$ be two convergent splitting, the first one weak regular and the second one regular. If $M_1^{-1} \geq M_2^{-1}$, then

$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1.$

3. Comparison Theorems

In this section, we compare such MGS method with the classical Gauss-Seidel method and the MGS method with the preconditioner $P_{U_1} = I + U$ ([11]), respectively.

To prove the theorems, we need some results.

We firstly prove that $A_{U_1} = M_{U_1} - N_{U_1}$ and $A_U = M_U - N_U$ are both regular and Gauss-Seidel convergent splitting.

For the preconditioner $P_{U_1} = I + U$ the preconditioned matrix $A_{U_1} = (I + U)A$ can be written as

$$A_{U_1} = M_{U_1} + N_{U_1} = (I - D - L - E) - (F + U^2).$$

In which D, E and F are defined as in section1. Hence, if $\sum_{K=i+1}^{n} a_{i,k}\, a_{k,i} \neq 1 (i = 1,2, \dots, n-1)$, Then, the MGS iterative matrix T_{U_1} for A_{U_1} can be defined by

$$T_{U_1} = M_{U_1}^{-1} N_{U_1} = (I - D - L - E)^{-1}(F + U^2)$$

as $(I - D - L - E)^{-1}$ exists. There is the following result:

Lemma 3.1. Let $A = I - L - U$ be a nonsingular M-matrix. Assume that $0 \leq \sum_{k=i+1}^{n} a_{i,k}\, a_{k,i} < 1, 1 \leq i \leq n-1$. Then $A_{U_1} = M_{U_1} - N_{U_1}$ is regular and Gauss-Seidel convergent.

Proof. We observe that when $0 \leq \sum_{k=i+1}^{n} a_{i,k}\, a_{k,i} < 1, 1 \leq i \leq n-1$, the diagonal elements of A_{U_1} are positive and $M_{U_1}^{-1}$ exists. It is known that (see ([18]) an L-matrix A is a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$. By taking such y, the fact that $I + U \geq 0$ implies$A_{U_1} y = (I + U)Ay > 0$. Consequently, the L-matrix A_{U_1} is a nonsingular M-matrix which means$A_{U_1}^{-1} \geq 0$. Since $0 \leq \sum_{k=i+1}^{n} a_{i,k}\, a_{k,i} < 1$ we have $(I - D)^{-1} \geq I$.

As strictly lower triangular matrix $L + E$ has nonnegative elements, by Neumann's series, the following inequality holds:

$$M_{U_1}^{-1} = [I + (I - D)^{-1}(L + E) + \{(I - D)^{-1}(L + E)\}^2 + \cdots$$

$$+\{(I - D)^{-1}(L + E)\}^{n-1}](I - D)^{-1} \geq 0$$

On the other hand, it is easy to see that $N_{U_1} = F + U^2 \geq 0$ .Thus,

$A_{U_1} = M_{U_1} - N_{U_1}$ is a regular and Gauss-Seidel convergent splitting by Definition 2.3 And Lemma 2.2. ∎

Theorem 3.2. Let A be a nonsingular M-matrix, assume that $0 \leq \sum_{k=i+1}^{n} a_{i,k}\, a_{k,i} < 1, 1 \leq i \leq n-1$ and$0 \leq \sum_{k=1}^{n-1} a_{n,k}\, a_{k,n} < 1$, then $A_U = M_U - N_U$ is regular and Gauss-Seidel convergent splitting.

