New exact solutions of Boiti-Leon-Manna-Pempinelli equation using extended F-expansion method

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Abstract: Using a computerized symbolic computation technique, we study the Boiti-Leon-Manna-Pempinelli equation using extended F-expansion method. It is shown that soliton solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions.

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1. Introduction

The nonlinear wave phenomena can be observed in various scientific fields, such as plasma physics, optical fibers, fluid dynamics, chemical physics, etc. Mathematical modelling of physical phenomena often leads to nonlinear partial differential equations NLPDEs. The exact solutions of these NLPDEs plays an important role in the understanding of nonlinear phenomena. In the past decades, many methods were developed for finding exact solutions of NLPDEs as the inverse scattering method [1, 2], Hirota's bilinear method [3], Bäcklund transformations [4], similarity transformation method [5], homogeneous balance method [6], tanh function methods [7, 8], Jacobi and Weierstrass elliptic function method [9]-[11], Exp-Function method [12], $\frac{G'}{G}$-expansion method [13, 14], F-expansion method [15]-[17], etc.

Although Porubov et al. [18]-[20] have obtained some exact periodic solutions to some nonlinear wave equations, they use the Weierstrass elliptic function and involve complicated deducing. A Jacobi elliptic function (JEF) expansion method, which is straightforward and effective, was proposed for constructing periodic wave solutions for some nonlinear evolution equations. The essential idea of this method is similar to the tanh method by replacing the tanh function with some JEFs such as sn, cn and dn. For example, the Jacobi periodic solution in terms of sn may be obtained by applying the sn-function expansion. Many similarly repetitious calculations have to be done to search for the Jacobi doubly periodic wave solutions in terms of cn and dn [21]. Recently F-expansion method [22]- [28] was proposed to obtain periodic wave solutions of NLEEs, which can be thought of as a concentration of JEF expansion since F here stands for everyone of JEFs.

Gilson et al. [29] derived (2+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation while they study (2 + 1)-dimensional generalization of the AKNS shallow-water wave equation through the bilinear approach.

$$u_{yt}+u_{xxxy}-3u_{xx}u_y-3u_xu_{xy}=0, \tag{1}$$

The (2+1)-dimensional BLMP equation has been studied in many papers such as, in [30] Bäcklund transformation of potential BLMP system with aid of symbolic computation, given solution of the potential BLMP system with three arbitrary functions. The bilinear form for the BLMP equation is obtained using the binary Bell polynomials in [31]. New solutions include rational function solutions, double-twisty function solutions, Jacobi oval function solutions and triangular cycle solutions of BLMP equation are obtained in [32]. Using exponential function, some new exact solutions of BLMP equation are obtained[33].Using the binary Bell polynomials, the bilinear form for the BLMP equation is obtained in [34]. In this paper, we apply the extended F-expansion (EFE) method with symbolic computation to Eq. (1) for constructing their interesting Jacobi doubly periodic wave solutions. It is shown that soliton solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions. In addition the algorithm that we use here also a computerized method, in which generating an algebraic system.

2. Materials and Methods (Extended F-expansion method)

In this section, we introduce a simple description of the EFE method, for a given partial differential equation

$$G(u,u_x,u_y,u_z,u_t,u_{xy},\ldots)=0. \qquad (2)$$

We like to know whether travelling waves (or stationary waves) are solutions of Eq. (2). The first step is to unite the independent variables x, y, z and t into one particular variable through the new variable

$$\zeta=\alpha x+\beta y+\gamma z+vt, \quad u(x,y,t)=U(\zeta),$$

where v is wave speed. Thus Eq. (2) can be reduced to an ordinary differential equation(ODE)

$$G(U,U',U'',U''',\ldots)=0. \qquad (3)$$

Our main goal is to derive exact or at least approximate solutions, if possible, for this ODE. For this purpose, we shall consider U as the expansion in the form,

$$u(x,y,z,t)=U(\zeta)=\sum_{i=0}^{N}a_iF^i+\sum_{i=1}^{N}a_{-i}F^{-i}, \qquad (4)$$

where

$$F'=\sqrt{A+BF^2+CF^4}, \qquad (5)$$

the highest degree of $\frac{d^pU}{d\zeta^p}$ is taken as

$$O\left(\frac{d^pU}{d\zeta^p}\right)=N+p, \qquad p=1,2,3,\cdots, \qquad (6)$$

$$O\left(U^q\frac{d^pU}{d\zeta^p}\right)=(q+1)N+p, \qquad (7)$$
$$q=0,1,2,\cdots,p=1,2,3,\cdots.$$

