On a Subclass of Analytic Functions Related with Janowski Functions

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Abstract. In this paper, we introduce a subclass of analytic functions by using the well-known Hadamard product along with Janowski functions. Some inclusion results, radius of univalence and other interesting properties of this class are discussed.
[Arif M, Mustafa S and Khan K. **On a Subclass of Analytic Functions Related with Janowski Functions.** *Life Sci J* 2012;9(4):3756-3762] (ISSN:1097-8135). http://www.lifesciencesite.com. 557

Key Words: Convolution, Carlson and Shaffer operator, close-to-convex functions.
2010 AMS Subject Classification: 30C45, 30C10.

1. Introduction

Let A be the class of analytic functions f

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \qquad (1.1)$$

defined in the open unit disk $E=\{z:|z|<1\}$. The class A is closed under the convolution or (Hadamard product)

$$(f*g)(z)=\sum_{n=2}^{\infty}a_nb_n z^{n+1},\ a_0=b_0=1,\ z\in E,$$

where

$$f(z)=\sum_{n=0}^{\infty}a_nz^{n+1} \text{ and } g(z)=\sum_{n=0}^{\infty}b_nz^{n+1}.$$

In particular, we consider the convolution with incomplete beta function $\phi(d,c,z)$, related to Gauss hypergeometric function [1] by

$$\phi(d,c,z)=z\ {}_2F_1(1,d,c,z)=\sum_{n=0}^{\infty}\frac{(d)_n}{(c)_n}z^{n+1}, \qquad (1.2)$$

$$z\in E,\ c\neq 0,-1,-2,\ldots,$$

where $(d)_n$ denotes the Pochhammer symbol given by

$$(d)_n=\begin{cases}1, & n=0,\ d\in\mathbb{C}\setminus\{0\},\\ a(a+1),\cdots(a+n-1), & n\in\mathbb{N}.\end{cases}$$

Note that $\phi(d,1,z)=\frac{z}{(1-z)^d}$ and $\phi(2,1,z)$ is the Koebe function, see [2].

Carlson and Shaffer [3] defined a convolution operator on A involving an incomplete beta function as

$$L(d,c)f=\phi(d,c,z)*f,\ f\in A,\ z\in E. \qquad (1.3)$$

It follows from (1.2) and (1.3) that,

$$z(L(d,c)f(z))'=dL(d+1,c)f(z)-(d-1)L(d,c)f(z). \qquad (1.4)$$

If $d=0,-1,-2,\ldots,$ then $L(d,c)f$ is a polynomial. If $d\neq 0,-1,-2,\ldots,$ then application of the root test shows that the infinite series for $L(d,c)f$ has the same radius of convergence as that for f because

$$\lim_{n\to\infty}\left|\frac{(d)_n}{(c)_n}\right|^{\frac{1}{n}}=1.$$

Hence $L(d,c)$ maps A into itself. $L(d,d)$ is the identity and if $d\neq 0,-1,-2,\cdots,$ then $L(d,c)$ has a continuous inverse $L(c,d)$ and is a one-to-one mapping of A onto itself. $L(d,c)$ provides a convenient representation of differentiation and integration. If $g(z)=zf'(z)$, then $g=L(2,1)f$ and $f=L(1,2)g$.

In [4], Janowski introduced the class $P[A,B]$, for A and B, $-1\leq B<A\leq 1$, if and only if for $z\in E$, a function p, analytic in E with $p(0)=1$ belongs to the class $P[A,B]$ if

$$p(z)=\frac{1+Aw(z)}{1+Bw(z)},$$

where $w(z)$ is an analytic function in E, with $w(0)=0$, and $|w(z)|<1$.