Proof. We observe that when $0 \leq \sum_{k=i+1}^{n} a_{i,k}\, a_{k,i} < 1, 1 \leq i \leq n-1$ and $0 \leq \sum_{k=1}^{n-1} a_{n,k}\, a_{k,n} < 1$, the diagonal elements of A_U are positive and M_U^{-1} exists. Similar to the proof of Theorem 3.1, We can show that

$A_U = (I + U + R)A$ is a nonsingular M-matrix when A is a nonsingular M-matrix. Thus, $A_U^{-1} \geq 0$. When $0 \leq \sum_{k=i+1}^{n} a_{i,k}\, a_{k,i} < 1, 1 \leq i \leq n-1$ and $0 \leq \sum_{k=1}^{n-1} a_{n,k}\, a_{k,n} < 1$, we have $D + \acute{D} < I$, so that $(I - D - \acute{D}) \geq 0$. Hence,

$$M_U^{-1} = [\left(I - D - \acute{D}\right) - \left(L - R + E + \acute{E}\right)]^{-1}$$

$$= [I - \left(I - D - \acute{D}\right)^{-1}\left(L - R + E + \acute{E}\right)]^{-1}\left(I - D - \acute{D}\right)^{-1}$$

$$= \{I + \left(I - D - \acute{D}\right)^{-1}\left(L - R + E + \acute{E}\right) + [\left(I - D - \acute{D}\right)^{-1}\left(L - R + E + \acute{E}\right)]^2 + \cdots$$

$$+[I - \left(I - D - \acute{D}\right)^{-1}\left(L - R + E + \acute{E}\right)]^{n-1}\}\left(I - D - \acute{D}\right)^{-1} \geq 0$$

It is easy to see that $N_U = F + U^2 \geq 0$.

Therefore, $A_U = M_U - N_U$ is a regular and Gauss-Seidel convergent splitting by Definition 2.3 And Lemma 2.2. ∎

Theorem 3.3. Let A be a nonsingular M-matrix. Then under the assumptions of Theorem 3.2, the following inequality holds:

$$\rho(T_U) \leq \rho(T) < 1$$

where, $T = (I - L)^{-1}$ U is the iterative matrix of the classical Gauss-Seidel method for $A = I - L - U$.

Proof. Since A is a nonsingular M-matrix, the classic Gauss-Seidel splitting $A = (I - L) - U$ of A is clearly regular and convergent.

For $M_U = I - D - L - E + R - \acute{D} - \acute{E}$ and $N_U = F + U^2$ by Theorem 3.2 we know that $A_U = M_U - N_U$ is a Gauss-Seidel convergent splitting.

To compare $\rho(T_U)$ with (T) , we have

$$A = (I + U + R)^{-1} M_U - (I + U + R)^{-1} N_U.$$

If we take $M_1 = (I + U + R)^{-1}M_U$ and $N_1 = (I + U + R)^{-1}N_U$, then $\rho(M_1^{-1}N_1) < 1$ since $M_U^{-1}N_U = M_1^{-1}N_1$. Also, we have

$$M_1^{-1} = M_U^{-1}(I + U + R)$$

$$= (I - D - L - E + R - \acute{D} - \acute{E})^{-1}(I + U + R)$$

$$\geq (I - D - L - E + R - \acute{D} - \acute{E})^{-1}$$

$$= [I - \left(I - D - \acute{D}\right)^{-1}\left(L - R + E + \acute{E}\right)]^{-1}\left(I - D - \acute{D}\right)^{-1}$$

$$\geq [I - \left(I - D - \acute{D}\right)^{-1}\left(L - R + E + \acute{E}\right)]^{-1}$$

$$\geq (I - L)^{-1},$$

If follows from Lemma 2.4 that $\rho(M_1^{-1}N_1) \leq \rho(M^{-1}N) < 1$. Hence,

$\rho(M_U^{-1}N_U) \leq \rho(M^{-1}N) < 1$, i.e., $\rho(T_U) \leq \rho(T) < 1$.

Next, we give a comparison theorem between the MGS method with the preconditioners $\boldsymbol{P_U}$ and $\boldsymbol{P_{U_1}}$, respectively.