Where A, B and C are constants, and N in Eq. (3) is a positive integer that can be determined by balancing the nonlinear term(s) and the highest order derivatives. Normally N is a positive integer, so that an analytic solution in closed form may be obtained. Substituting Eqs. (2)- (5) into Eq. (3) and comparing the coefficients of each power of $F(\zeta)$ in both sides lead to get an over-determined system of nonlinear algebraic equations with respect to v, a_0, a_1, $\cdots$. Solving the over-determined system of nonlinear algebraic equations by use of Mathematica. The relations between values of A, B, C and corresponding JEF solution $F(\zeta)$ of Eq. (4) are given in table 1. Substitute the values of A, B, C and the corresponding JEF solution $F(\zeta)$ chosen from table 1 into the general form of solution, then an ideal periodic wave solution expressed by JEF can be obtained.

Table 1: Relation between values of (A,B,C) and corresponding F

A	B	C	$F(\zeta)$
1	-1-m^2	m^2	sn(ζ) or cd(ζ) $=\frac{cn(\zeta)}{dn(\zeta)}$
1-m^2	2m^2-1	-m^2	cn(ζ)
m^2-1	2-m^2	-1	dn(ζ)
m^2	-1-m^2	1	ns(ζ) $\frac{1}{sn(\zeta)}$ or dc(ζ)= $\frac{dn(\zeta)}{cn(\zeta)}$
-m^2	2m^2-1	1-m^2	nc(ζ)= $\frac{1}{cn(\zeta)}$
-1	2-m^2	m^2-1	nd(ζ)= $\frac{1}{dn(\zeta)}$
1	2-m^2	1-m^2	sc (ζ)= $\frac{sn(\zeta)}{cn(\zeta)}$
1	2m^2-1	-m^2(-1-m^2)	sd(ζ)= $\frac{sn(\zeta)}{dn(\zeta)}$
1-m^2	2-m^2	1	cs(ζ)= $\frac{cn(\zeta)}{sn(\zeta)}$
-$m^2\times$(1-m^2)	2m^2-1	1	ds(ζ)= $\frac{dn(\zeta)}{sn(\zeta)}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	ns(ζ)+cs (ζ)
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{2}$	nc(ζ)+sc (ζ)
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	ns(ζ)+ds (ζ)
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	sn(ζ)+$ics(\zeta)$

Where sn (ζ), cn (ζ) and dn (ζ) are the JE sine function, JE cosine function and the JEF of the third kind, respectively. Also

$$cn^2(\zeta)=1-sn^2(\zeta), \quad dn^2(\zeta)=1-m^2sn^2(\zeta), \qquad (8)$$

with the modulus m $(0<m<1)$.

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.,
$sn\zeta \to tanh\zeta,\ cn\zeta \to sech\zeta,\ dn\zeta \to sech\zeta,$
when $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.,
$sn\zeta \to sin\zeta,\ cn\zeta \to cos\zeta$ and $dn \to 1$.

3. Results

3.1. Boiti-Leon-Manna-Pempinelli equation

In this section, we will apply extended method to study the (2+1)-dimensional BLMP equation (1)

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} = 0. \qquad (9)$$

If we use $\zeta = \alpha x + \beta y + vt$, $u(x,y,t)=U(\zeta)$ carries PDE (9) into the ODE

$$\beta v U'' + \beta\alpha^3 U^{(4)} - 6\beta\alpha^2 U' U'' = 0, \qquad (10)$$

where by integrating once we obtain, upon setting the constant of integration to zero,

$$\beta v U' + \beta\alpha^3 U^{(3)} - 3\beta\alpha^2 (U')^2 = 0, \qquad (11)$$

Setting $u' = v$, Eq. (11) becomes

$$\beta v V + \beta\alpha^3 V'' - 3\beta\alpha^2 V^2 = 0. \qquad (12)$$

Balancing the term V'' with the term V^2 we obtain $N = 2$ then

$$V(\zeta) = a_0 + a_1 F + a_{-1}F^{-1} + a_2 F^2 + a_{-2}F^{-2},$$
$$F' = \sqrt{A + BF^2 + CF^4}. \qquad (13)$$

Substituting Eq. (13) into Eq. (12) and comparing the coefficients of each power of F in both sides lead to get an over-determined system of nonlinear algebraic equations with respect to v, a_i, $i = 1,\ -1,\ -2,\ 2$. Solving the over-determined system of nonlinear algebraic equations by use of Mathematica, we obtain three groups of constants:

1. $$a_{-1} = a_1 = 0,\ a_0 = -\frac{2\alpha(B + \sqrt{B^2\alpha^2 + 12AC})}{3},\ a_2 = 2C\alpha,\ a_{-2} = 2A\alpha,$$
and $v = \pm 4\alpha^3\sqrt{B^2 + 12AC}$, (14)

2. $$a_{-1} = a_1 = a_2 = 0,\ a_0 = -\frac{2\alpha(B + \sqrt{B^2\alpha^2 - 3AC})}{3},\ a_{-2} = 2A\alpha,$$
and $v = \pm 4\alpha^3\sqrt{B^2 - 3AC}$, (15)

3. $$a_{-1} = a_1 = a_{-2} = 0,\ a_0 = -\frac{2\alpha(B + \sqrt{B^2\alpha^2 - 3AC})}{3},\ a_2 = 2C\alpha, \qquad (16)$$
and $v = \pm 4\alpha^3\sqrt{B^2 - 3AC}$.