We note that $P[-1,1]=P$, the class of functions with positive real part consists of functions

p with $\operatorname{Re} p(z) > \alpha$. Also it can be clearly seen that $P[A,B] \subset P(\beta)$, $\beta = \frac{1-A}{1-B}$, therefore, we have

$$p(z) = (1-\beta)p_1(z) + \beta, \ p_1 \in P. \quad (1.5)$$

Let $P_k[\alpha, A, B]$ denote the class of functions p that are analytic in E with $p(0)=1$ and are represented by

$$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \quad (1.6)$$

where $p_1, p_2 \in P[\alpha, A, B]$, $-1 \le B < A \le 1$, $0 \le \alpha < 1$, $k \ge 2$. It is clear that

$$P_k[\alpha, A, B] \subset P_k[A,B] \subset P_k(\beta),$$

$$\beta = \tfrac{1-A_1}{1-B_1}, \ A_1 = (1-\beta)A + \beta B$$

and $P_k[1,-1] = P_k$ [5]. For $k=2$, the class $P_k[\alpha, A, B] = P_2[\alpha, A, B]$ was introduced by [6].

We will assume throughout our discussion, unless otherwise stated, that $d \ne 0, -1, -2, \cdots$, $c \ne 0, -1, -2, \cdots$, $-1 \le B < A \le 1$, $0 \le \alpha < 1$ and $z \in E$.

By using a linear operator $L(d,c)$, we define the following analytic classes.

Definition 1.1. Let $f \in A$, $z \in E$. Then $f \in C_{d,c}[\alpha, A, B]$, if and only if

$$\frac{(z(L(d,c)f(z)')'}{(L(d,c)f(z))'} \in P[\alpha, A, B]. \quad (1.7)$$

From (1.7), it is clear that
$f \in C_{d,c}[\alpha, A, B] \Leftrightarrow L(d,c)f \in C[\alpha, A, B] \subset C.$

Similarly, $f \in S^*_{d,c}[\alpha, A, B]$, if and only if,

$$L(d,c)f \in S^*[\alpha, A, B] \subset S^*.$$

Definition 1.2. Let $f \in A$. Then $f \in Q_k^{d,c}[\beta, \alpha, A, B]$, if and only if, for $\beta \ge 0$,

$$\left\{(1-\beta)\frac{z(L(d,c)f(z))'}{(L(d,c)g(z))} + \beta\frac{(z(L(d,c)f(z)')'}{(L(d,c)g(z))'}\right\} \in P_k[\alpha, A, B], \quad (1.8)$$

for some $g \in C_{d,c}[\alpha, A, B]$.

By varying the parameters c, d, α, β, A and B in the class $Q_k^{d,c}[\beta, \alpha, A, B]$, we get different subclasses of analytic and univalent functions studied earlier by various authors, see for example [7-12].

2. Preliminary Lemmas

Lemma 2.1 [13]. Let $0 < d \le c$ and $d \ge 2$ or $c + d \ge 3$. Then the function

$$\varphi(d,c)(z) = \sum_{n=0}^{\infty} \frac{(d)_n}{(c)_n} z^{n+1}, \ z \in E$$

is in the class C of convex univalent functions.

Lemma 2.2 [13]. Let $f \in C$ and $g \in S^*$. Then for every function F analytic in E with $F(0)=1$, we have

$$\frac{(f * Fg)}{(f * g)}(E) \subset \overline{co}(F(E)),$$

where $\overline{co}(F(E))$ denotes the closed convex hull of $F(E)$.

Lemma 2.3 [14]. Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $\Psi(u,v)$ be a complex-valued function satisfying the following conditions:

1) $\Psi(u,v)$ is continuous in a domain $D \subset \mathbb{C}^2$,
2) $(1,0) \in D$ and $\operatorname{Re}(\Psi(1,0)) > 0$,
3) $\operatorname{Re}(\Psi(iu_2, v_1) \le 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \le \frac{-1}{2}(1+u_2^2)$.

If $p(z) = 1 + \sum\limits_{m=1}^{\infty} c_m z^m$ is a function analytic in E, such that $(p(z), zp'(z)) \in D$, $z \in E$ and $\operatorname{Re}(\Psi(p(z), zp'(z)) > 0$ for $z \in E$, then $\operatorname{Re} p(z) > 0$ in E.

3. Main Results

In this section, some properties of the class $Q_k^{d,c}[\beta, \alpha, A, B]$ such as inclusion results, second coefficient bound, its invariant property under convolution with the convex function, covering theorem and radius problem will be discussed.

