Folding and Differential Equations of Some Curves in Minkowski Space

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Abstract: In this paper we will introduce a new connection between folding and differential equations of some curves in Minkowski space. The concept of folding on some curves in Minkowski space will be characterized by using differential equations. New types of linear ordinary differential equations are introduced. Theorems governing this connection are obtained.

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1. Introduction and definitions

As is well known, the theory of folding is always one of interesting topics in Euclidian and Non-Euclidian space and it has been investigated from the various viewpoints by many branches of topology and differential geometry [2, 3, 4, 5, 6, 8,14, 19]. Minkowski space represents one of the most intriguing and emblematic discoveries in the history of geometry. Although if it were introduced for a purely geometrical purpose, they came into prominence in many branches of mathematics and physics. This association with applied science and geometry generated synergistic effect: applied science gave relevance to Minkowski space and Minkowski space allowed formalizing practical problems[15, 16, 17, 18, 20].

Most folding problems are attractive from a pure mathematical standpoint, for the beauty of the problems themselves. The folding problems have close connections to important industrial applications. Linkage folding has applications in robotics and hydraulic tube bending. Paper folding has application in sheet-metal bending, packaging, and air–bag folding. Also, used folding to solve difficult problems related to shell structures in civil engineering and aero space design, namely buckling instability. Isometric folding between two Riemannian manifold may be characterized as maps that send piecewise geodesic segments to a piecewise geodesic segments of the same length. For a topological folding the maps do not preserves lengths, i.e. A map $\Im: M \to N$, where M and N are c^{∞} –Riemannian manifolds of dimension m, n respectively is said to be an isometric folding of M into N, iff for any piecewise geodesic path $\gamma: J \to M$, the induced path $\Im \circ \gamma: J \to N$ is a piecewise geodesic and of the same length as γ. If $\Im$ does not preserve length, then $\Im$ is a topological folding [1, 7, 9, 10, 11, 12, 13].

2. Main results

Our aim in this position is to establish types of theorems which describe different new types of folding and differential equations of some curves in three and four dimensional Minkowski space.

Let $L = p + tv$, where $p, v \in \mathbb{R}^3$ be the straight line in E_1^3 , and $f_1: L \to L$ be a folding from L into itself such that $f_1(L) = (p_1 + t^* v_1, p_2 + t^* v_2, p_3 + t^* v_3)$, $a \le t^* < \infty$, the point $(p_1 + a v_1, p_2 + a v_2, p_3 + a v_3)$ is a singular point of L or the set of points of L where f_1 fails to be differentials. The system of differential equations of this folding will be

$\dot{x} + \dot{y} + \dot{z} + k = 0, \ddot{x} + \ddot{y} + \ddot{z} = 0$

Which are defined on (a, ∞). Also, if $f_2(L) = (p_1 + t^* v_1, p_2 + t^* v_2, p_3 + t^* v_3)$, $a \le t^* \le b$, $a < b$, then, f_2 is a type of folding of $f_2(L)$ into itself with two singular points$(p_1 + a v_1, p_2 + a v_2, p_3 + a v_3)$, $(p_1 + b v_1, p_2 + b v_2, p_3 + b v_3)$.

Moreover, if $f_3(L) = (p_1 + t v_1, p_2 + t v_2, p_3 + t v_3)$, then, f_3 is a type of folding of $f_3(L)$ into itself without singular points.

By changing the value of $t \in (-\infty, \infty)$, we have an infinite types of foldings with a singular point, with two singular points, or without singular point like f_1, $f_2$and f_3 respectively. This foldings preserve the curvature.

From the above discussion we obtain the following theorem.

Theorem 1. Let f_i be a folding from L into itself by changing in the value of $t \in (-\infty, \infty)$ then f_i, $\forall i \in \mathbb{N}$, have the system of linear ordinary differential equations.

If $g: L \to L$ be the folding of L into itself such that $g(L) = (p_1 + t v_1, p_2 + t^n v_2, 0)$, $n > 1$, $n \in \mathbb{N}$, then this folding doesn't preserve the curvature $K(g(L)) \neq 0$, since $\ddot{y} \neq 0$. The system of differential equations of $g(L)$ will be

$\dot{x} + \dot{y} + \dot{z} - v_1 - (y - p_2) \cdot \frac{n}{t} = 0$,
$\ddot{x} + \ddot{y} + \ddot{z} - (y - p_2) \cdot \frac{n(n-1)}{t^2} = 0, \ldots$,
$x^{(n)} + y^{(n)} + z^{(n)} - n! \cdot v_2 = 0$,
$x^{(n+1)} + y^{(n+1)} + z^{(n+1)} = 0$

Thus, the above result can be formulated in the following theorem.