Letting $\varphi = 0.5Am[(\alpha x + \beta y + vt)\,|\,m]$, $R = E[Am[(\alpha x + \beta y + vt)\,|\,m]\,|\,m]$, recalling $\zeta = \alpha x + \beta y + vt$, and using Eqs.(14)-(16), we can obtain the electrostatic potentials of Eq. (9) as follows:

$$u_1 = \frac{2}{3}[((2+(3-m)m)\alpha + \sqrt{(1+14m^2+m^4)\alpha^2})(\zeta) - \frac{3\alpha cn(\zeta)dn(\zeta)}{sn(\zeta)} - \frac{3(1+m)\alpha R dn(\zeta)}{\sqrt{1-msn^2(\zeta)}}],$$
$$v = \pm 4\alpha^3\sqrt{(12m^2 + (1+m^2)^2)}, \qquad (17)$$

$$u_2 = \frac{1}{3cn(\zeta)\sqrt{1-m+mcn^2(\zeta)}\sqrt{1-msn^2(\zeta)}} \times [(6m^2\alpha cn^2(\zeta)dn(\zeta)sn(\zeta) + 6\alpha dn^3(\zeta)sn(\zeta) - 2cn(\zeta)\sqrt{1-m+mcn^2(\zeta)} \times (\zeta(\alpha + m^2\alpha - \sqrt{(1+14m^2+m^4)\alpha^2})\sqrt{1-msn^2(\zeta)} - 3(1+m)\alpha(\zeta - R)dn(\zeta)))],$$
$$v = \pm 4\alpha^3\sqrt{(12m^2 + (1+m^2)^2)}, \qquad (18)$$

$$u_3 = -\frac{2}{3\sqrt{1-msn^2(\zeta)}}[\zeta((1+(-3+m)m\alpha - \sqrt{(1-16m^2+16m^4)\alpha^2})\sqrt{1-msn^2(\zeta)} + 3\alpha dn(\zeta)((-1+m^2)(\zeta) + (1+2m)R - (1+m)\sqrt{1-m+mcn^2(\zeta)}sc(\zeta))],$$
$$v = \pm 4\alpha^3\sqrt{(12m^2 + (-1-m^2)^2)} \qquad (19)$$

$$u_4 = \frac{1}{3\sqrt{1-m+mcn^2(\zeta)}\sqrt{1-msn^2(\zeta)}} \times [6\alpha dn(\zeta)(m(1+m)cn(\zeta)sn(\zeta) - (2+m)R\sqrt{1-m+mcn^2(\zeta)}) + 2\alpha(\zeta)\sqrt{1-m+mcn^2(\zeta)}\sqrt{1-msn^2(\zeta)} \times ((2-m^2) + \sqrt{16-16m^2+m^4})],$$
$$v = \pm 4\alpha^3\sqrt{12m^2 + (-1-m^2)^2} \qquad (20)$$