Theorem 3.1. If $0 < c \le d$, $d \ge \min[2, 3-c]$ and if $0 < a \le c$, $c \ge \min[1, 1-a]$, then

$$Q_k^{d,c}[\beta, \alpha, A, B] \subset Q_k^{d,a}[\beta, \alpha, A, B]$$

$$Q_k^{d,a}[\beta, \alpha, A, B] \subset Q_k^{c,d}[\beta, \alpha, A, B].$$

Proof. To prove the first inclusion, we proceed as follows:

(i) Let $f \in Q_k^{d,c}[\beta,\alpha,A,B]$. Now

$$(1-\beta)\frac{z(L(d,a)f(z))'}{L(d,a)g(z)}+\beta\frac{(z(L(d,a)f(z))')'}{(L(d,a)g(z))'}$$

$$=(1-\beta)\frac{z*[(\phi(d,c)*\phi(c,a)*f)]'}{(\phi(d,c)*\phi(c,a)*g)}+\beta\left[\frac{[(z(\phi(d,c)*\phi(c,a)*f)']'}{\phi(d,a)*\phi(c,a)*g)'}\right]$$

$$=(1-\beta)\left[\frac{\phi(c,a)*\frac{N}{D}(\phi(d,c)*g)}{(\phi(c,a)*\phi(d,c)*g}\right]+\beta\left[\frac{\phi(c,a)*\frac{N'}{D'}(\phi(d,c)*zg')}{\phi(c,a)*\phi(d,c)*g')}\right]. \quad (3.1)$$

Since, from Lemma 2.1, $\phi(c,a)$ is convex, therefore by using Lemma 2.3 to see that the right hand side of (3.1) is contained in the convex hull of the image of E under $[(1-\beta)\frac{N}{D}+\beta\frac{N'}{D'}]$ with

$N(z)=z(\phi(d,c)*f)'$,

$D(z)=\phi(d,c)*g\in C[\alpha,A,B]\subset C\subset S^*$.

This implies $f\in Q_k^{d,a}[\beta,\alpha,A,B]$ for $z\in E$ and the proof of (i) is complete.

The proof of the second inclusion is similar to that of the first part.

Theorem 3.2. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in Q_k^{d,c}[\beta,\alpha,A,B]$. Then

$$|a_2|\le\frac{(1-\alpha)(A-B)(c)(k+1+\beta)}{4d(1+\beta)}.$$

The equality, with $\alpha=0,\ A=1,\ B=-1,$ occurs for $f_0(z)$, given by

$$F_0(z)=L(d,c)f_0(z),\ G_0(z)=L(d,c)g_0(z)=\frac{z}{1-z},$$

$$p_0(z)=(1-\beta)\frac{zF_0'(z)}{G_0'(z)}+\beta\frac{(zF_0'(z))'}{G_0'(z)},$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+z}{1-z}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-z}{1+z}.$$

Proof. Let $L(d,c)g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ and $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$. Since $g\in C_{d,c}[\alpha,A,B]$ and $p\in P_k[\alpha,A,B]$, it easily follows from known results [4] that $|c_n|\le\frac{k}{2}(1-\alpha)(A-B)$, $n>1$ and

$$|b_2|\le\frac{(1-\alpha)(A-B)}{2}.$$

Now

$$(1-\beta)[z+\sum_{n=2}^{\infty}\frac{n(d)_{n-1}}{(c)_{n-1}}a_nz^n][1+\sum_{n=2}^{\infty}nb_nz^{n-1}]+\beta[1+\sum_{n=2}^{\infty}\frac{n^2(d)_{n-1}}{(c)_{n-1}}a_nz^{n-1}][z+\sum_{n=2}^{\infty}b_nz^n]$$

$$=[1+\sum_{n=2}^{\infty}c_nz^n][z+\sum_{n=2}^{\infty}b_nz^n][1+\sum_{n=2}^{\infty}nb_nz^{n-1}].$$