***Theorem* 3.4.** Let A be a nonsingular M-matrix. Then under the assumptions of Theorem **3.2** and $\boldsymbol{a_{n,j}\sum_{k=1}^{n-1} a_{n,k}a_{k,n} \leq \sum_{k=1}^{n-1} a_{n,k}a_{k,j}}$, $\boldsymbol{1 \leq j \leq n-1}$, we have

$$\boldsymbol{\rho(T_U) \leq \rho(T_{U_1}) < 1}$$

***Proof*.** For the matrices $\boldsymbol{M_{U_1}, M_U}$, $\boldsymbol{N_{U_1}}$ and $\boldsymbol{N_U}$ in the splitting of matrices $\boldsymbol{P_{U_1}A = M_{U_1} - N_{U_1}}$ and $\boldsymbol{P_U A = M_U - N_U}$, they can be expressed in the partitioned forms as follows:

$$\boldsymbol{M_{U_1} = I - D - L - E = \begin{pmatrix} \widehat{M} & 0 \\ u^T & 1 \end{pmatrix}},$$

$$\boldsymbol{M_U = M_{U_1} + RA = \begin{pmatrix} \widehat{M} & 0 \\ V^T & v_n \end{pmatrix}},$$

$$\boldsymbol{N_U = N_{U_1} = \begin{pmatrix} \widehat{N} & W \\ 0 & 0 \end{pmatrix}},$$

where

$\boldsymbol{\widehat{M} = (\widehat{m}_{i,j})}$

$$\boldsymbol{\widehat{m}_{i,j}} = \begin{cases} \boldsymbol{0}, & \boldsymbol{1 \leq i} < j \leq n-1, \\ \boldsymbol{1 - \sum_{k=i+1}^{n} a_{i,k}a_{k,i}}, & \boldsymbol{i = j}, \\ \boldsymbol{a_{i,j} - \sum_{k=i+1}^{n} a_{i,k}a_{k,j}}, & j < i \leq n-1, \end{cases}$$

$$\boldsymbol{u^T = (a_{n,1}, \dots, a_{n,n-1})},$$

$$\boldsymbol{V^T = (v_1, \dots, v_{n-1})}$$

$$\boldsymbol{v_j = a_{n,j} - \sum_{k=1}^{n-1} a_{n,k}a_{k,j} \; (1 \leq j \leq n-1)}$$

$$\boldsymbol{v_n = 1 - \sum_{k=1}^{n-1} a_{n,k}a_{k,n}}$$

$$\boldsymbol{W = (\omega_1, \dots, \omega_{n-1})^T}$$

$$\boldsymbol{\omega_i = -a_{i,n} + \sum_{k=i+1}^{n} a_{i,k}a_{k,n} \; (1 \leq i \leq n-1)}$$

and $\boldsymbol{\widehat{N} \geq 0}$ is an $\boldsymbol{(n-1) \times (n-1)}$ strictly upper triangular matrix.

Direct computation yields

$\boldsymbol{M_{U_1}^{-1} = \begin{pmatrix} \widehat{M}^{-1} & 0 \\ -u^T\widehat{M}^{-1} & 1 \end{pmatrix}}$ and

$$\boldsymbol{M_U^{-1} = \begin{pmatrix} \widehat{M}^{-1} & 0 \\ -v_n^{-1}V^T\widehat{M}^{-1} & v_n^{-1} \end{pmatrix}}$$

therefore ,

$$\boldsymbol{N_{U_1}M_{U_1}^{-1} = \begin{pmatrix} \widehat{T}_{U_1} & W \\ 0 & 0 \end{pmatrix} \geq 0}$$

and

$$\boldsymbol{N_U M_U^{-1} = \begin{pmatrix} \overline{T}_U & v_n^{-1}W \\ 0 & 0 \end{pmatrix} \geq 0}$$