$$
\begin{aligned}
u_5 = {} & \frac{1}{3\sqrt{1-msn^2(\zeta)}}[-6\alpha dn(\zeta)((2+m)R \\
& +dn(\zeta)(cs(\zeta)-(1+m)sc(\zeta))) \\
& +2((\zeta)((2-m^2)\alpha \\
& +\alpha\sqrt{16-16m^2+m^4})\sqrt{1-msn^2(\zeta)})], \\
& \nu=\pm 4\alpha^3\sqrt{12(1-m^2)+(2-m^2)^2}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
u_6 = {} & \frac{1}{(m-1)\sqrt{1-msn^2(\zeta)}}[6m^2\alpha(\zeta) \\
& -\frac{3\alpha dn(\zeta)}{\sqrt{1-msn^2(\zeta)}}((m-1)(\zeta)+R+\frac{cn(\zeta)dn(\zeta)}{sn(\zeta)}) \\
& -3m(1+m^2)\alpha dn(\zeta)((m-1)(\zeta) \\
& +R-\frac{mcn(\zeta)sn(\zeta)}{\sqrt{1-m+mcn^2(\zeta)}})], \\
& \nu=\pm 4\alpha^2\sqrt{12m^2(1+m^2)+(-1+2m^2)^2}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
u_7 = {} & \frac{1}{dn(\zeta)sn(\zeta)\sqrt{1-msn^2(\zeta)}}[0.5m\alpha Rsn^3(\zeta) \\
& +0.5\alpha cn(\zeta)dn^2(\zeta)\sqrt{1-1msn^2(\zeta)} \\
& +sn(\zeta)(-0.5\alpha R+(\zeta)(0.833333\alpha \\
& -0.66667m^2\alpha+0.66667\sqrt{(1-m^2+m^4)\alpha^2}) \\
& \times dn(\zeta)\sqrt{1-msn^2(\zeta)}+\alpha dn^2(\zeta)(0.5\zeta-1.5R \\
& +\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}(-0.353553cs(\zeta) \\
& -0.707107ns(\zeta)+0.707107tan(\varphi))))], \\
& \nu=\pm 4\alpha^3\sqrt{0.75+(0.5-m^2)^2}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
u_8 = {} & \frac{2}{3}[(0.5+0.5m^2)(\zeta)\alpha+(\zeta)\alpha\sqrt{1.75(1+m^4)-2.5m^2} \\
& +((2.121320343559643(-1+m^2)\alpha dn(\zeta)nc(\zeta)) \\
& \times\frac{\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}}{(m-1)\sqrt{1-msn^2(\zeta)}} \\
& +\frac{(1.5(0.5-0.5m^2)\alpha dn(\zeta)2R}{(m-1)\sqrt{1-msn^2(\zeta)}} \\
& -\frac{1.4142135623731sc(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}}{(m-1)\sqrt{1-msn^2(\zeta)}} \\
& +\frac{3(0.5-0.5m^2)\alpha}{m-1}(-\frac{dn(\zeta)sn(\zeta)}{cn(\zeta)}+m(\zeta)-\zeta \\
& +\frac{mR(-1+1/m+cn^2(\zeta))}{dn(\zeta)\sqrt{1-msn^2(\zeta)}}) \\
& +\frac{1}{((m-1)\sqrt{1-msn^2(\zeta)}(cos(\varphi)+sin(\varphi)))} \\
& \times(0.7071067811865475(0.75(1-m^2)\alpha dn(\zeta) \\
& \times(2\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)} \\
& \times(cos(\varphi)-sin(\varphi)) \\
& +1.4142135623731(m-1)(\zeta) \\
& \times(cos(\varphi)+sin(\varphi)) \\
& +2.8284271247461903R \\
& \times(cos(\varphi)+sin(\varphi))))], \\
& \nu=\pm 4\alpha^3\sqrt{12(0.5-0.5m^2)(0.25-0.25m^2)+(0.5+0.5m^2)^2},
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
u_9 = {} & \frac{2}{3}[(0.5+0.5m^2)(\zeta)\alpha+(\zeta)\sqrt{(1.75-2.5m^2+1.75m^4)\alpha^2} \\
& +\frac{\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}}{(m-1)\sqrt{1-msn^2(\zeta)}} \\
& \times(-2.12132+2.12132m^2)\alpha dn(\zeta)nc(\zeta) \\
& +\frac{1.5(0.5-0.5m^2)\alpha dn(\zeta)2R}{(m-1)\sqrt{1-msn^2(\zeta)}} \\
& -\frac{1.41421sc(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)})}{(m-1)\sqrt{1-msn^2(\zeta)}} \\
& +\frac{3\alpha(0.5-0.5m^2)}{m-1}(-\frac{dn(\zeta)sn(\zeta)}{cn(\zeta)}+(m-1)(\zeta) \\
& +\frac{mR(1-m+mcn^2(\zeta))}{dn(\zeta)\sqrt{1-msn^2(\zeta)}})) \\
& +\frac{0.707107(0.75-0.75m^2)\alpha dn(\zeta)}{(m-1)\sqrt{1-msn^2(\zeta)}(cos(\varphi)+sin(\varphi))} \\
& \times(2\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)} \\
& \times(cos(\varphi)-sin(\varphi)) \\
& +1.41421(m-1)(\varphi)(cos(\varphi)+sin(\varphi)) \\
& +2.82843R(\cos(\varphi)+sin(\varphi)))], \\
& \nu=\pm 4\alpha^3\sqrt{(0.75m^2+(-1+0.5m^2)^2}
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
u_{10} = {} & \frac{2}{3}[(2-m^2+\sqrt{1-m^2+m^4})(\zeta)\alpha \\
& +\frac{3(1+m)\alpha dn(\zeta)(R-dn(\zeta)sc(\zeta))}{\sqrt{1-msn^2(\zeta)}}], \\
& \nu=\pm 4\alpha^3\sqrt{3(m^2-1)+(2-m^2)^2},
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
u_{11} = {} & \frac{2}{3}[(3m-1-m^2+\sqrt{1-m^2+m^4})(\zeta)\alpha \\
& -\frac{3m\alpha R\sqrt{1-msn^2(\zeta)}}{dn(\zeta)}], \\
& \nu=\pm 4\alpha^3\sqrt{-3m^2+(-1-m^2)^2},
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
u_{12} = {} & \frac{2}{3}[(3m-1-m^2+\sqrt{1-m^2+m^4})(\zeta)\alpha \\
& -\frac{3\alpha cn(\zeta)dn(\zeta)}{sn(\zeta)}-\frac{3\alpha R\sqrt{1-msn^2(\zeta)}}{dn(\zeta)}], \\
& \nu=\pm 4\alpha^3\sqrt{-3m^2+(-1-m^2)^2},
\end{aligned}
\tag{28}
$$