Equating coefficient of z^2 on both sides, we have

$$\frac{[2(1-\beta)d+4\mu\beta]}{c}a_2=-2(1-\beta)b_2-\beta b_2+3b_2+c_1,$$

or

$$\frac{2d(1+\beta)}{c}|a_2|\le(1+\beta)|b_2|+|c_1|.$$

Using the coefficient bounds of $|b_2|$ and $|c_1|$ as given earlier, we have

$$\frac{2d(1+\beta)}{c}|a_2|\le\frac{(1+\beta)(1-\alpha)(A-B)}{2}+\frac{k}{2}(1-\alpha)(A-B)$$

$$=\frac{1}{2}(1-\alpha)(A-B)[k+1+\beta].$$

That is

$$|a_2|\le\frac{(1-\alpha)(A-B)(c)(k+1+\beta)}{4d(1+\beta)}.$$

This completes the proof.

Special Case

We note that for $A=1,\ B=-1,\ \alpha=0$, we have coefficient bound for the class $B_2^{\alpha}(a,c)$, see [15].

The next result shows that the class $Q_k^{d,c}[\beta,\alpha,A,B]$ is closed under convolution with convex functions in E.

Theorem 3.3. Let $f\in Q_k^{d,c}[\beta,\alpha,A,B]$ and let

Ψ be convex univalent in E. Then $(\Psi * f) \in Q_k^{d,c}[\beta,\alpha,A,B]$ in E.

Proof. We first show, that for $g \in C_{d,c}[\alpha,A,B]$, $(\Psi * g)$ also belongs to $C_{d,c}[\alpha,A,B]$, which implies that

$L(d,c)g \in C[\alpha,A,B] \subset C.$

So

$(\Psi * L(d,c)g) = L(d,c)(\Psi * g) \in C[\alpha,A,B].$

Let

$$N(z) = z(L(d,c)f(z))',$$
$$D(z) = L(d,c)g(z), \ D \in C$$

and therefore starlike in E. Since

$$\frac{z(L(d,c)(\Psi * f)'(z)}{L(d,c)(\Psi * g)(z)} = \frac{\Psi * \frac{N}{D}(L(d,c)g)(z)}{(\Psi * L(d,c)g)(z)},$$

and so it is in the convex hull of the image of E under $\frac{N}{D}$. Similarly,

$$\frac{(z(L(d,c)(\Psi * f)')'(z)}{(L(d,c)(\Psi * g))'(z)}$$

is in the convex hull of E under $\frac{N}{D}$. Since $P_k[\alpha,A,B]$ is a convex set, it follows that

$$\left\{(1-\beta)\frac{N}{D} + \beta\frac{N'}{D'}\right\} \in P_k[\alpha,A,B]$$

and consequently

$$(\Psi * f) \in Q_k^{d,c}[\beta,\alpha,A,B] \quad \text{for} \quad z \in E.$$

Remark 3.1. (i) It follows from Theorem 3.3 that the class $Q_k^{d,c}[\beta,\alpha,A,B]$ is invariant under the integral operators defined by $f_i(z), \ i = 1,2,3,4$

.

(i) $f_1(z) = (\phi_2 * f)(z) = \frac{2}{z}\int_0^z f(t)dt, \phi_1(z) = -\frac{2}{z}[z + \log(1-z)],$

(ii) $f_2(z) = (\phi_1 * f)(z) = \int_0^z \frac{f(t)}{t}dt, \ \phi_2(z) = -\log(1-z),$

(iii) $f_3(z) = (\phi_4 * f)(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt, \phi_3(z) = \sum_{n=1}^{\infty}\frac{1+c}{n+c}z^n, (\operatorname{Re} c > -1)$

(iv) $f_4(z) = (\phi_3 * f)(z) = \int_0^z \frac{f(t)-f(tx)}{t-tx}dt, |x| \le 1, \phi_4(z) = \frac{1}{1-x}\log\frac{1-xz}{1-z}$

Since $\phi_i \in C$, the proof is immediate when we apply Theorem 3.3.