Theorem 2. Let $g(L) = p + t^n v, n > 1, n \in \mathbb{N}$, be a folding from L into itself and any folding homeomorphic to this type of folding, then $g(L)$ produce the $(n+1)$ - order system of ordinary differential equations.

Theorem 3. If $H: L \to L$ be the folding of L into itself such that $H(L) = p + t^{\frac{1}{n}} v, n > 1, n \in \mathbb{N}$, and any folding homeomorphic to this type of folding , then the corresponding system of ordinary differential equations is given by

$$\left.\begin{aligned} x^{(m)} &= \frac{(1-n)(1-2n)(1-3n)\cdots(1-(m-1)n)}{n^m}((x - p_1) \cdot t^{-m}) \quad y^{(m)} = \frac{(1-n)(1-2n)(1-3n)\cdots(1-(m-1)n)}{n^m}((y - p_2). t^{-m}), \\ z^{(m)} &= \frac{(1-n)(1-2n)(1-3n)\cdots(1-(m-1)n)}{n^m}((z - p_3) \cdot t^{-m}) \end{aligned}\right\}$$

, $m \in \mathbb{N}$.

Now, we consider the following types of folding on a circle $S^1 = p + r(cost, sint, 0), r \neq 0$, in 3- dimensional Minkowski space E_1^3 .

Let $f_1: S^1 \to S^1$ be a folding from S^1 into itself such that $f_1(S^1) = p + r(cost^*, sint^*, 0), 0 \leq t^* < \pi$, with one singular point. Also, let $f_2: S^1 \to S^1$ where $f_2(S^1) = p + r(cost^*, sint^*, 0)$, $\frac{\pi}{2} \leq t^* \leq \pi$, then f_2 is a kind of a folding from S^1 into itself with two singular points.

Again, $f_3: S^1 \to S^1$ is a folding from S^1 into itself such that $f_3(S^1) = p + r(cost, sint, 0)$, f_3 is a folding without singular point. Then by changing in the value of $t \in (0, 2\pi)$, we get an infinite number of foldings which can be represented by this differential equations:

$\dot{x} + \dot{y} + \dot{z} - x + p_1 + y - p_2 = 0$
$\ddot{x} + \ddot{y} + \ddot{z} + x + y - p_1 - p_2 = 0$
$\dddot{x} + \dddot{y} + \dddot{z} + x - y + p_2 - p_1 = 0$

So we can state the following theorem.

Theorem 4. All types of the foldings of the circle S^1 into itself given by changing in the value of $t \in (0, 2\pi)$ have the system of linear ordinary differential equations.

Now, let $g_1: S^1 \to S^1$ where $g_1(S^1) = p + r(cos\frac{t}{n}, sin\frac{t}{n}, 0)$, $n \in \mathbb{N}$, is folding from the circle S^1 into itself. Then the system of differential equations of $g_1(S^1)$ is:

$\dot{x} = \frac{-y}{n} + \frac{p_2}{n}$, $\quad \dot{y} = \frac{x}{n} - \frac{p_1}{n}$, $\quad \dot{z} = 0$
$\ddot{x} = \frac{-x}{n^2} + \frac{p_1}{n^2}$, $\quad \ddot{y} = \frac{-y}{n^2} + \frac{p_2}{n^2}$, $\quad \ddot{z} = 0$
$\dddot{x} = \frac{y}{n^3} - \frac{p_2}{n^3}$, $\quad \dddot{y} = \frac{-x}{n^3} + \frac{p_1}{n^3}$, $\quad \dddot{z} = 0$
$x^{(4)} = \frac{x}{n^4} - \frac{p_1}{n^4}$, $\quad y^{(4)} = \frac{y}{n^4} - \frac{p_2}{n^4}$, $\quad z^{(4)} = 0$

