Sensitivity analysis of Markov chains for M/G/1queueing systems

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Abstract: In this paper, we consider the problems of sensitivity analysis and estimates of the steady-state performance for an M/G/1 queueing system. By studying its embedded Markov chain, we give the sensitivity formulas expressed by the potentials of the embedded Markov chain. note thatNon-Markov-type queueing systems are often used as mathematical models in studying some practical engineering problems, such as communication networks. we study an M/G/1 queue with equal classesof customers, the server selects a customer to serve from among all customers waiting in the system with equal probability,Specifically, if there are n customers waiting in the system when the server selects a customer to serve, each customer is selected with probability $\frac{1}{n}$. we give the sensitivity formulas expressed by the potentials of the embedded Markov chain. Based on the performance potential theory and these formulas, we propose an algorithm to compute system potentials and performance derivatives for M/G/1 queueing systems.

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1. Introduction

A queueing system is one of the most fundamental models for many discrete event dynamic systems (DEDSs). Many practical systems can be well described by Markov chains. The Markov chain model has been widely used in queueing systems, communication systems, manufacturing systems, and reliability studies. General systems can often be sufficiently approximated by Markov chains via the well known“method of stages”. Performance analysis of DEDS is a practical problem. We often need to compare different designs or to find the best designs, which is usually done based on a given performance measure. Markov chains are often used as mathematical models of natural phenomena, with transition probabilities defined in terms of parameters that are of interest in the scientific question at hand.Based on the performance potential theory, we studied the problems of sensitivity analysis of the steady-state performance for some Markov-typequeueing systems and gave the sensitivity formulas of the performance (see [1–4]). The state evolvement of these queueing systems usually can be described as a Markov process. But in many theoretic and practical problems, a non-Markov- type queueing system is often used as a mathematical model. Its state process can be described as a general stochastic process, such as semi-Markov process. In this paper, we study the problems of the performance sensitivity analysis for M/G/1queueing systems which are special semi-Markov processes. Though the state evolvement of the M/G/1queueing system can be described as a semi-Markov process, the system has no memory at some special time. In other words, we can use a Markov chain (called embedded Markov chain) to describe most characters of the system, and the semi-Markov process which is used to describe its state evolvement has the same steady-state probabilities as its embedded Markov chain (see [5]).Many queueing problems can be modeled by Markov chains of *M/G/1* type. One of the major problems in Markov chains is the computation of the probability invariant vectorπ, that is, the solution of the infinite system $\pi= P\pi$. Many quantities of physical interest, such as the steady-state distribution of probability, when this exists, can be expressed conveniently in terms of the fundamental matrix G of the Markov chain. This paper studies the classification problem of the Markov process of M/G/1 type with a tree structure, which is a special case of the Markov process of matrix M/G/1 type with a tree structure introduced in (see[6]). A simple criterion is found for a complete classification of the Markov process of M/G/1 type with a treestructur.Gajratet al. (see[7]) studied a Markov process of random strings which is a hybrid model of the M/G/1 type and the GI/M/1 type Markov processes with a tree structure. They obtained necessary and sufficient conditions for the Markov process to be positive recurrent, null recurrent, or transient, in terms of positive solutions of a finite system of polynomial equations.we only consider the steady-state performance sensitivity, we can discuss it by studying its embedded Markov chain. In view of this, we give the steady-state performance sensitivity formulas which are based on the potential of the embedded Markov chain.The remainder of this paper is organized as follows. In Section 2, we discuss the problems of the performance sensitivity analysis for Markov chains.Derivativeestimation is an important problem in performanceanalysis of discrete event dynamic systems. Derivative estimation of stationary performance measures is difficult since it generally requires the consistency of estimators. In Section 3, we give some results for M/G/1queueing systems.In Section 4, we giveMarkov chain of M/G/1 type with a tree structure.

2. Sensitivity analysis of Markov chains

Sensitivity analysis is an important way to quantify the effects of changes in these parameters on the behavior of the chain. Many properties of Markov chains can be written as simple matrix expressions, and hence matrix calculus is a powerful approach to sensitivity analysis. Perturbationanalysis is an important tool for understanding howthose parameters determine the properties of the chain, and in predicting how changes in the environment(*sensulato*)will change the outcome. For applications, the analyses should be easily computable, and flexible enough to handle a variety ofdependent variables. In this section,We first study the sensitivity analysis of the steady-state performance for a class of Markov processes.We will discuss the problems of the sensitivity analysis of the steady-state performance for a Markov chain.