(ii) Let D_1, D_2 be the linear operators defined on the class A, as follows:

$$D_1(f) = zf', \ [2], \ D_2(f) = \frac{(zf)'}{2}, \ [16].$$

Both of these operators can be written as $D_i(f) = h_i * f$, where

$$h_1(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2},$$

and

$$h_2(z) = \sum_{n=1}^{\infty}\frac{(n+1)}{2}z^n = \frac{z - \frac{z^2}{2}}{(1-z)^2}.$$

We note that the radius of convexity of h_1 and h_2 are

$$r_c(h_1) = 2-\sqrt{3} \text{ and } r_c(h_2) = \frac{1}{2}.$$

Thus, it follows from Theorem 3.3 that, if $f \in Q_k^{d,c}[\beta,\alpha,A,B]$, then

$$D_1(f) \in Q_k^{d,c}[\beta,\alpha,A,B]$$

for $|z| < 2-\sqrt{3}$ and

$D_2(f) \in Q_k^{d,c}[\beta,\alpha,A,B]$ for $|z| < \frac{1}{2}$.

In the next result, we shall use the notation

$$Q_k^{d,c}[\beta,\alpha,1,-1] = Q_k^{d,c}(\beta,\alpha).$$

Theorem 3.4. For $\beta \ge 0$,

$$Q^{d,c}(\beta,\alpha) \subset Q_k^{d,c}(0,\alpha_1),$$

$$\alpha_1 = \frac{2\alpha + \beta\delta_1}{2+\beta\delta_1}, \ \delta_1 = \frac{\operatorname{Re} h_0(z)}{|h_0(z)|}, \qquad (3.2)$$

and

$$h_0(z) = \frac{z(L(d,c)g(z))'}{L(d,c)g(z)}, \ L(d,c)g(z) \in C.$$

Proof. The case $\beta = 0$ is trivial, so we suppose $\beta > 0$. Let $f \in Q_k^{d,c}(\beta,\alpha)$ and let

$$\frac{z(L(d,c)f(z))'}{L(d,c)g(z)} = h(z)$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \qquad (3.3)$$

for some $L(d,c)g \in C(\alpha) \subset C \subset S^*$.

Differentiate (3.3) logarithmically, we have

$$\frac{(z(L(d,c)f(z))')'}{(L(d,c)g(z))'} = h(z) + \frac{zh'(z)}{h_0(z)}$$

with $h_0 = \dfrac{z(L(d,c)g(z))'}{L(d,c)g(z)} \in P(\alpha)$.

Since $f \in Q_k^{\lambda,\mu}(\beta,\alpha)$, we have

$$(1-\beta)h(z)+\beta\left(h(z)+\frac{zh'(z)}{h_0(z)}\right)$$
$$=\left(h(z)+\frac{\beta zh'(z)}{h_0(z)}\right)\in P_k(\alpha),$$

and using convolution technique, see [17,18], it follows that, for $z \in E$, $i=1,2$

$$\left(h_i(z)+\frac{\beta zh_i'(z)}{h_0(z)}\right)\in P(\alpha).$$

Writing $h_i(z)=(1-\alpha_1)p_i(z)+\alpha_1$, we have, from (3.4)

$$\left[(1-\alpha_1)p_i(z)+\alpha_1-\alpha+\beta(1-\alpha_1)\frac{zp_i'(z)}{h_0(z)}\right]\in P.$$

We now form the functional $\Psi(u,v)$, by choosing $u=p_i(z)$, $v=zp_i'(z)$,

$$\Psi(u,v)=(1-\alpha_1)u+(\alpha_1-\alpha)+\beta\frac{(1-\alpha_1)v}{h_0(z)}.$$

The first two conditions of Lemma 2.3 are clearly satisfied. We verify (iii) as follows:

$$\operatorname{Re}\Psi(iu_2,v_1)=(\alpha_1-\alpha)+\frac{\beta(1-\alpha_1)v_1\operatorname{Re}h_0(z)}{|h_0(z)|^2}$$
$$=(\alpha_1-\alpha)+\beta(1-\alpha_1)v_1\delta_1,\ \delta_1=\frac{\operatorname{Re}h_0(z)}{|h_0(z)|^2}.$$

