Adomian decomposition method for solving the Kuramoto-Tsuzuki equation

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Abstract: In this paper the Adomian decomposition method (ADM) and modified Adomian decomposition method (MADM) are used to solve the homogenous and inhomogeneous Kuramoto-Tsuzuki equations. ADM approximate solution, which is obtained as a series has a reasonable residual error. MADM gets the exact solution of inhomogeneous Kuramoto-Tsuzuki equation. Comparison of the results with those of ADM, MADM and finite difference scheme shows the accuracy of the ADM and MADM methods.

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1. Introduction

The inhomogeneous Kuramoto-Tsuzuki equation is as follows

$$\frac{\partial\omega}{\partial t} = (1+ic_1)\frac{\partial^2\omega}{\partial x^2} + \omega - (1+ic_2)|\omega|^2\omega + \psi(x,t), (x,t) \in (0,1)\times(0,T], \qquad (1\text{-}1)$$

with the initial condition

$$\omega(x,0) = \omega_0(x), x \in [0,1], \qquad (1\text{-}2)$$

and homogeneous boundry conditions

$$\frac{\partial\omega}{\partial x}(0,t) = 0, \frac{\partial\omega}{\partial x}(1,t) = 0, t \in (0,T] \qquad (1\text{-}3)$$

where c_1 and c_2 are two real constants, $\omega(x,t)$ is an unknown complex function, $\omega_0(x)$ and $\psi(x,t)$ are given complex functions.

Equation (1-1) describes the behavior of many two-component systems in a neighborhood of the bifurcation point [9]. Reaction-diffusion type equations have been applied in the study of broad class of nonlinear processes, including a well-known synergetic model [3,10]. The problem of constructing and validating difference schemes for these classes of problems has been in detail taken up in [7,8]. A finite element Galerkin method had been discussed in [11,12]. Tsertsvadze studied in [13] the convergence of difference schemes for the Kuramoto-Tsuzuki equation and for systems of reaction-diffusion type. In this paper, we use the ADM and MADM to solve equation (1-1).

2. Adomian Decomposition Method

The decomposition procedure of Adomian was first proposed by the American mathematician, G. Adomian (1923-1996) and has been applied already to a wide class of stochastic and deterministic problems in science and engineering. It is based on the search for a solution in the form of a series and on decomposing the nonlinear operator into a series in which the terms are calculated recursively using Adomian polynomials [2].

In this method, the nonlinear function in the equation is decomposed into terms of special polynomials called *Adomian's polynomials* and then the terms of the solution which is regarded as a series, are determined recurrently.
Consider the differential equation

$$F(u) = g(x), x \in \Omega, \qquad (2\text{-}1)$$

where F represents a general nonlinear ordinary differential operator involving both linear and nonlinear parts and $g(x)$ is a given function. Equation (2-1) can be written as

$$Lu + Ru + Nu = g(x), \qquad (2\text{-}2)$$

where L is an easily invertible operator, which is usually taken as the highest-ordered derivative, R is the reminder of the linear operator, and N is the nonlinear term in $F(u)$.
Applying the inverse operator L^{-1} to (2-2) yields

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu, \qquad (2\text{-}3)$$

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and therefore

$$u = c + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \qquad (2\text{-}4)$$

where c is the integration constant and satisfies $Lc = 0$. Based on the ADM, the solution of Eq. (2-1) is regarded as

$$u = \sum_{n=0}^{\infty} u_n, \qquad (2\text{-}5)$$

and the nonlinear term Nu is decomposed as follows

$$Nu = \sum_{n=0}^{\infty} A_n, \qquad (2\text{-}6)$$

where the components A_n are Adomian's polynomials which are calculated by the formula

$$A_n(u_0, \dots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left[\sum_{k=0}^{\infty} \lambda^k u_k\right]\right]_{\lambda=0}, n \geq 0. \qquad (2\text{-}7)$$

Substituting (2-6) and (2-5) in (2-4) results

$$\sum_{n=0}^{\infty} u_n = c + L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \qquad (2\text{-}8)$$

Now according to the decomposition procedure of Adomian, we can obtain the components u_ns by the following recurrent relation

$$u_0 = c + L^{-1}g, \qquad (2\text{-}9)$$

$$u_n = -L^{-1}Ru_{n-1} - L^{-1}A_{n-1}, \quad n \geq 1. \qquad (2\text{-}10)$$

The n-term approximation of the solution is defined as $\theta_n = \sum_{k=0}^{n} u_k$ and $u = \lim_{n\to\infty}\theta_n$. As we know, the more terms added to the approximate solution, the more accurate it would be.