$$\begin{aligned} u_{13} = &\frac{2}{3}[(\sqrt{1-m^2+m^4}-(1+m^2))(\zeta)\alpha \\ &-\frac{3\alpha dn(\zeta)}{\sqrt{1-msn^2(\zeta)}}\times((\zeta)-R \\ &+\frac{sc(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}}{\sqrt{2}})], \\ &v=\pm 4\alpha^3\sqrt{-3m^2+(-1-m^2)^2}, \end{aligned} \tag{29}$$

$$\begin{aligned} u_{14} = &\frac{2}{3}[(-(1+m^2)+\sqrt{1-m^2+m^4})(\zeta)\alpha \\ &-\frac{3m(\zeta)\alpha dn(\zeta)}{\sqrt{1-msn^2(\zeta)}}-\frac{3m\alpha R dn(\zeta)}{\sqrt{1-msn^2(\zeta)}}+ \\ &\frac{3\sqrt{2}m^2\alpha cn(\zeta)dn(\zeta)sn(\zeta)}{\sqrt{1-msn[x,m]^2}\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}}], \\ &v=\pm 4\alpha^3\sqrt{-3m^2+(-1-m^2)^2}, \end{aligned} \tag{30}$$

$$\begin{aligned} u_{15} = &\frac{2}{3}[(3m-3m^2-1+2m^2+\sqrt{1-m^2+m^4})(\zeta)\alpha \\ &-\frac{3m\alpha R(1-m+mcn^2(\zeta))}{dn(\zeta)\sqrt{1-msn^2(\zeta)}}], \\ &v=\pm 4\alpha^3\sqrt{3m^2(1-m^2)+(-1+2m^2)^2}, \end{aligned} \tag{31}$$

$$\begin{aligned} u_{16} = &\frac{2}{3}[((2-m^2)+\sqrt{1-m^2+m^4})(\zeta)\alpha \\ &-\frac{3m\alpha R dn(\zeta)}{\sqrt{1-msn^2(\zeta)}}], \\ &v=\pm 4\alpha^3\sqrt{(2-m^2)^2+3(-1+m^2)}, \end{aligned} \tag{32}$$

$$\begin{aligned} u_{17} = &\frac{2}{3}[(2-m^2+\sqrt{1-m^2+m^4})(\zeta)\alpha \\ &+\frac{3(1+m)\alpha cn(\zeta)sn(\zeta)}{dn(\zeta)}+\frac{3(1+m)\alpha dn(\zeta)R}{\sqrt{1-msn^2(\zeta)}}], \\ &v=\pm 4\alpha^3\sqrt{(2-m^2)^2+3(-1+m^2)}, \end{aligned} \tag{33}$$

$$\begin{aligned} u_{18} = &\frac{2}{3}[(2-m^2+\sqrt{1-m^2+m^4})(\zeta)\alpha \\ &+\frac{2(1+m)\alpha dn(\zeta)sn(\zeta)}{cn(\zeta)}+\frac{2(1+m)\alpha dn(\zeta)R}{\sqrt{1-msn^2(\zeta)}}], \\ &v=\pm 4\alpha^3\sqrt{3m^2(1-m^2)+(1-2m^2)^2}, \end{aligned} \tag{34}$$

$$\begin{aligned} u_{19} = &\frac{2}{3}[(2m^2-1+\sqrt{1-7m^2+m^4})(\zeta)\alpha \\ &-\frac{3m(1+m^2)\alpha dn(\zeta)}{(m-1)\sqrt{2(1-msn^2(\zeta))}}(\sqrt{2}(m-1)(\zeta) \\ &+\sqrt{2}R-\frac{cn(\zeta)sn(\zeta)}{\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}})], \\ &v=\pm 4\alpha^3\sqrt{(1-2m^2)^2-3m^2(1+m^2)}, \end{aligned} \tag{35}$$