Consider a positive recurrent, irreducible Markov processY $= \{Y_t; t \geq 0\}$with a state space$\Phi = \{0, 1, \ldots\}$and an infinitesimal generator A= $[a_{ij}]$, satisfying Ae =0.Where$a_{ii} \leq 0$; $a_{jj} \geq 0$, $i \neq j$, $i, j \in \Phi$, $e = (1, 1, \ldots)^\tau$the"τ "representsthetranspose.We assume that Y has standard transfer probabilities$P_{ij}(t), i, j \in \Phi$, According to the Markov theory, Y has a unique steady-state probability measure$\boldsymbol{\pi}(\boldsymbol{i})and\boldsymbol{\pi}(\boldsymbol{i}) > \mathbf{0}$, for any $i \in \Phi$. Denote the steady-state probability measure by a row vector$\pi = (\pi(0), \pi(1), \ldots)$, then we have$\pi A = 0, \pi e = 1$.

Definition 1. $d_{\mathrm{ij}} = E\{\int_0^{T^{\{i,j\}}} \left[f\left(Y_t^{\{j\}}\right) - f\left(Y_t^{\{i\}}\right)\right] dt, i, j \in \Phi$is called a perturbationrealization factor of the Markovprocess Y with respect to the performance function f ; the matrix$D^{(\mathrm{f})} = [d_{ij}]$is called a realization matrix.

From the definition, we have$(D^{(\mathrm{f})})^\tau = -D^{(f)}$ (see[5])

For any i, j$\in \Phi$,let$Y^{\{i\}} = \{Y_t : Y_0 = i; t \geq 0andY^{\{j\}} = \{Y_t : Y_0 = j; t \geq 0\}$be two Markov processeswith the initial state i and j respectively. We suppose that$Y^{\{i\}}$and$Y^{\{j\}}$are independent for any i, j$\in \Phi$.Define$Z^{\{i,j\}} = (Y^{\{i\}}, Y^{\{j\}})$, then$Z^{\{i,j\}}$is a Markov process with a state space$\Phi \times \Phi$. $Z^{\{i,j\}}$is positive recurrent and irreducible since$Y^{\{i\}}$and$Y^{\{j\}}$are.Thus, the first passage time from any state to any other state has a finite mean. Let S={(K, K): K $\in \Phi$ }, define$T^{\{i,j\}}$= inf{t : t $\geq 0, Z^{\{i,j\}} \in S$ }.

$$d_{\mathrm{ij}} = \lim_{\mathrm{T}\to\infty}\{E[\int_0^{\mathrm{T}} f\left(Y_t^{\{j\}}\right) dt] - \mathrm{E}[\int_0^{\mathrm{T}} \int f\left(Y_t^{\{i\}}\right) dt]\} j \in \Phi. \quad (1)$$

Proof. (See [8]).

From Lemma 1, we have

$$d_{\mathrm{ij}} = d_{\mathrm{ik}} + d_{\mathrm{kj}}, \; i, j, k \in \Phi \quad (2)$$

This property is similar to that of the potential energy in physics. In view of this, we may pick up any integer$K^* \in \Phi$and any real number c and define a quantity xi for any i$\in \Phi$, i $\neq K^*$, as follows:

$$x_{K^*} = c \quad , \qquad x_i = x_{K^*} + d_{K^* i}. \quad (3)$$

Definition 2. The potential vector $x^{(f)}$satisfies the Poisson equation

$$\mathrm{Ax} = -\mathrm{f} + \eta f e.$$

Now we suppose that the infinitesimal generator A is a differentiable function with respect to the parameter θ on the interval J $\subset R$, that is to say, all of $a_{ij} = a_{ij}(\theta)$, i, j $\in \Phi$ are differentiable functions with respect toθ.We also assume that f depends on θ and is a differentiable function with respect to θ. Then from (4) we have

$$\frac{dA}{d\theta} x^{(f)} + A \frac{dx^{(f)}}{d\theta} + \frac{df}{d\theta} = \frac{d\eta f}{d\theta} e.$$

Multiplying both sides with π on the left and noting thatπA = 0, $\pi e = 1$, we get

$$\frac{d\eta f}{d\theta} = \pi \frac{dA}{d\theta} x^{(f)} + \pi \frac{df}{d\theta}. \quad (5)$$