where $\boldsymbol{\widehat{T}_{U_1} = \widehat{N}\widehat{M}^{-1} - Wu^T\widehat{M}^{-1}}$ and $\boldsymbol{\overline{T}_U = \widehat{N}\widehat{M}^{-1} - Wv_n^{-1}V^T\widehat{M}^{-1}}$. Since both the lower-right corner of $\boldsymbol{N_U M_U^{-1}}$ and $\boldsymbol{N_{U_1}M_{U_1}^{-1}}$ have zeros, $\boldsymbol{\rho(N_U M_U^{-1})}$ and $\boldsymbol{\rho(N_{U_1}M_{U_1}^{-1})}$ exist in $\boldsymbol{\overline{T}_U}$ and $\boldsymbol{\widehat{T}_{U_1}}$, respectively. That is, $\boldsymbol{\rho(N_U M_U^{-1}) = \rho(\overline{T}_U)}$ and $\boldsymbol{\rho(N_{U_1}M_{U_1}^{-1}) = \rho(\widehat{T}_{U_1})}$. By simple computation, we know that $\boldsymbol{\overline{T}_U \leq \widehat{T}_{U_1}}$ under the assumption $\boldsymbol{a_{n,j} - \sum_{k=1}^{n-1} a_{n,k}a_{k,n} \leq \sum_{k=1}^{n-1} a_{n,k}a_{k,j}}$, $\boldsymbol{1 \leq j \leq n-1}$. Hence by Lemma **2.1** , we have

$$\boldsymbol{\rho(N_U M_U^{-1}) = \rho(\overline{T}_U) \leq \rho(\widehat{T}_{U_1}) = \rho(N_{U_1}M_{U_1}^{-1})}.$$

Therefore , by Lemma **2.3** we immediately know that

$$\boldsymbol{\rho(M_U^{-1}N_U) = \rho(N_U M_U^{-1}) \leq \rho(N_{U_1}M_{U_1}^{-1}) = \rho(M_{U_1}^{-1}N_{U_1})}$$, which means that $\boldsymbol{\rho(T_U) \leq \rho(T_{U_1})}$.

4. Comparison Theorems

In this section, we discuss a comparison with P_U and P_R. The comparison result show that the preconditioner P_U is better than P_R for sloving the preconditioned linear system (2).

***Theorem* 4.1.** Let A be a nonsingular M-matrix. If $0 \le \sum_{k=i+1}^{n} a_{i,k}a_{k,i} < 1, 1 \le i \le n-1$ and $0 \le \sum_{k=1}^{n-1} a_{n,k}a_{k,n} < 1$ and $0 \le a_{i,k_i}a_{k_i,i} < 1, 1 \le i \le n-1$, then $\rho(T_U) \le \rho(T_R) < 1$

***Proof*.** For $M_U = I - D - L - E + R - \acute{\acute{D}} - \acute{\acute{E}}$ and $N_U = F + U^2$ by Theorem 3.2 we know that $A_U = P_U A = M_U - N_U$ is a Gauss-Seidal convergent splitting. For $M_R = I - \acute{D} - L - \acute{E} + R - \acute{\acute{D}} - \acute{\acute{E}}$ and $N_R = U - S_{max} + \acute{F} + S_{max}U$ that $\acute{D}, \acute{E}$ and $\acute{F}$ are respectivly the diagonal, strictly lower triangular and strictly upper triangular parts of $S_{max}L$, and $\acute{\acute{D}}$ and $\acute{\acute{E}}$ are the diagonal, strictly lower triangular parts of $R(L+U)$, respaectivly. From [4] we know that $A_R = P_R A = M_R - N_R$ is a Gauss-Seidel convergant splitting. To compare $\rho(T_U)$ with $\rho(T_R)$, we consider the following splitting of A:

$$A_U = P_U A = M_U - N_U$$

$$(I + U + R)A = M_U - N_U$$

$$A = (I + U + R)^{-1}M_U - (I + U + R)^{-1}N_U$$

that we take $M_1 = (I + U + R)^{-1}M_U$ and $N_1 = (I + U + R)^{-1}N_U$

and

$$A_R = P_R A = M_R - N_R$$

$$(I + S_{max} + R)A = M_R - N_R$$

$$A = (I + S_{max} + R)^{-1}M_R - (I + S_{max} + R)^{-1}N_R$$

If we take $M_2 = (I + S_{max} + R)^{-1}M_R$ and $N_2 = (I + S_{max} + R)^{-1}N_R$, then $\rho(M_1^{-1}N_1) < 1$ and $\rho(M_2^{-1}N_2) < 1$ since $M_U^{-1}N_U = M_1^{-1}N_1$ and $M_R^{-1}N_R = M_2^{-1}N_2$.