$$\begin{aligned} u_{20} = &\frac{2}{3}[(2-m^2+\sqrt{1-m^2+m^4})(\zeta)\alpha \\ &-\frac{3\alpha dn(\zeta)}{2\sqrt{2(1-msn^2(\zeta))}}(2R \\ &+\sqrt{2}cs(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)})], \\ &v=\pm 4\alpha^3\sqrt{3(m^2-1)+(2-m^2)^2}, \end{aligned} \tag{36}$$

$$\begin{aligned} u_{21} = &\frac{2}{3}[(2m^2-1+\sqrt{1-m^2+m^4})(\zeta)\alpha \\ &-\frac{3\alpha dn(\zeta)}{2\sqrt{1-msn^2(\zeta)}}(2(m-1)(\zeta)+2R \\ &+\sqrt{2}cs(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)})], \\ &v=\pm 4\alpha^3\sqrt{3m^2(1-m^2)+(-1+2m^2)^2}, \end{aligned} \tag{37}$$

$$\begin{aligned} u_{22} = &\frac{2}{3}[(0.83333-0.66667(m^2-\sqrt{0.0625-1.m^2+m^4}))(\zeta)\alpha \\ &-\frac{0.5\alpha cn(\zeta)dn(\zeta)}{sn(\zeta)}-\frac{0.5\alpha R\sqrt{1-msn^2(\zeta)}}{dn(\zeta)}-\frac{1}{\sqrt{1-msn^2(\zeta)}} \\ &\times(0.707107dn(\zeta)ns(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)} \\ &-0.25\alpha dn(\zeta)(2R \\ &+1.41421cs(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}))], \\ &v=\pm 4\alpha^3\sqrt{(0.5-m^2)^2-0.1875}, \end{aligned} \tag{38}$$

$$\begin{aligned} u_{23} = &\frac{2}{3}[(2-m^2+\sqrt{1.25m^2-0.125-0.125m^4})(\zeta)\alpha-\frac{1}{(m-1)\sqrt{1-msn^2(\zeta)}} \\ &\times((2-2m^2)\alpha dn(\zeta)nc(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)} \\ &+1.5(0.5-0.5m^2)\alpha dn(\zeta)(2R \\ &-1.41421sc(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}) \\ &+3(0.5-0.5m^2)\alpha(\frac{R(1-m+mcn^2(\zeta))}{dn(\zeta)}-\frac{dn(\zeta)sn(\zeta))\sqrt{1-msn^2(\zeta)}}{cn(\zeta)}))], \\ &v=\pm 4\alpha^3\sqrt{3(0.5m^2-0.5)(0.25-0.25m^2)+(0.5+0.5m^2)^2}, \end{aligned} \tag{39}$$

$$u_{24} = \frac{2}{3}[(0.833333m^2 - 0.666667(1-\sqrt{1-1.1875m^2+0.25m^4}))(\zeta)\alpha$$
$$-\frac{0.25(m^2)\alpha dn(\zeta)}{\sqrt{1-msn^2(\zeta)}}(2(m-1)(\zeta)+2R$$
$$+1.41421cs(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)})$$
$$-m^2\alpha cs(\zeta)-\frac{0.5m^2\alpha cn((\zeta)dn(\zeta)}{sn(\zeta)}-\frac{0.5m^2\alpha R\sqrt{1-msn^2(\zeta)}}{dn(\zeta)})],$$

$$v=\pm 4\alpha^3\sqrt{(1-0.5m^2)^2-0.1875m^2}, \tag{40}$$

$$u_{25} = \frac{2}{3}[(0.333333m^2+0.5m-0.666667(1-\sqrt{1-m^2+0.0625m^4}))(\zeta)\alpha$$
$$-\frac{dn(\zeta)}{\sqrt{1-msn^2(\zeta)}}(i\,m^{\frac{3}{2}}\arctan(\frac{1.41421\sqrt{m}sn(\zeta)}{\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}}$$
$$-\frac{0.5m^2\alpha R(1-sn^2(\zeta))}{dn^2(\zeta)})$$
$$+0.35m^2cs(\zeta)\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}+\frac{m^2}{2}R)],$$