If $g_2: S^1 \to S^1$ be a folding from S^1 into itself such that $g_2(S^1) = p + r(cos\frac{t}{n}, -sin\frac{t}{n}, 0), n \in \mathbb{N}$, This folding can be represented by this linear ordinary differential equations:

$\dot{x} = \frac{y}{n} - \frac{p_2}{n}$, $\quad \dot{y} = \frac{-x}{n} + \frac{p_1}{n}$, $\quad \dot{z} = 0$
$\ddot{x} = \frac{-x}{n^2} + \frac{p_1}{n^2}$, $\quad \ddot{y} = \frac{-y}{n^2} + \frac{p_2}{n^2}$, $\quad \ddot{z} = 0$
$\dddot{x} = \frac{-y}{n^3} + \frac{p_2}{n^3}$, $\quad \dddot{y} = \frac{x}{n^3} - \frac{p_1}{n^3}$, $\quad \dddot{z} = 0$
$x^{(4)} = \frac{x}{n^4} - \frac{p_1}{n^4}$, $\quad y^{(4)} = \frac{y}{n^4} - \frac{p_2}{n^4}$, $\quad z^{(4)} = 0$

Generally, we will arrive the following theorem.

Theorem 5. If $g(S^1)$ be a folding from S^1 into itself such that $g(S^1) = p + r(cos\frac{(-1)^m t}{n}, sin\frac{(-1)^m t}{n}, 0)$, $n, m, q \in \mathbb{N}$, and m is odd or even then, $(x^{(2q-1)}, y^{(2q-1)}, z^{(2q-1)})$ is the same as $(-x^{(2q-1)}, -y^{(2q-1)}, -z^{(2q-1)})$ and, $(x^{(2q)}, y^{(2q)}, z^{(2q)})$ is the same as $(x^{(2q)}, y^{(2q)}, z^{(2q)})$.

Again, if, $h_1 : S^1 \to S^1$ is a folding from S^1 into itself defined by $h_1(S^1) = p + m\, r(cos\, t, sin\, t, 0)$, $m \in \mathbb{N}$. This folding can be represented by:

$\dot{x} = -y + p_2$, $\quad \dot{y} = x - p_1$, $\quad \dot{z} = 0$
$\ddot{x} = -x + p_1$, $\quad \ddot{y} = -y + p_2$, $\quad \ddot{z} = 0$
$\dddot{x} = y - p_2$, $\quad \dddot{y} = -x + p_1$, $\quad \dddot{z} = 0$
$x^{(4)} = x - p_1$, $\quad y^{(4)} = y - p_2$, $\quad z^{(4)} = 0$

Also, if $h_2 : S^1 \to S^1$ where $h_2(S^1) = p + \frac{1}{n} r(cos\, t , sin\, t, 0), n \in \mathbb{N}$, is a folding from S^1 into itself. Then the system of linear ordinary differential equations of $h_2(S^1)$ is:

$\dot{x} = -y + p_2$, $\quad \dot{y} = x - p_1$, $\quad \dot{z} = 0$
$\ddot{x} = -x + p_1$, $\quad \ddot{y} = -y + p_2$, $\quad \ddot{z} = 0$
$\dddot{x} = y - p_2$, $\quad \dddot{y} = -x + p_1$, $\quad \dddot{z} = 0$
$x^{(4)} = x - p_1$, $\quad y^{(4)} = y - p_2$, $\quad z^{(4)} = 0$

If $h_3: S^1 \to S^1$ is a folding from S^1 into itself such that $h_3(S^1) = p - m\, r(cos\, t, sin\, t, 0)$, $m \in \mathbb{N}$. Also, the linear ordinary differential equations of $h_3(S^1)$ is:

$\dot{x} = -y + p_2$, $\quad \dot{y} = x - p_1$, $\quad \dot{z} = 0$
$\ddot{x} = -x + p_1$, $\quad \ddot{y} = -y + p_2$, $\quad \ddot{z} = 0$
$\dddot{x} = y - p_2$, $\quad \dddot{y} = -x + p_1$, $\quad \dddot{z} = 0$
$x^{(4)} = x - p_1$, $\quad y^{(4)} = y - p_2$, $\quad z^{(4)} = 0$