Now, for $v_1 \le \frac{-1}{2}(1+u_2^2)$, we have

$$\operatorname{Re}\Psi(iu_2,v_1)\le(\alpha_1-\alpha)-\frac{1}{2}\beta\delta_1(1-\alpha_1)(1+u_2^2),$$
$$=\frac{1}{2}\left[(2\alpha_1-2\alpha)-\beta\delta_1(1-\alpha_1)-\beta\delta_1(1-\alpha_1)u_2^2\right]$$
$$=\frac{1}{2}(L+Mu_2^2),$$

where

$$L=2(2\alpha_1-2\alpha)-\beta\delta_1(1-\alpha_1),$$
$$M=\beta\delta_1(1-\alpha_1)<0\ .$$

Therefore $\operatorname{Re}\Psi(iu_2,v_1)\le 0$, if $L\le 0$ and this gives us α_1 as defined by (3.2) . Applying Lemma 2.7, we conclude that $\operatorname{Re}p_i(z)>0$ in E, $i=1,2$ and so $h_i\in P(\alpha_1)$. Consequently $h\in P_k(\alpha_1)$. This completes the proof that $f\in Q_k^{\lambda,\mu}(0,\alpha_1)$.

Theorem 3.5. For $0\le\beta_2<\beta_1$,

$$Q_k^{d,c}(\beta_1,\alpha)\subset Q_k^{d,c}(\beta_2,\alpha).$$

Proof. For $\beta_2=0$, the proof is immediate from Theorem 3.5 Therefore we let $\beta_2>0$ and $f\in Q_k^{d,c}(\beta_1,\alpha)$. There exist two functions, $H_1,H_2\in P_k(\alpha)$ such that, for $g\in C_{d,c}(\alpha)$,

$$H_2(z)=\left\{(1-\beta_1)\frac{z(L(d,c)f(z))'}{L(d,c)g(z)}+\beta_1\frac{(z(L(d,c)f(z))')'}{(L(d,c)g(z))'}\right\},$$

$$H_1(z)=\frac{z(L(d,c)f(z))'}{L(d,c)g(z)}\in P_k(\alpha),\ \text{by Theorem 3.4.}$$

We use the fact that $P_k(\alpha)$ is a convex set and

$$\left\{(1-\beta_2)\frac{z(L(d,c)f(z))'}{L(d,c)g(z)}+\beta_2\frac{(z(L(d,c)f(z))')'}{(L(d,c)g(z))'}\right\}$$
$$=\frac{\beta_2}{\beta_1}H_1(z)+\left(1-\frac{\beta_2}{\beta_1}\right)H_2(z),$$

we obtain the required result that $f\in Q_k^{d,c}(\beta_2,\alpha)$.

Remark 3.2. Since $\phi(d,c)(z)$ is convex for $0<c\le d$, $c+d\ge 2$ by virtue of Lemma 2.1 and the class of close-to-convex functions is closed under convex convolution, we have

$$Q_2^{d,c}(\beta,\alpha)\subset Q_2^{d,c}(0,\alpha)\subset K(\alpha)\subset K\subset S.$$

We now derive a covering theorem.

Theorem 3.6. Let for $0<c<d$, $d+c\ge 2$, $f\in Q_2^{d,c}(\beta,\alpha)$. If B is the boundary of the image of E under f, then every point of B is at distance at least

$$\left[\frac{2\mu(1+\beta)}{(1-\alpha)(1+\lambda)(3+\beta)+4\mu(1+\beta)}\right]$$

from the origin.

Proof. Let $c\ne 0$, be any complex number such that $f(z)\ne c$ for $z\in E$. Then, by

Remark 3.2, $f(z)=\dfrac{cf(z)}{c-f(z)}$ is univalent in E. Let f be given by (1.1). Then

$$\frac{cf(z)}{c-f(z)}=z+(a_2+\frac{1}{c})z^2+\cdots,$$

and hence

$$\left|a_2+\frac{1}{c}\right|\leq 2.$$

Now using triangular inequality and coefficient bound of $|a_2|$ for class $Q_k^{d,c}[\beta,\alpha,A,B]$ as proved in Theorem 3.2 in the above, we have

$$|c|\geq\frac{2d(1+\beta)}{(1-\alpha)(c)(3+\beta)+4c(1+\beta)}$$

which is the required result.