Then $A = M_1 - N_1 = M_2 - N_2$ are two convergant splittings.

Since matrices $L, D, \acute{D}, E, \acute{E}, R, \acute{\acute{D}}$ and $\acute{\acute{E}}$ are positive and $D \ge \acute{D}$ and $E \ge \acute{E}$, we have $-D \le -\acute{D}$ and $-E \le -\acute{E}$. Then the following inequality holds:

$$I - D - L - E \le I - \acute{D} - L - \acute{E}$$

and we have:

$$I - D - L - E + R - \acute{\acute{D}} - \acute{\acute{E}} \le I - \acute{D} - L - \acute{E} + R - \acute{\acute{D}} - \acute{\acute{E}}.$$

Therefore

$$(I - D - L - E + R - \acute{\acute{D}} - \acute{\acute{E}})^{-1} \ge (I - \acute{D} - L - \acute{E} + R - \acute{\acute{D}} - \acute{\acute{E}})^{-1}$$

Also, $P_U = I + U + R$ and $P_R = I + S_{max} + R$ are positive matrices and we have

$$I + U + R \ge I + S_{max} + R \qquad (6)$$

from (5) and (6) the following relation holds:

$$(I - D - L - E + R - \acute{\acute{D}} - \acute{\acute{E}})^{-1}(I + U + R)$$

$$\ge (I - \acute{D} - L - \acute{E} + R - \acute{\acute{D}} - \acute{\acute{E}})^{-1}(I + S_{max} + R)$$

and we know that

$$M_1^{-1} = (I - D - L - E + R - \acute{\acute{D}} - \acute{\acute{E}})^{-1}(I + U + R)$$

and

$$M_2^{-1} = (I - \acute{D} - L - \acute{E} + R - \acute{\acute{D}} - \acute{\acute{E}})^{-1}(I + S_{max} + R)$$

Then, from (7), $M_1^{-1} \ge M_2^{-1}$ it follows from Lemma 2.4 that

$\rho(M_1^{-1}N_1) \le \rho(M_2^{-1}N_2) < 1$. Hence, $(M_U^{-1}N_U) \le \rho(M_R^{-1}N_R) < 1$, i.e., $\rho(T_U) \le \rho(T_R) < 1$.
■

5. Numerical Examples

***Example* 5.1.** Consider the following matrix,

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

by computation, we have

$\rho(M^{-1}N) = 0.460779 > \rho(M_U^{-1}N_U) = 0.156956$
and

$\rho(M_{U_1}^{-1}N_{U_1}) = 0.186007 > \rho(M_U^{-1}N_U) = 0.156956$ and

$\rho(M_R^{-1}N_R) = \mathbf{0.257251} > \rho(M_U^{-1}N_U) = \mathbf{0.156956}$.

***Example* 5.2.** Let the coefficient matrix A given by

$$A = \begin{pmatrix} 1 & 0 & -0.1 & -0.2 & 0 & 0 & -0.4 & -0.1 & -0.1 \\ -0.1 & 1 & 0 & 0 & -0.3 & -0.1 & -0.1 & 0 & -0.2 \\ -0.1 & -0.2 & 1 & 0 & -0.1 & 0 & -0.3 & 0 & 0 \\ 0 & -0.1 & -0.1 & 1 & 0 & -0.1 & -0.4 & 0 & -0.1 \\ -0.2 & 0 & -0.1 & 0 & 1 & 0 & -0.4 & -0.1 & -0.1 \\ -0.1 & 0 & 0 & -0.1 & 0 & 1 & -0.3 & 0 & -0.2 \\ -0.2 & -0.2 & 0 & -0.1 & 0 & 0 & 1 & -0.2 & -0.1 \\ -0.1 & 0 & 0 & -0.2 & -0.2 & -0.1 & 0 & 1 & -0.3 \\ 0 & 0 & -0.1 & -0.2 & 0 & 0 & -0.1 & -0.3 & 1 \end{pmatrix}$$

Obviously, from numerical results, we have $\rho(T_U) \le \rho(T_R)$ and

$\rho(T_U) \le \rho(T_{U_1}) \le \rho(T)$, we have $\rho(T_U) =$ 0.414255 , $\rho(T_R)$=0.478073 , $\rho(T_{U_1}) =$ 0.421223 and $\rho(T) = 0.670704$.

