# Hasimoto Surfaces

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**Abstract:** The purpose of the present work is to construct a Hasimoto surface from its fundamental form coefficients via numerical integration of Gauss-Weingarten equations and fundamental theorem of surfaces.
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## 1. Introduction

There are a catalogue of surfaces that can be described by integrable equations such as surfaces of constant negative Gaussian curvature, surfaces of constant mean curvature, minimal surfaces, affine spheres. This paper continues the program by adding Hasimoto surfaces to the catalouge. These surfaces are obtained by evolving a regular space curve $x$ in $R^3$ as it evolves over time according to this evolution equation:

$$x_t(s,t) = x_s \wedge x_{ss} = \kappa(s,t)b \qquad (1)$$

this is an evolution of the curve in its binormal direction with velocity equal to its curvature. Eq.1 Known as the vortex filament flow or Localized Induction Equation (LIE). Here, $\mathbf{x}(s,t)$ is a position vector for a point on the curve, $t$ is the time, $s$ is the arc--length parameter, $\kappa$ is the curvature of $\mathbf{x}$, $\mathbf{b}$ is the unit binormal and the subscripts indicate the differentiation with respect to the indicated variables. The subject of how space curves evolve in time is of great interest and has been investigated by many authors. Hasimoto [1] showed that the evolution of a thin vortex filament regarded as a moving space curve can be mapped to the nonlinear Schrodinger equation. Rick Mukherjee and Radha Balakrishnan [2] are studied moving curves of the sine-Gordon equation. Nassar, et al [3, 4, 5, 6] studied evolution of plane curves, motion of hyper surfaces and evolution of space curves in $R^n$. The authors in [7] constructed Hasimoto surface via integration for Frenet--Serret equations using fundamental existence and uniqueness theorem for space curves. Here we take another method different from them, the outline of this method is to construct six fundamental quantities $\{g_{11}, g_{12}, g_{22}, L_{11}, L_{12}, L_{22}\}$ for Hasimoto surface after then we integrate Gauss-Weingarten equations numerically. Since the surface generated form the evolution of the curve so in the next section we introduce geometry of space curve evolution.

Our Goal**.** The goal of this paper is to construct Hasimoto surfaces and display it via numerical integration of Gauss-Weingarten equations.

The article is organized as follows**.** In section 2 we introduce geometry of space curve evolution, derive (CNPDEs) which formulates the problem directly in terms of the curvatures and get exact solution for them. In section 3, we reconstruct the curve from its curvatures. In section 4, we introduce differential geometry of surfaces. In section 5, we study the geometric properties of Hasimoto surfaces. In section 6, we reconstruct the surface from its fundamental forms via numerical integration of Gauss-Weingarten equations and display it.

## 2. General Curve Evolution

If $\mathbf{x} = \mathbf{x}(s,t)$ is the position vector of a curve $C$ moving in space, then the unit tangent, principal normal and binormal vectors ,which are denoted by $\{t,n,b\}$ respectively vary along $C$ according to the well$-$known Serret$-$Frenet relations [8]

$$\begin{aligned} t_s &= \kappa n, \\ n_s &= -\kappa t + \tau b, \\ b_s &= -\tau n, \end{aligned} \qquad (2)$$

where $s$ measures arc length along $C$, $\kappa$ is its curvature and $\tau$ its torsion. The general temporal evolution in which the triad $\{t,n,b\}$ remains orthonormal adopts the form [9],

$$\begin{aligned} t_t &= \alpha n + \beta b, \\ n_t &= -\alpha t + \gamma b, \\ b_t &= -\beta t - \gamma n. \end{aligned} \qquad (3)$$

Here $\alpha$ , $\beta$ and $\gamma$ are geometric parameters which are generally functions of $s$ and $t$ . These describe the evolution in $t$ of the Frenet frame $\{t, n, b\}$ on the curve. For non$-$stretching curves, the triad must satisfy the compatibility conditions

$$t_{ts} = t_{st}, \quad n_{ts} = n_{st}, \quad b_{ts} = b_{st}. \tag{4}$$

Using Eqs.(2) and (3) , the compatibility conditions become

$$\begin{aligned} \kappa_t &= \alpha_s - \tau\beta, \\ \tau_t &= \gamma_s + \kappa\beta, \\ \beta_s &= \kappa\gamma - \tau\alpha. \end{aligned} \tag{5}$$