$$v=\pm 4\alpha^3\sqrt{(1-0.5m^2)^2-0.1875m^4}, \tag{41}$$

$$u_{26} = \frac{2}{3}[(0.5m^2+0.5+\sqrt{1.25m^2-0.125(1+m^4)})(\zeta)\alpha$$
$$-\frac{0.707107(0.75-0.75m^2)\alpha dn(\zeta)}{(m-1)\sqrt{1-msn^2(\zeta)}(cos(\varphi)+sin(\varphi))}$$
$$\times(1.41421(m-1)(\zeta)(cos(\varphi)+sin(\varphi))$$
$$+2.82843R(cos(\varphi)+sin(\varphi))$$
$$+2\sqrt{2-m+mcn^2(\zeta)-msn^2(\zeta)}$$
$$\times(cos(\varphi)-sin(\varphi)))],$$
$$v=\pm 4\alpha^3\sqrt{3(0.5m^2-0.5)(0.25-0.25m^2)+(0.5+0.5m^2)^2}, \tag{42}$$

$$u_{27} = \frac{2}{3}[(0.33333(m^2+2)+\sqrt{1-1.1875m^2+0.25m^4})(\zeta)\alpha$$
$$-\frac{\alpha dn(\zeta)}{m^2\sqrt{1-msn^2(\zeta)}}((\zeta)-0.5m(\zeta)$$
$$-R-0.707107\sqrt{2-m+mcos(4\varphi)}cs(\zeta)$$
$$+\frac{cs(\zeta)\sqrt{1-msn^2(\zeta)}}{dn(\zeta)}))],$$
$$v=\pm 4\alpha^3\sqrt{(1-0.5m^2)^2-0.1875m^2}, \tag{43}$$

3.2. Soliton Solutions

Some solitary wave solutions can be obtained, if the modulus m approaches to 1 in Eqs. (17)-(43), letting $\chi = \alpha x + \beta y$, we obtain

$$u_{28} = \frac{2\alpha}{3}(8(\chi\pm 16\alpha^3 t)-3\coth(\chi\pm 16\alpha^3 t)-3\tanh(\chi\pm 16\alpha^3 t)), \tag{44}$$

$$u_{29} = \frac{2\alpha}{3}(2(\chi\pm 16\alpha^3 t)-3\tanh(\chi\pm 4\alpha^3 t)), \tag{45}$$

$$u_{30} = \frac{2\alpha}{3}(2(\chi\pm 16\alpha^3 t)-3\coth(\chi\pm 4\alpha^3 t)), \tag{46}$$

$$u_{31} = \frac{2\alpha}{3}(3(\chi\pm 20\alpha^3 t)-3\coth(\chi\pm 20\alpha^3 t)+\frac{3}{2}\sinh(2(\chi\pm 20\alpha^3 t))), \tag{47}$$

$$u_{32} = \frac{4\alpha}{3}(\chi\pm 4\alpha^3 t)-\alpha\coth(0.5(\chi\pm 4\alpha^3 t))-\alpha\tanh(0.5(\chi\pm 4\alpha^3 t)), \tag{48}$$

$$u_{33} = \frac{10\alpha}{3}(\chi\pm\alpha^3 t)+\alpha sech(0.5(\chi\pm\alpha^3 t))$$
$$\times(5.55112\times 10^{-17}\cosh(0.5(\chi\pm\alpha^3 t))-\sinh(0.5(\chi\pm\alpha^3 t))), \tag{49}$$

$$u_{34} = \frac{2\alpha}{3}(-2+i\sqrt{5})(\chi\pm i\sqrt{5}\alpha^3 t)+\frac{3}{2}\sinh(2(\chi\pm i\sqrt{5}\alpha^3 t))), \tag{50}$$

$$u_{35} = \frac{4\alpha}{3}(\alpha x+\beta y\pm 4\alpha^3 t)+\alpha\cosh(2(\chi\pm 4\alpha^3 t))+\alpha\sinh(2(\chi\pm 4\alpha^3 t)), \tag{51}$$

$$u_{36} = \frac{10\alpha}{3}(\chi\pm\alpha^3 t)+2.i\,\alpha\arctan(\tanh(0.5(\chi\pm\alpha^3 t)))$$
$$+0.25\alpha\coth(0.5(\chi\pm\alpha^3 t))+0.25\alpha\tanh(0.5(\chi\pm\alpha^3 t))$$
$$-0.5\alpha\tanh(\chi\pm\alpha^3 t), \tag{52}$$