Proof. (See [5])

The results about Markov processes can be easily translated into those of Markov chains. Consider a positiverecurrent, irreducible Markov chain X = $\{X_n; n \geq 0\}$ on a state space $\Phi = \{0, 1, \ldots\}$ with a transition probability matrix P = $[P_{ij}]$. X has a unique steady-state probability measure. Denoting the steady-state probability by a row vector $\pi = (\pi(0), \pi(1), \ldots)$, then we have $\pi e = 1$ and

$$\mathrm{Pe} = \mathrm{e}, \qquad \pi P = \pi. \quad (6)$$

Now we suppose that the transition probability matrix P = $[P_{ij}]$ is differentiable with respect to the parameter θ on the interval J $\subset R$, that is to say, all of $P_{ij} = P_{ij}(\theta)$, i, j $\epsilon\Phi$ are differentiable functions with respect to θ. We also assume that f depends on θ and is a differentiable function with respect to θ .

Let A = P – I, where I is a unit matrix of infinite dimension. Obviously A can be used as an infinitesimalgenerator of a Markov process. Consider a Markov process Y = $\{Y_t; t \geq 0\}$ on the state space Φ with the infinitesimal generator A, whose embedded Markov chain is X. Y is positive recurrent, irreducible since X is. And Y has the same steady-state probability vector as X. Thus Y and X have the same steady-state performance measure ηf.From (5), we can get

$$\frac{d\eta f}{d\theta} = \pi\frac{dP}{d\theta}x^{(f)} + \pi\frac{df}{d\theta},$$

(7)

Where $x^{(f)}$ is the potential vector of the Markov chain X, that is to say $x^{(f)}$ satisfies the Poisson equation

(P – I) x = = -f $+\eta fe$

(8)

3. M/G/1 queueing systems

Many queueing problems can be modeled by Markov chains of *M/G/*1 type, that is, Markov chains whose probability transition matrix has the structure

$$P = \begin{bmatrix} B_1 & A_0 & & 0 & \\ B_2 & A_1 & A_0 & & \\ B_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \qquad (9)$$

Where B_{i+1} and A_i(i≥ 0) are K× Knonnegative matrices such that $\sum_{i=1}^{\infty} B_i$ and $\sum_{i=0}^{\infty} A_i$ are columnstochastic. One of the major problems in Markov chains is the computation of the probability invariant vector π, that is, the solution of the infinite system $\pi P = \pi. \|\pi\| = 1$.For the matrices of structure (9), the computation of πcan be reduced to the computation of the minimal nonnegative solution *G* of the nonlinear matrix equation

X = $\sum_{i=0}^{\infty} X^i A_i$,

(10)

Where x K× Kmatrix. In the case when the matrix *P* is irreducible and positive recurrent, Eq. (10)has a unique nonnegative solution *G*, which is column stochastic. We will assume that 1 is a simpleeigenvalue and the only eigenvalue of *G* of modulus 9.

In this section, we will give some results about M/G/1 queueing systems. Consider an M/G/1 queueing system, the arriving of the customer is a Poisson process with the intensity λ, the service time of the customers is independent and has the same distribution G. We suppose that the service time is independent of the arriving process and the service discipline is FCFS. Let

0< $\frac{1}{\mu} \equiv \int_0^{\infty} x dG(x) < \infty$

(11)

where µ is the mean service rate of the system, satisfying $\rho = \frac{\lambda}{\mu} < 1$.Denote the number of the customers in the system after the recent customer left at time t by Y_t, then Y = $\{Y_t; t \geq 0\}$ is a semi-Markov process on the state space

$$Q(t) = \begin{bmatrix} p_0(t) & p_1(t) & p_2(t) & \cdots & \cdots & \cdots \\ q_0(t) & q_1(t) & q_2(t) & \cdots & \cdots & \cdots \\ 0 & q_0(t) & q_1(t) & \cdots & \cdots & \cdots \\ \cdots & 0 & q_0(t) & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

(12)

Where

$$q_k(t) = \int_0^t \frac{e^{-\lambda x}(\lambda x)^k}{k!} dG(x), \quad k = 0, 1, ...$$

$p_k(t) = \int_0^t q_k(t$ – x) $\lambda e^{-\lambda x} dx, \quad k = 0, 1, ...$

(14)