If the velocity vector $v = \mathbf{x}_t$ of a moving curve $C$ has the decomposition

$$\frac{\partial \mathbf{x}}{\partial t} = \lambda \mathbf{t} + \mu \mathbf{n} + \nu \mathbf{b}, \tag{6}$$

then imposition of the condition $\mathbf{x}_{ts} = \mathbf{x}_{st}$ yields

$$\begin{aligned} 0 &= \lambda_s - \mu\kappa, \\ \alpha &= \lambda\kappa + \mu_s - \nu\tau, \\ \beta &= \mu\tau + \nu_s. \end{aligned} \tag{7}$$

where $\lambda, \mu, \nu$ as functions of s and t, correspond to the normal, binormal and tangent projections of the velocity. Below we restrict our attention to a purely local form for these velocities as in the form,

$$\begin{aligned} \lambda &= \lambda(\kappa, \kappa_s, ..., \tau, \tau_s, ...), \\ \mu &= \mu(\kappa, \kappa_s, ..., \tau, \tau_s, ...), \\ \nu &= \nu(\kappa, \kappa_s, ..., \tau, \tau_s, ...). \end{aligned}$$

The dynamical equations for the curvature $\kappa$ and the torsion $\tau$ of the evolving curve $C$ may now be expressed in terms of the components of velocity $\lambda, \mu, \nu$ by substitution of Eq.(7) in to Eq.(5) to obtain

$$\begin{cases} \kappa_t = (\lambda\kappa + \mu_s - \nu\tau)_s - (\mu\tau + \nu_s)\tau \\ \tau_t = \dfrac{1}{\kappa}((\mu\tau + \nu_s)_s + \tau(\lambda\kappa + \mu_s - \nu\tau)) + (\mu\tau + \nu_s)\kappa \end{cases} \tag{8}$$

where

$$\gamma = \frac{1}{\kappa}[(\mu\tau + \nu_s)_s + \tau(\lambda\kappa + \mu_s - \nu\tau)] \tag{9}$$

Eqs.(8) link the curves with non linear partial differential equations. For a given $\lambda, \mu, \nu$ the above coupled nonlinear partial differential equations (CNPDES) determined the motion of the curve . Now let a curve $C$ moving in the space according to

$$\frac{\partial \mathbf{x}}{\partial t} = \kappa \mathbf{b}. \tag{10}$$

Known as the vortex filament flow or localized induction equation (LIE).

For this motion we have

$$\{\lambda, \mu, \nu\} = \{0, 0, \kappa\} \tag{11}$$

From Eqs.(5), (7) the velocities of the moving frame are

$$\begin{aligned} \alpha &= -\kappa\tau, \\ \beta &= \kappa_s, \\ \gamma &= \frac{\kappa_{ss} - \kappa\tau^2}{\kappa}. \end{aligned} \tag{12}$$

(5)

The evolution of the moving frame w.r.to $t$ is given from Eq.(3)

$$\begin{aligned} t_t &= -\kappa\tau n + \kappa_s b, \\ n_t &= \kappa\tau t + \frac{\kappa_{ss} - \kappa\tau^2}{\kappa} b, \\ b_t &= -\kappa_s t - \frac{\kappa_{ss} - \kappa\tau^2}{\kappa} n. \end{aligned} \tag{13}$$

and from Eq.(8) The evolution equations for curvature and torsion are

$$\begin{aligned} \kappa_t &= -2\kappa_s\tau - \kappa\tau_s, \\ \tau_t &= \kappa\kappa_s - 2\tau\tau_s + (\frac{\kappa_{ss}}{\kappa})_s. \end{aligned} \tag{14}$$

A (CNPDES) (14) was formulated by Da Rios in [10]. We used MATHEMATICA package software (computational software program used in scientific, engineering, mathematical fields and other areas of technical computing) for solving the system Eq.(14) which apply the tanh$-$and sech$-$methods [11]. Thus the above system has a solution in the form,

$$\kappa = 2c_2 \operatorname{sech}(c_1 t + c_2 s + c_3), \quad \tau = -c_1/2c_2. \tag{15}$$

where $c_1, c_2, c_3$ are arbitrary real constant.