The following result deals with the converse case of Theorem 3.4.

Theorem 3.7. Let $f\in Q_k^{d,c}(0,\alpha)$ in E. Then $f\in Q_k^{d,c}(\beta,\alpha)$ for $|z|<r_\beta$, where

$$r_\beta=\frac{1}{1+2\beta},\qquad(3.5)$$

This result is sharp.

Proof. Since $f\in Q_k^{d,c}(0,\alpha)$, there exists a function $L(d,c)g(z)\in C\subset S^*(\frac{1}{2})$, such that

$$\frac{z(L(d,c)f(z))'}{L(d,c)g(z)}=(1-\alpha)h(z)+\alpha,\ h\in P_k.\qquad(3.6)$$

Differentiating (3.6), we have

$$(z(L(d,c)f(z))')'=(L(d,c)g(z))(1-\alpha)h'(z)$$
$$+\{(1-\alpha)h(z)+\alpha\}(L(d,c)g(z))'$$

Since $Q_k^{d,c}(0,\alpha)$, we have

$$\frac{1}{1-\alpha}\left\{(1-\beta)\frac{z(L(d,c)f(z))'}{L(d,c)g(z)}+\beta\frac{(z(L(d,c)f(z)')'}{(L(d,c)g(z))'}-\alpha\right\}$$
$$=h(z)+\frac{\beta zh'(z)}{h_0(z)},$$

where $h_0(z)=\frac{z(L(d,c)g(z))'}{L(d,c)g(z)}\in P(\frac{1}{2})$.

Now using the same convolution technique, we have

$$\left(h(z)*\frac{\phi_\beta(z)}{z}\right)=\left\{h_1(z)+\frac{\beta zh_1'(z)}{h_0(z)}\right\}$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z)+\frac{\beta zh_1'(z)}{h_0(z)}\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z)+\frac{\beta zh_2'(z)}{h_0(z)}\right],$$

where $h_1,h_2\in P$.

From this it follows that

$$\left\{h_i(z)+\frac{\beta zh_i'(z)}{h_0(z)}\right\}\in P\text{ for }i=1,2.$$

Using well-known results [2] for the class P and $P(\frac{1}{2})$ that

$$\left|zh_i'(z)\right|\leq\frac{2r\operatorname{Re}h_i(z)}{1-r^2},\ \operatorname{Re}h_i(z)\geq\frac{1-r}{1+r},$$

$$\operatorname{Re}h_0(z)>\frac{1}{1-r},$$

we have

$$\operatorname{Re}\left\{h_i(z)+\frac{\beta zh_i'(z)}{h_0(z)}\right\}\geq\operatorname{Re}\left\{h_i(z)-\left|\frac{\beta zh_i'(z)}{h_0(z)}\right|\right\},\qquad(3.7)$$
$$\geq\operatorname{Re}h_i(z)\left\{1-\frac{2\beta r}{1-r}\right\}$$
$$=\operatorname{Re}h_i(z)\left\{\frac{1-(1+2\beta)r}{1-r}\right\}.\qquad(3.8)$$

From (3.7), it is clear that

$$\operatorname{Re}\left\{h_i(z)+\frac{\beta zh_i'(z)}{h_0(z)}\right\}\geq 0,\ \text{for } r<r_\beta$$

and consequently it follows that

$$\left\{h(z)+\frac{\beta zh'(z)}{h_0(z)}\right\}\in P_k,\ \text{for } |z|<r_\beta.$$

Hence $f\in Q_k^{\lambda,\mu}(\beta,\alpha)$, where r_β is given (3.5).

We obtain the sharpness of this result by taking

$$h_i(z)=\frac{1+z}{1-z},\ h_0(z)=\frac{1}{1-z}.$$

This completes the proof.

Conclusion:

We introduce a subclass of analytic functions by using Hadamard product along with Janowski functions. Some inclusion results, radius of univalence and other interesting properties of this class are discussed.

In future work we intend to use formal approaches to prove theorem based on developed formal tools. The formal methods have many application in real life problems [19-32].

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10/20/2012