3.3. Triangular Periodic Solutions

Some trigonometric function solutions can be obtained, if the modulus m approaches to zero in Eqs. (17)-(43)

$$u_{37} = -2\alpha\cot(\chi\pm 4\alpha^3 t), \tag{53}$$

$$u_{38} = 2\alpha\tan(\chi\pm 4\alpha^3 t), \tag{54}$$

$$u_{39} = 2\alpha\tan(\chi\pm 4\alpha^3 t)-2\alpha\cot(\chi\pm 4\alpha^3 t), \tag{55}$$

$$u_{40} = \alpha\tan(0.5(\chi\pm 4\alpha^3 t))-\alpha\cot(0.5(\chi\pm 4\alpha^3 t)), \tag{56}$$

$$u_{41} = 0.784783\alpha(\chi\pm 5.3\alpha^3 t)$$
$$+\frac{4\alpha\sin(0.5(\chi\pm 5.3\alpha^3 t))}{\cos(0.5(\chi\pm 5.3\alpha^3 t))-\sin(0.5(\chi\pm 5.3\alpha^3 t))}$$
$$+\frac{2\alpha\sin(0.5(\chi\pm 5.3\alpha^3 t))}{\cos(0.5(\chi\pm 5.3\alpha^3 t))+\sin(0.5(\chi\pm 5.3\alpha^3 t))}, \tag{57}$$

$$u_{42} = 0.0625(\chi \pm 4\alpha^3 t)\alpha - 0.03125\alpha\sin(2(\chi \pm 4\alpha^3 t)), \tag{58}$$

$$u_{43} = (-0.666667 + 0.235702i)\alpha(\chi \pm 1.414212i\alpha^3 t) + \frac{4\alpha\sin(0.5(\chi \pm 1.414212i\alpha^3 t))}{\cos(0.5(\chi \pm 1.414212i\alpha^3 t)) - \sin(0.5(\chi \pm 1.414212i\alpha^3 t))}, \tag{59}$$

The modulus of solitary wave solutions u_1 and u_2 are displayed in figures 1 and 2 respectively, with values of parameters listed in their captions.

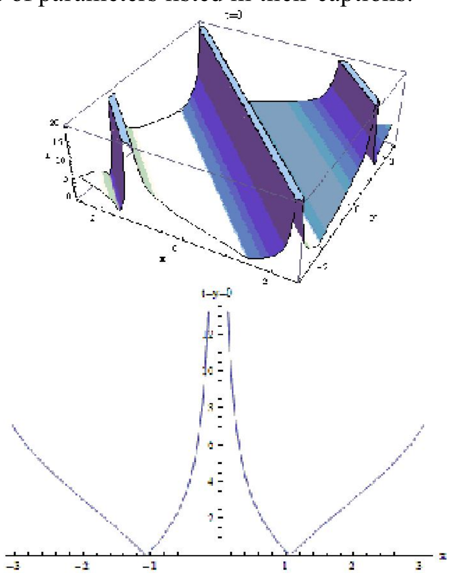

Fig. 1 The modulus of solitary wave solution u_1 (Eq. 17) where $\alpha = \beta = 1$, $m = 0.5$ and $t = 0$.

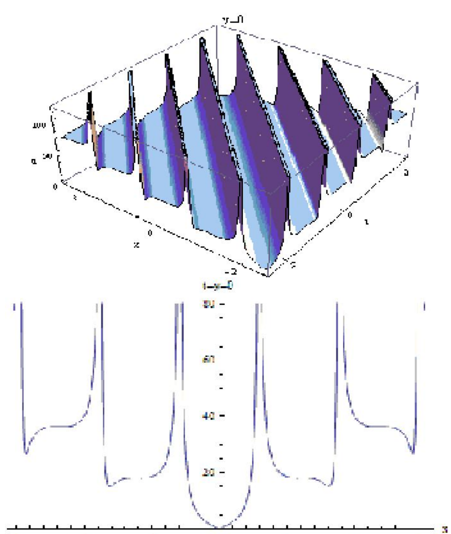

Fig. 2 The modulus of solitary wave solution u_2 (Eq. 18) where $\alpha = \beta = \pi$, $m = 0.5$ and $t = 0$.

4. Discussion

The investigation of exact solutions is the key of understanding the nonlinear physical phenomena. It is known that many physical phenomena are often described by NLPDEs. Many methods for obtaining exact travelling solitary wave solutions to NLPDEs have been proposed. By introducing appropriate transformations and using extended F-expansion method, we have been able to obtain in a unified way with the aid of symbolic computation system-mathematica, a series of solutions including single and the combined Jacobi elliptic function. Moreover, it is shown that soliton solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions. For $m \to 1$, the above solutions are obtained using the hyperbolic and extended hyperbolic functions method. Where $m \to 0$, these solutions are equivalent to these obtained using the triangular and etended triangular functions method.

5. Conclusion

By introducing appropriate transformations and using extended F-expansion method, we have been able to obtain in a unified way with the aid of symbolic computation system-mathematica, a series of solutions including single and the combined Jacobi elliptic function. Also, extended F-expansion method is shown that soliton solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions. When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions and given the solutions by the extended hyperbolic functions methods. When $m \to 0$, the Jacobi functions degenerate to the triangular functions and given the solutions by extended triangular functions methods.

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