## 3. Constructing a Curve from the Curvature and Torsion

One of the basic problems in geometry is to determine exactly the geometric quantities which distinguish one figure from another. For example, line segments are uniquely determined by their lengths, circles by their radii, triangles by side-angle-side, etc. It turns out that this problem can be solved in general for sufficiently smooth regular curves. We will see that a regular curve is uniquely determined by two scalar quantities, called curvature and torsion, as functions of the natural parameter.

**Theorem 3.1 (Fundamental existence and uniqueness theorem for space curves)** *[12] Let $\kappa(s)$ and $\tau(s)$ be arbitrary continuous functions on $a \le s \le b$. then there exists, except for position in space , one and only one space curve C for which*

*$\kappa(s)$ is the curvature, $\tau(s)$ is the torsion and s is a natural parameter along C.*

The figure 1 in the section 6 represent snapshot of the evolving space curve obtained by solving the Frenet − Serret Eqs. (2) for a specified curvature and torsion using Mathematica [13]. Any moving space curve can be studied from two perspectives,namely, the shape of the curve and the evolution of the curve .At every fixed $t$, we clearly have a representation of the corresponding static space curve at that instant.

## 4. Differential Geometry of Surfaces

Let $x = x(s,t)$ denote the position vector of a generic point P on a surface S in $R^3$. Then, the vectors $x_s$ and $x_t$ are tangential to $S$ at $P$, at such points at which they are linearly independent,

$$N = \frac{x_s \wedge x_t}{|x_s \wedge x_t|} \tag{16}$$

determines the unit normal vector to $S$. The first and second fundamental forms (abbreviated FFF, SFF) on $S$ are defined respectively by

$$\begin{aligned} I &= \langle dx.dx\rangle = g_{11}ds^2 + 2g_{12}dsdt + g_{22}dt^2 \\ II &= \langle -dx.dN\rangle = L_{11}ds^2 + 2L_{12}dsdt + L_{22}dt^2 \end{aligned} \tag{17}$$

where $g_{ij}$ and $L_{ij}$ are given by

$$\begin{aligned} g_{11} &= \langle x_1.x_1\rangle, \quad g_{12} = \langle x_2.x_1\rangle, \quad g_{22} = \langle x_2.x_2\rangle \\ L_{11} &= \langle x_{11}.N\rangle, \quad L_{12} = \langle x_{12}.N\rangle, \quad L_{22} = \langle x_{22}.N\rangle. \end{aligned} \tag{18}$$

where $\langle,\rangle$ is the Euclidean scaler product. The Gauss and Weingerten equations give us the rate of change of $(X_1, X_2, n)$ associated with the surface $S$, which take the form [8]

$$\begin{aligned} x_{ss} &= \Gamma^1_{11}x_s + \Gamma^2_{11}x_t + L_{11}\mathbf{N}. \\ x_{st} &= \Gamma^1_{12}x_s + \Gamma^2_{12}x_t + L_{12}\mathbf{N}. \\ x_{tt} &= \Gamma^1_{22}x_s + \Gamma^2_{22}x_t + L_{22}\mathbf{N}. \end{aligned} \tag{19}$$

$$\begin{aligned} N_s &= \frac{g_{12}L_{12} - g_{22}L_{11}}{g}x_s + \frac{g_{12}L_{11} - g_{11}L_{12}}{g}x_t. \\ N_t &= \frac{g_{12}L_{22} - g_{22}L_{12}}{g}x_s + \frac{g_{12}L_{12} - g_{11}L_{22}}{g}x_t \end{aligned} \tag{20}$$

where

$$g = g_{11}g_{22} - g_{12}^2 \tag{21}$$

The quantities $\Gamma^k_{ij}$ are called the Christoffel symbols of the second kind and given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial u^i}g_{lj} + \frac{\partial}{\partial u^j}g_{il} + \frac{\partial}{\partial u^l}g_{ij}\right), i,j,k,l = 1,2 \tag{22}$$

where ( $g^{ij}$ ) is the inverse of ( $g_{ij}$ ). In the above, the Einstein convention of summation over repeated indices has been adopted. The Gaussian curvature $\kappa_g$ and the mean curvature $\kappa_m$ are

$$\kappa_g = \frac{L}{g} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \tag{23}$$

$$\kappa_m = \frac{L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}}{2g}. \tag{24}$$