Convergence of Adomian decomposition scheme was established by many authors using fixed point theorem, see for example [4,5,6].

For inhomogeneous equations a simple strategy is used to increases the convergent rate. In performing the ADM, we can expand $g(x) = g_1(x) + g_2(x)$ and then we use the following substitution in (2-3) to get the exact solution

$$u_0 = c + L^{-1}g_1,$$
$$u_1 = L^{-1}g_2 - L^{-1}Ru_0 - L^{-1}A_0, \qquad (2\text{-}11)$$
$$u_n = -L^{-1}Ru_{n-1} - L^{-1}A_{n-1}, n \geq 2,$$

where c is computed with the initial conditions of the problem. the rate of convergence depends on choosing the functions g_1 and g_2. Usually they have chosen in such a way that $u_1 = u_2 = \cdots = 0$. This method is refereed as modified Adomian decomposition method (MADM).

3. Illustrations

In this section, we present some examples of Kuramoto-Tsuzuki equation. We compare the results with the exact solution.

Example 1: Consider the homogenous equation ([14])

$$\frac{\partial \omega}{\partial t} = (1+i)\frac{\partial^2 \omega}{\partial x^2} + \omega - (1+i)|\omega|^2\omega, \qquad (3\text{-}1)$$

with the initial and boundary conditions

$$\omega(x,0) = \frac{\sqrt{3}}{2}\exp\left(i\frac{x}{2}\right), \qquad (3\text{-}2)$$

$$\omega(x,t) = \omega(x+4\pi, t), t \in (0,T], \qquad (3\text{-}3)$$

and the exact solution is

$$\omega(x,t) = \frac{\sqrt{3}}{2}\exp\left(i\left(\frac{x}{2} - t\right)\right). \qquad (3\text{-}4)$$

Eq. (3-1) suggests the linear operator

$$L\omega = \frac{\partial \omega}{\partial t}, \qquad (3\text{-}5)$$

with the property $L(c) = 0$,

$$R\omega = -(1+i)\frac{\partial^2 \omega}{\partial x^2} - \omega, \qquad (3\text{-}6)$$

and the nonlinear operator as

$$N\omega = (1+i)|\omega|^2\omega = (1+i)\omega^2\overline{\omega}. \qquad (3\text{-}7)$$

The inverse operator L^{-1}, is as follows

$$L^{-1}\omega = c + \int_0^t \omega(\tau)d\tau, \qquad (3\text{-}8)$$

the initial condition yields that $c = 0$.

To solve Eq. (3-1) by means of ADM, we choose the initial approximation

$$\omega_0(x,t) = \omega(x,0) = \frac{\sqrt{3}}{2}\exp\left(i\frac{x}{2}\right), \quad (3\text{-}9)$$

then according to (2-9) we get

$$\omega_n(x,t) = \int_0^t \left[(1+i)\frac{\partial^2\omega_{n-1}(x,\tau)}{\partial x^2} + \omega_{n-1}(x,\tau) - (1+i)A_{n-1}(x,\tau)\right]d\tau, \quad (3\text{-}10)$$

where $A_n(x,t)$ is the nth Adomian polynomial. According to (2-7) A_n is obtain as

$$A_n = \sum_{k=0}^{n} \omega_{n-k} \sum_{s=0}^{k} \omega_{k-s}\overline{\omega}_s. \quad (3\text{-}11)$$

For example

$$\begin{aligned}
A_0 &= \omega_0^2\overline{\omega}_0,\\
A_1 &= \omega_0^2\overline{\omega}_1 + 2\omega_0\omega_1\overline{\omega}_0,\\
A_2 &= 2\omega_2\omega_0\overline{\omega}_0 + \omega_1^2\overline{\omega}_0 + 2\omega_1\omega_0\overline{\omega}_1 + \omega_0^2\overline{\omega}_2,\\
A_3 &= 2\omega_3\omega_0\overline{\omega}_0 + 2\omega_2\omega_1\overline{\omega}_0 + 2\omega_2\omega_0\overline{\omega}_1 + \omega_1^2\overline{\omega}_1 + 2\omega_1\omega_0\overline{\omega}_2 + \omega_0^2\overline{\omega}_3,\\
A_4 &= 2\omega_4\omega_0\overline{\omega}_0 + 2\omega_3\omega_1\overline{\omega}_0 + 2\omega_3\omega_0\overline{\omega}_1 + \omega_2^2\overline{\omega}_0 + 2\omega_2\omega_1\overline{\omega}_1 + 2\omega_2\omega_0\overline{\omega}_2\\
&\quad + \omega_1^2\overline{\omega}_2 + 2\omega_1\omega_0\overline{\omega}_3 + \omega_0^2\overline{\omega}_4,\\
&\vdots
\end{aligned} \quad (3\text{-}12)$$