Now, if $h_4 : S^1 \to S^1$ such that $h_4(S^1) = p - \frac{1}{n} r(\cos t, \sin t, 0)$, $n \in \mathbb{N}$, is a folding from S^1 into itself. Then $h_4(S^1)$ have the linear ordinary differential equations:

$$\dot{x} = -y + p_2, \quad \dot{y} = x - p_1, \quad \dot{z} = 0$$
$$\ddot{x} = -x + p_1, \quad \ddot{y} = -y + p_2, \quad \ddot{z} = 0$$
$$\dddot{x} = y - p_2, \quad \dddot{y} = -x + p_1, \quad \dddot{z} = 0$$
$$x^{(4)} = x - p_1, \quad y^{(4)} = y - p_2, \quad z^{(4)} = 0$$

So we can state the following theorem:

Theorem 6. If $h(S^1) = p + l\,r(\cos t, \sin t, 0)$ were $l \in (\mathbb{Q}^{+} \cup \mathbb{Q}^{-}) - \{0\}$ is a folding from the circle S^1 into itself then $h(S^1)$ have the system of linear ordinary differential equations. Also, the system of differential equations of $h(S^1)$when $l \in \mathbb{Q}^{+}$ is the same as the system of differential equations of $h(S^1)$when $l \in \mathbb{Q}^{-}$.

Now, let $f_1: H \to H$ be a folding of hyperbola $H = p + r\ (0, \sinh t, \cosh t)$, $r > 0$, $p \in \mathbb{R}^3$, in Minkowski 3-space E_1^3 into itself such that $f_1(H) = p + r\ (0, \sinh t^*, \cosh t^*)$, $c \le t^* < \infty$, this folding induces singular point $(p_1, p_2 + r \sinh c, p_3 + r \cosh c)$.

Also, f_2: $H \to H$ is a folding from H into itself such that $f_2(H) = p + r\ (0, \sinh t^*, \cosh t^*)$, $a \le t^* \le b$, $a < b$, then $f_2(H)$ is a kind of a folding from H into itself with two singular points $(p_1, p_2 + r\sinh a, p_3 + r\cosh a)$ and $(p_1, p_2 + r \sinh b, p_3 + r \cosh b)$. The system of linear ordinary differential equations are:

$$\dot{x} = 0, \quad \dot{y} = z - p_3, \quad \dot{z} = y - p_2$$
$$\ddot{x} = 0, \quad \ddot{y} = y - p_2, \quad \ddot{z} = z - p_3$$

Then we get the following theorem:

Theorem 7. Let f be the folding of H into itself by changing in the value of $t \in (-\infty, \infty)$ then f produce the 2 -order system of ordinary differential equations.

Now, if g: $H \to H$ is the folding of H into itself such that $g(H) = p + r\ (0, -\sinh t, \cosh t)$, then the system of differential equations of $g(H)$is:

$$\dot{x} = 0, \quad \dot{y} = -z + p_3, \quad \dot{z} = -y + p_2$$
$$\ddot{x} = 0, \quad \ddot{y} = y - p_2, \quad \ddot{z} = z - p_3$$

Also, let M_1: $H \to H$ where $M_1(H) = p + lr\ (0, \sinh t, \cosh t)$, $l \in \mathbb{Q}^+$, is the folding from H into itself, then the system of linear ordinary differential equations of $M_1(H)$is:

$$\dot{x} = 0, \quad \dot{y} = z - p_3, \quad \dot{z} = y - p_2$$
$$\ddot{x} = 0, \quad \ddot{y} = y - p_2, \quad \ddot{z} = z - p_3$$

Again, let M_2: $H \to H$ be the folding from H into itself such that $M_2(H) = p - lr\ (0, \sinh t, \cosh t)$, $l \in \mathbb{Q}^+$, This folding can be represented by this system of differential equations:

$$\dot{x} = 0, \quad \dot{y} = z - p_3, \quad \dot{z} = y - p_2$$
$$\ddot{x} = 0, \quad \ddot{y} = y - p_2, \quad \ddot{z} = z - p_3$$

Thus, the above result can be formulated in the following theorem:

Theorem 8. If $M\ (H) = p + (-1)^m\ lr\ (0, \sinh t, \cosh t)$, $m = 1, 2$, is a folding from H into itself and any folding homeomorphic to this type of folding, then the system of ordinary differential equations at m = 1 is the same as the system of ordinary differential equations at m = 2.