where $g = det(g_{ij}), L = det(L_{ij})$ . The compatibility conditions $(\mathbf{x}_{ss})_t = (\mathbf{x}_{st})_s$ and $(\mathbf{x}_{st})_t = (\mathbf{x}_{tt})_s$ applied to the linear Gauss system (19) produce the Gauss and Mainardi-Codazzi system

$$\begin{aligned} L = g_{11}((\Gamma^1_{22})_s - (\Gamma^1_{12})_t + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{12} \\ - \Gamma^2_{12}\Gamma^1_{22}) + g_{12}((\Gamma^2_{22})_s - (\Gamma^2_{12})_t + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12}) \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{\partial L_{11}}{\partial t} - \frac{\partial L_{12}}{\partial s} &= L_{11}\Gamma^1_{12} + L_{12}(\Gamma^2_{12} - \Gamma^1_{11}) - L_{22}\Gamma^2_{11} \\ \frac{\partial L_{12}}{\partial t} - \frac{\partial L_{22}}{\partial s} &= L_{11}\Gamma^1_{22} + L_{12}(\Gamma^2_{22} - \Gamma^1_{12}) - L_{22}\Gamma^2_{12} \end{aligned} \tag{26}$$

**Theorem 4.1 ( Fundamental existence and uniqueness theorem Of Surfaces)** *[12] Let $g_{11}$, $g_{12}$ and $g_{22}$ be functions of $s$ and $t$ of class $C^2$ and let $L_{11}, L_{12}$ and $L_{22}$ be functions of $s$ and $t$ of class $C^1$ all defined on an open set containing $(s_0, t_0)$ such that for all (s, t),*

(i)

$$g_{11}g_{22} - g_{12^2} > 0, \qquad g_{11} > 0, \qquad g_{22} > 0$$

(ii) $g_{11}, g_{12}, g_{22}, L_{11}, L_{12}, L_{22}$ satisfy the compatibility equations (25),(26) Then there exists a patch $X = X(s,t)$ of class $C^3$ defined in a neighborhood of $(s_0, t_0)$ for which $g_{11}, g_{12}, g_{22}, L_{11}, L_{12}, L_{22}$ are the first and second fundamental coefficients. The surface represented by $\mathbf{X} = \mathbf{X}(s,t)$ is unique except for position in space.

## 5. Geometric Properties of Hasimoto Surfaces

For Hashimoto surfaces $\mathbf{x} = \mathbf{x}(s,t)$ the tangent vectors are

$$\mathbf{x}_s = t, \qquad \mathbf{x}_t = \kappa b$$

The coefficients of the FFF are

$$\begin{aligned} I &= \langle d\mathbf{x}.d\mathbf{x} \rangle \\ &= \langle (\mathbf{x}_s ds + \mathbf{x}_t dt).(\mathbf{x}_s ds + \mathbf{x}_t dt) \rangle \\ &= ds^2 + \kappa^2 dt^2 \end{aligned}$$

so that

$$\boxed{g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \kappa^2.} \qquad (28)$$

The unit normal to $\mathbf{x}$ :

$$\mathbf{N} = \frac{x_s \wedge x_t}{| x_s \wedge x_t |} = -\mathbf{n} \qquad (29)$$

The coefficients of the SFF are

$$\begin{aligned} II &= \langle -d\mathbf{X}.d\mathbf{N} \rangle \\ &= \langle (\mathbf{t}ds + \kappa \mathbf{b} dt).(\mathbf{n}_s ds + \mathbf{n}_t dt) \rangle \end{aligned}$$

$$= -\kappa ds^2 + 2\kappa\tau ds dt + (\kappa_{ss} - \kappa\tau^2) dt^2$$

where we use the time evolution equations of the triad $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ with respect to $s$ and $t$ respectively Eqs.(2), (13) to obtain

$$\boxed{L_{11} = -\kappa, \quad L_{12} = \kappa\tau, \quad L_{22} = \kappa_{ss} - \kappa\tau^2.} \qquad (31)$$