From (3-9), (3-10) and (3-11) we successively obtain

$$\begin{aligned}
\omega_0(x,t) &= \frac{\sqrt{3}}{2}exp\left(i\frac{x}{2}\right),\\
\omega_1(x,t) &= \frac{\sqrt{3}}{2}exp\left(i\frac{x}{2}\right)(-it),\\
\omega_2(x,t) &= \frac{\sqrt{3}}{2}exp\left(i\frac{x}{2}\right)\frac{(-it)^2}{2!},\\
\omega_3(x,t) &= \frac{\sqrt{3}}{2}exp\left(i\frac{x}{2}\right)\frac{(-it)^3}{3!},\\
\omega_4(x,t) &= \frac{\sqrt{3}}{2}exp\left(i\frac{x}{2}\right)\frac{(-it)^4}{4!},\\
&\vdots
\end{aligned} \quad (3\text{-}13)$$

and so on. Therefore, we get the solution

$$\omega = \frac{\sqrt{3}}{2}exp\left(i\frac{x}{2}\right)\left[1 + (-it) + \frac{(-it)^2}{2!} + \frac{(-it)^3}{3!} + \frac{(-it)^4}{4!} + \cdots\right]. \quad (3\text{-}14)$$

Obviously (3-14) is the Taylor expansion of $\omega(x,t) = \frac{\sqrt{3}}{2}\exp\left(i(\frac{x}{2} - t)\right)$, which is the exact solution of the Eq. (3-1).

Example 2: Consider the inhomogeneous equation ([14])

$$\frac{\partial\omega}{\partial t} = (1+i)\frac{\partial^2\omega}{\partial x^2} + \omega - (1+i)|\omega|^2\omega + \psi(x,t), \quad (3\text{-}15)$$

where

$$\psi(x,t) = [\pi^2 - \sin^2\pi x + i(2t + 2\pi^2 - 3\cos^2\pi x)]e^{it^2}\cos\pi x, \quad (x,t) \in (0,1)\times(0,T], \quad (3\text{-}16)$$

with the initial and boundary conditions

$$\omega(x,0) = \cos\pi x, x \in [0,1], \quad (3\text{-}17)$$

$$\frac{\partial\omega}{\partial x}(0,t) = 0, \frac{\partial\omega}{\partial x}(1,t) = 0, t \in (0,T], \quad (3\text{-}18)$$

whose exact solution is

$$\omega(x,t) = e^{it^2}\cos\pi x. \quad (3\text{-}19)$$

We choose L, R and N like (3-5), (3-6) and (3-7). To solve Eq. (3-15) by means of MADM, we choose $\psi(x,t) = \psi_1(x,t) + \psi_2(x,t)$ where

$$\begin{aligned}
\psi_1(x,t) &= 2tie^{it^2}\cos\pi x,\\
\psi_2(x,t) &= [\pi^2 - \sin^2\pi x + i(2\pi^2 - 3\cos^2\pi x)]e^{it^2}\cos\pi x.
\end{aligned} \quad (3\text{-}20)$$

Using (2-11) we get

$$\omega_0(x,t) = L^{-1}\psi_1(x,t) + \omega(x,0) = e^{it^2}\cos\pi x,$$
$$\omega_1(x,t) = L^{-1}\psi_2(x,t) - L^{-1}Ru_0(x,t) - L^{-1}A_0(x,t) = 0 \qquad (3\text{-}21)$$
$$\omega_n(x,t) = -L^{-1}Ru_{n-1}(x,t) - L^{-1}A_{n-1}(x,t) = 0, n = 2,3,\dots,$$

then obviously we get the exact solution

$$\omega(x,t) = \sum_{n=0}^{\infty}\omega_n(x,t) = e^{it^2}\cos\pi x. \qquad (3\text{-}22)$$

Example 3: Consider the inhomogeneous initial-boundary nonlinear equation system ([1])

$$\frac{\partial\omega}{\partial t} = (1+i)\frac{\partial^2\omega}{\partial x^2} - (1+i)|\omega|^2\omega + \psi(x,t), \qquad (3\text{-}23)$$

where

$$\psi(x,t) = (1+i)\exp\left(-\frac{(i+1)t}{10}\right)\sin\pi x\left[-\frac{1}{10} + \pi^2 + \exp\left(-\frac{2t}{10}\right)\sin^2\pi x\right], \quad (x,t)\in(0,1)\times(0,T], \qquad (3\text{-}24)$$

with the initial and boundary conditions

$$\omega(x,0) = \sin\pi x, x\in[0,1], \qquad (3\text{-}25)$$

$$\omega(0,t) = \omega(1,t) = 0, t\in(0,T], \qquad (3\text{-}26)$$

whose exact solution is

$$\omega(x,t) = \exp\left(-\frac{(i+1)t}{10}\right)\sin\pi x. \qquad (3\text{-}27)$$