References

[1] A. D. Gunawardena, S. K. Jain, L. Snyder, Modified Itrative Methods For Consistent Linear Systams, Linear Algebra Appl. **154 – 156(1991)123 – 143**.

[2] H. Kotakemori, K. Harada, M. Morimoto, H. Niki, A Comparison Theorem For The Iterative Mathod With The Preconditioner $(I + S_{max})$, J. Comput. Appl. Math. **145(2002)373 – 378**.

[3] J. P. Milaszewicz, Impriving Jacobi And Gauss-Seidel Iterations, Linear Algabra Appl. **93(1987)161 – 170**.

[4] B. Zheng, S. X. Miao, Two New Modified Gauus-Seidel Methods For Linear System With M-Matrices, J. Comput. Appl. Math. **233(2009)922 – 930**.

[5] T. Kohono, H. Kotakemori, H. Niki, Improving Th Modified Gauss-Seidel Method For Z-Matrices, Linear Algabra Appl. **267(1997)113 – 123**.

[6] H. Kotakemori, H. Niki, N. Okamoto, Accerated Iterative Method For Z-Matrices, J. Comput. Appl. Math. **75(1996)87 – 97**.

[7] H. Niki, K. Harada, M. Morimoto, M. Sakakihara, The Survey Of Preconditioners Used For Accelerating The Rate Of Convergence In The Gauss-Seidel Method, J. Comput. Appl. Math. **164 – 165(2004)587 – 600**.

[8] W. Li, A Note On The Preconditioned Gauss-Seidal (GS) Method For Linear Systems, J. Comput. Appl. Math. **182(2005)81 – 90**.

[9] J. H. Yun, A Note On Preconditioned AOR Method For L-Matrices, J. Comput. Appl. Math. **220(2008)13 – 16**.

[10] Z. I. Wozniki, Nonnegative Splitting Theory, Japan J. Industrial Appl. Math. **11(1994)289 – 342**.

[11] M. Usui, H. Niki, T. Kohno, Adaptive Gauss-Seidal Method For Linear Systyems, Int. J. Comput. Math. **51(1994)119 – 125**.

[12] H. Wang, Y.T. Li, A. New Preconditioned AOR Iterative Method For L-Matrices, J. Comput. Appl. Math. **229(2009)47 – 53**.

[13] H. Niki, T. Kohono, M. Morimoto, The Preconditioned Gauss-Seidal Method Faster Than The SOR Method, J. Comput. Appl. Math. **219(2008)59 – 71**.

[14] R. S. Varga, Matrix Iterative Analysis, Pre Ntice, Englewood Cliffs, Nj, **1981**.

[15] F. Zhang, Matrix Theory, Springer, **1999**.

[16] D. M. Young, Itreative Solution Of Large Linear System, Academic Press, New York, **1971**.

[17] W. Li, W. W. Sun, Modified Gauss-Seidal Type Methods And Jacobi Type Methods FOR Z-Matrices, Linear Algebra Appl. **317(2000)227 – 240**.

[18] A. Berman, R. J. Plemmons, Nonnegative Matrices In The mathematical Sciences, Academic Press, New York, **1979**.

[19] Y. Z. Song, Comparisons Of Nonnegative Splittings Of Matrices, Linear Algebra Appl. **154 – 156(1991)433 – 455**.

[20] H. Schneider, Theorems On M-Splittings Of A Singular M-Matrix Which Dapend On Graph Structure, Linear Algebra Appl. **58(1984)407 – 424**.