It is easily verified that the Eqs. (28), (31) with (15) satisfies the compatibility conditions (25),(26) which are determined the surface up to its position in space as we shall show in the next section. The Gaussian curvature $\kappa_g$ and the mean curvature $\kappa_m$ for Hasimoto surface are

$$\kappa_g = \frac{-\kappa_{ss}}{\kappa} \qquad (32)$$

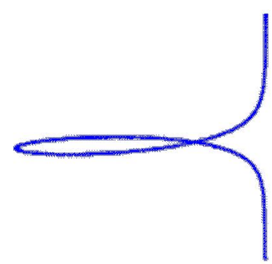
$$\kappa_m = \frac{1}{2\kappa}\left(\frac{\kappa_{ss}}{\kappa} - \kappa^2 - \tau^2\right) \qquad (33)$$

## 6. Geometric Visualization of the Hasimoto Surfaces and its Generator

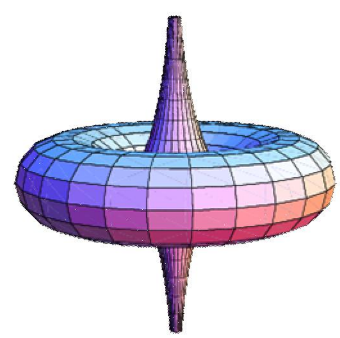
We recall that a curve in $E^3$ is uniquely determined by two local invariant quantities, curvature and torsion, as functions of arc length. Similarly, a surface in $E^3$ is uniquely determined by certain local invariant quantities called the first and second fundamental forms. Now by using Fundamental Theorem Of Surfaces 4.1 which states that the sextuplet $\{g_{11}, g_{12}, g_{22}, L_{11}, L_{12}, L_{22}\}$ determines the surface $S$ up to its position in space. The surfaces below generated by evolution of space curve obtained via solving the Gauss-Weingarten equations (19,20) for a specified the coefficients of FFF and SFF using Mathematica [14].

(27)

Figure 1: Hasimoto surface corresponding to (30)
$(\kappa = 2sech(s + 2t), \tau = -1)$



(a) $(\kappa = 2sech(s), \tau = 0)$



(b) surface generated by evolution of the above curve
Figure 2: Hasimoto surface and its generator

## 7. Conclusions

we constructed the Hasimoto surface from its fundamental form coefficients via numerical integration of Gauss-Weingarten equations and fundamental theorem of surfaces.

**References**


[1] H. Hasimoto, J. Fluid Mech. 51 (1972) 477.
[2] Rick Mukherjee, Radha Balakrishnan, Moving curves of the sine−Gordon equation: New links, Phys. Lett. A 372 (2008) 6347.
[3] Nassar H. Abdel-All, M. A. Abdel-Razek, H. S. Abdel-Aziz, A. A. Khali, Geometry of evolving plane curves problem via lie group analysis, studies in mathematical scinces, 2 (2011) 51.
[4] Nassar H. Abdel-All, M. A. A. Hamad, M. A. Abdel-Razek, A. A. Khalil, Computation of Some Geometric Properties for New Nonlinear PDE Models, Applied Mathematics, 2 (2011) 666.
[5] Nassar H. Abdel-All and M. T. Al-dossary, Motion of hyper surfaces,Assuit univ. Journal of Math. and comuter science 40 (2011) 91.
[6] Nassar H. Abdel-All and M. T. Al-dossary, Evolution of space curves in $R^n$, Accepted for publication in Journal of Applied Mathematics, (2011).
[7] Peter J. VassiliouIan, G. Lisle, pp.(38), Geometric approaches to differential equations, Cambridge University Press 2000.
[8] M. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, 1976.
[9] C. Rogers and W. K. Schief, Backlund and Darboux Transformations Geometry and Modern Application in Soliton Theory, Cambridge University press, Cambridge, 2002.
[10] L. S. Da Rios, Rend. Circ. Mat. Palermo, 22 (1906) 117.
[11] D. Baldwina, Ü. Göktas, W. Hereman, L. Hong, R.S. Martino, J.C. Miller, Symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDEs, Journal of Symbolic Computation 37 (2004) 669
[12] Martin M Lipschutz, Theory and problems of differential geometry Schaum's outline series, 1969.
[13] A. Gray, Modern Differential Geometry of Curves and Surfaces with Mathematica, CRC, New York, 1998.
[14] Yoshihiko Tazawa, Theory of Curves and Surfaces, An Introduction to Classical Differential Geometry by Mathematica 1999 in Japanese. An English outline: Experiments in the Theory of Surfaces, IMS 1999.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