Denote X = $\{X_n; n \geq 0\}$ as the embedded Markov chain of the semi-Markov process Y, where X_n means the Customer numbers staying in the system when the nth customer left. Let

$a_k = \lim_{t\to\infty} q_k(t), k = 0, 1, ...$

be the probability of k customers arriving in the interval when one is being served, it is independent of the one being served, then $\rho = \sum_{k=0}^{\infty} k\, a_k$ is the mean of arriving customer numbers when one is being served. It can be proved that the embedded Markov chain X is positive recurrent, irreducible and aperiodic under the condition of $\rho < 1$, so is the semi-Markov process Y too and X has a unique steady-state probability vector $\pi = (\pi(0), \pi(1), ...)$, satisfying $\pi(k) > 0, k\epsilon\Phi$, $\pi e = 1$ and

$\pi(P - I) = 0$

(16)

Where

$$P = \lim_{t\to\infty} Q(t) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & \cdots \\ a_0 & a_1 & a_2 & \cdots & \cdots & \cdots \\ 0 & a_0 & a_1 & \cdots & \cdots & \cdots \\ \cdots & 0 & a_0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad (17)$$

is the transition probability matrix of the embedded Markov chain X. It can be proved that $\lim_{n\to\infty} p_k(t) = \lim_{n\to\infty} q_k(t) = a_k, k = 0, 1, \ldots$.

The semi-Markov process Y has a unique steady-state probability vector, and its steady-state probability vector is also π (see [9]). For the same performance function f $:\Phi \to R$, Y and X have the same steady-state performance measure $\eta f = \pi f$, so we can discuss the steady-state performance sensitivity of the system by its embedded Markov chain X. We suppose that the distribution G of the customer service time is adifferentiable function with respect to the parameter θ in the interval $J \subset R$ and the performance function f is dependent on parameters λ and θ. Moreover, it is a differentiable function with respect to λ and θ too. Therefore, the sensitivity formulas of the steady-state performance ηf with respect to λ and θ can be obtained by (7) in an M/G/1queueing system. (see[5])

4. Markov chain of M/G/1 type with a tree structure

In this section, a discrete time Markov chain of M/G/1 type with a tree structure is defined. This Markov chain is a special case of the Markov chain of matrix M/G/1 type with a tree structure introduced in (see[10]). Continuous time Markov processes of M/G/1 type with a tree structure will be defined.

First, we define a K-ary tree. A K-ary tree is a tree in which each node has a parent and K children, exceptthe root node of the tree. The root node of the tree is denoted as 0. Strings of integers between 1 and K areused to represent nodes of the tree. For instance, the kth child of the root node has a representation of k.The jth child of node k has a representation of kj.(see[11]).
Let $\aleph = \{\boldsymbol{J} : \boldsymbol{J} = \boldsymbol{K_1 K_2} \ldots \boldsymbol{K_n}, \boldsymbol{1} \le \boldsymbol{K_i} \le \boldsymbol{K}, \boldsymbol{1} \le \boldsymbol{i} \le \boldsymbol{n}, \boldsymbol{n} > 0\} \cup \{\boldsymbol{0}\}$.Any string $\mathbf{J} \epsilon \aleph$ is a node in the K-ary tree. The length of a string J is defined as the number of integers in the string and is denoted by $|\boldsymbol{J}|$.When $\mathbf{J} = \mathbf{0}, |\boldsymbol{J}| = \mathbf{0}$. The following two operations related to strings in $\aleph$ are used in this paper.
Consider a discrete time Markov chain $\{X_n; n \ge 0\}$ for which X_n takes values in $\aleph$. X_n is referred to asthe node of the Markov chain at time n. To be called a (homogenous) Markov chain of M/G/1 type with a tree structure, X_n transits at each step to either the parent node of the current node or a descendent of the parent node. All possible transitions and theircorresponding probabilities are given as follows. Suppose that $J = K_1 \ldots K_{|J|}$ and $X_n = J + K$.

7. Conclusions

In this paper, we study the problem of the steady-state performance sensitivity analysis in M/G/1 queueing systems. For an M/G/1 queueing system, the semi-Markov process which is used to describe the system state evolution and its embedded Markov chain have the same steady-state probabilities. Since we only consider the steady-state performance sensitivity analysis, we can do it by studying the embedded Markov chain and give the sensitivity formulas expressed by the potential of the embedded Markov chain.

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