We choose L and N like (3-5) and (3-6) and $R = (1+i)\frac{\partial^2}{\partial x^2}$. To solve Eq. (3-15) by means of MADM, we choose

$$\psi_1(x,t) = -\frac{(1+i)}{10}\exp\left(-\frac{(i+1)t}{10}\right)\sin\pi x,$$
$$\psi_2(x,t) = \psi(x,t) = (1+i)\exp\left(-\frac{(i+1)t}{10}\right)\sin\pi x\left[\pi^2 + \exp\left(-\frac{2t}{10}\right)\sin^2\pi x\right]. \qquad (3\text{-}28)$$

Using (2-11) we get

$$\omega_0(x,t) = \exp\left(-\frac{(i+1)t}{10}\right)\sin\pi x, \qquad (3\text{-}29)$$

$$\omega_n(x,t) = 0, n = 1,2,\dots, \qquad (3\text{-}30)$$

then obviously we get the exact solution

$$\omega(x,t) = \sum_{n=0}^{\infty}\omega_n(x,t) = \exp\left(-\frac{(i+1)t}{10}\right)\sin\pi x. \qquad (3\text{-}31)$$

Example 4: Consider the inhomogeneous initial-boundary nonlinear equation system ([1])

$$\frac{\partial\omega}{\partial t} = (1+i)\frac{\partial^2\omega}{\partial x^2} - (1+i)|\omega|^2\omega + \psi(x,t), \qquad (3\text{-}32)$$

where

$$\psi(x,t) = \exp\left((1+i)\left(\frac{t}{5}-1\right)^2\right)\left[\frac{2}{5}\left(\frac{t}{5}-1\right) + \pi^2 + \exp\left(2\left(\frac{t}{5}-1\right)^2\right)\sin^2\pi x + \frac{2i}{5}\left(\frac{t}{5}-1\right)\right]\sin\pi x, \quad (x,t)\in(0,1)\times(0,T], \qquad (3\text{-}33)$$

with the initial and boundary conditions

$$\omega(x,0) = \exp(i+1)\sin\pi x, x\in[0,1], \qquad (3\text{-}34)$$

$$\omega(0,t) = \omega(1,t) = 0, t\in(0,T] \qquad (3\text{-}35)$$

whose exact solution is

$$\omega(x,t) = \exp\left((1+i)\left(\frac{t}{5}-1\right)^2\right)\sin\pi x. \qquad (3\text{-}36)$$

We choose L and N like (3-5) and (3-6) and $R = (1+i)\frac{\partial^2}{\partial x^2}$. To solve Eq. (3-15) by means modified ADM, we choose

$$\psi_1(x,t) = \frac{2}{5}\left(\frac{t}{5}-1\right)\exp\left((1+i)\left(\frac{t}{5}-1\right)^2\right), \qquad (3\text{-}37)$$

$$\psi_2(x,t) = \exp\left((1+i)\left(\frac{t}{5}-1\right)^2\right) \times \left[\pi^2 + \exp\left(2\left(\frac{t}{5}-1\right)^2\right)\sin^2\pi x + \frac{2i}{5}\left(\frac{t}{5}-1\right)\right]\sin\pi x,. \quad (3\text{-}38)$$

Using (2-11) we get

$$\omega_0(x,t) = \exp\left((1+i)\left(\frac{t}{5}-1\right)^2\right)\sin\pi x, \quad (3\text{-}39)$$

$$\omega_n(x,t) = 0, n = 1,2,..., \quad (3\text{-}40)$$

then obviously we get the exact solution

$$\omega(x,t) = \sum_{n=0}^{\infty} \omega_n(x,t) = \exp\left((1+i)\left(\frac{t}{5}-1\right)^2\right)\sin\pi x. \quad (3\text{-}41)$$

4. Conclusions

In this paper, Adomian decomposition method and modified Adomian decomposition method have successfully used for solving the homogeneous and inhomogeneous Kuramoto-Tsuzuki equations respectively. We provide the exact solutions for Kuramoto-Tsuzuki equations whereas finite difference schemes proposed in [14] and [1] requires more computations and get low accurate approximations.

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