In this position, consider the hyperhelix in Minkowski 4-space defined as $H^2 = \left(a \cos\left(\frac{r}{\sqrt{a^2r^2+b^2}} t\right), a \sin\left(\frac{r}{\sqrt{a^2r^2+b^2}} t\right), b \cos\left(\frac{1}{\sqrt{a^2r^2+b^2}} t\right), b \sin\left(\frac{1}{\sqrt{a^2r^2+b^2}} t\right)\right)$. Now, we introduce types of foldings of the hyperhelix H^2

Let f_1: $H^2 \to H^2$ where $f_1(H^2) = \left(a \cos\left(\frac{r}{\sqrt{a^2r^2+b^2}} t^*\right), a \sin\left(\frac{r}{\sqrt{a^2r^2+b^2}} t^*\right), b \cos\left(\frac{1}{\sqrt{a^2r^2+b^2}} t^*\right), b \sin\left(\frac{1}{\sqrt{a^2r^2+b^2}} t^*\right)\right)$, $0 \le t^* < \sqrt{a^2r^2 + b^2}\,\pi$, is a folding from H^2 into itself.

Also, let f_2: $H^2 \to H^2$ such that $f_2(H^2) = \left(a \cos\left(\frac{r}{\sqrt{a^2r^2+b^2}} t^*\right), a \sin\left(\frac{r}{\sqrt{a^2r^2+b^2}} t^*\right), b \cos\left(\frac{1}{\sqrt{a^2r^2+b^2}} t^*\right), b \sin\left(\frac{1}{\sqrt{a^2r^2+b^2}} t^*\right)\right)$, $\sqrt{a^2r^2 + b^2}\,\frac{\pi}{2} \le t^* \le \sqrt{a^2r^2 + b^2}\,\frac{3\pi}{2}$, be a folding from H^2 into itself. This type of folding with singular points and this folding can be represented by:

$$\dot{x}_1 = -\frac{r}{\sqrt{a^2r^2+b^2}} x_2, \quad \dot{x}_2 = \frac{r}{\sqrt{a^2r^2+b^2}} x_1,$$
$$\dot{x}_3 = -\frac{1}{\sqrt{a^2r^2+b^2}} x_4, \quad \dot{x}_4 = \frac{1}{\sqrt{a^2r^2+b^2}} x_3,$$
$$\ddot{x}_1 = -\frac{r^2}{a^2r^2+b^2} x_1, \quad \ddot{x}_2 = -\frac{r^2}{a^2r^2+b^2} x_2,$$
$$\ddot{x}_3 = -\frac{1}{a^2r^2+b^2} x_3, \quad \ddot{x}_4 = -\frac{1}{a^2r^2+b^2} x_4$$

Then we get the following theorem

Theorem 9. Let f be the folding of H^2 into itself by changing in the value of t, $n \in \mathbb{N}$, then the corresponding system is given by:

$$x_1^{(2n-1)} = (-1)^n \frac{r^{2n-1}}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_2,$$
$$x_2^{(2n-1)} = (-1)^{n+1} \frac{r^{2n-1}}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_1,$$
$$x_3^{(2n-1)} = (-1)^n \frac{1}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_4,$$

$x_4{}^{(2n-1)} = (-1)^{n+1} \frac{1}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_3.$

$x_1{}^{(2n)} = (-1)^n \frac{r^{2n}}{(a^2r^2+b^2)^n} x_1,$

$x_2{}^{(2n)} = (-1)^n \frac{r^{2n}}{(a^2r^2+b^2)^n} x_2 ,$

$x_3{}^{(2n)} = (-1)^n \frac{1}{(a^2r^2+b^2)^n} x_3,$

$x_4{}^{(2n)} = (-1)^n \frac{1}{(a^2r^2+b^2)^n} x_4 .$

Now, Let $g: H^2 \to H^2$ be a folding from H^2 into itself such that $g(H^2) = (a\cos\left(\frac{r}{m\sqrt{a^2r^2+b^2}}t\right), a\sin\left(\frac{r}{m\sqrt{a^2r^2+b^2}}t\right), b\cos\left(\frac{1}{m\sqrt{a^2r^2+b^2}}t\right), b\sin\left(\frac{1}{m\sqrt{a^2r^2+b^2}}t\right))$, where $m, n \in \mathbb{N}$. Then the system of linear ordinary differential equations of $g(H^2)$ is:

$\dot{x_1} = -\frac{r}{m\sqrt{a^2r^2+b^2}} x_2,$ $\dot{x_2} = \frac{r}{m\sqrt{a^2r^2+b^2}} x_1 ,$

$\dot{x_3} = -\frac{1}{m\sqrt{a^2r^2+b^2}} x_4,$ $\dot{x_4} = \frac{1}{m\sqrt{a^2r^2+b^2}} x_3,$

$\ddot{x_1} = -\frac{r^2}{m^2(a^2r^2+b^2)} x_1,$ $\ddot{x_2} = -\frac{r^2}{m^2(a^2r^2+b^2)} x_2 ,$

$\ddot{x_3} = -\frac{1}{m^2(a^2r^2+b^2)} x_3,$ $\ddot{x_4} = -\frac{1}{m^2(a^2r^2+b^2)} x_4, \ldots,$

$x_1{}^{(2n-1)} = (-1)^n \frac{r^{2n-1}}{m^{2n-1}(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_2,$

$x_2{}^{(2n-1)} = (-1)^{n+1} \frac{r^{2n-1}}{m^{2n-1}(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_1,$

$x_3{}^{(2n-1)} = (-1)^n \frac{1}{m^{2n-1}(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_4,$

$x_4{}^{(2n-1)} = (-1)^{n+1} \frac{1}{m^{2n-1}(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_3,$

$x_1{}^{(2n)} = (-1)^n \frac{r^{2n}}{m^{2n}(a^2r^2+b^2)^n} x_1,$

$x_2{}^{(2n)} = (-1)^n \frac{r^{2n}}{m^{2n}(a^2r^2+b^2)^n} x_2,$

$x_3{}^{(2n)} = (-1)^n \frac{1}{m^{2n}(a^2r^2+b^2)^n} x_3,$

$x_4{}^{(2n)} = (-1)^n \frac{1}{m^{2n}(a^2r^2+b^2)^n} x_4.$

Generally, we will arrive the following theorem

Theorem 10. If g is the folding of H^2into itself by the change in the angles, then $g(H^2)$ produce the n - order system of ordinary differential equations.

Also, let $M: H^2 \to H^2$ be a folding from H^2 into itself defined as $M(H^2) = (c\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}t\right), c\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}t\right), d\cos\left(\frac{1}{\sqrt{a^2r^2+b^2}}t\right), d\sin\left(\frac{1}{\sqrt{a^2r^2+b^2}}t\right))$, where $c \neq a$ and $d \neq b, n \in \mathbb{N}$. This folding can be represented by this system of differential equations:

$\dot{x_1} = -\frac{r}{\sqrt{a^2r^2+b^2}} x_2,$ $\dot{x_2} = \frac{r}{\sqrt{a^2r^2+b^2}} x_1 ,$

$\dot{x_3} = -\frac{1}{\sqrt{a^2r^2+b^2}} x_4,$ $\dot{x_4} = \frac{1}{\sqrt{a^2r^2+b^2}} x_3,$

$\ddot{x_1} = -\frac{r^2}{a^2r^2+b^2} x_1,$ $\ddot{x_2} = -\frac{r^2}{a^2r^2+b^2} x_2 ,$

$\ddot{x_3} = -\frac{1}{a^2r^2+b^2} x_3,$ $\ddot{x_4} = -\frac{1}{a^2r^2+b^2} x_4, \ldots,$

$x_1{}^{(2n-1)} = (-1)^n \frac{r^{2n-1}}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_2,$

$x_2{}^{(2n-1)} = (-1)^{n+1} \frac{r^{2n-1}}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_1 ,$

$x_3{}^{(2n-1)} = (-1)^n \frac{1}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_4,$

$x_4{}^{(2n-1)} = (-1)^{n+1} \frac{1}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_3,$

$x_1{}^{(2n)} = (-1)^n \frac{r^{2n}}{(a^2r^2+b^2)^n} x_1 ,$

$x_2{}^{(2n)} = (-1)^n \frac{r^{2n}}{(a^2r^2+b^2)^n} x_2 ,$

$x_3{}^{(2n)} = (-1)^n \frac{1}{(a^2r^2+b^2)^n} x_3,$

$x_4{}^{(2n)} = (-1)^n \frac{1}{(a^2r^2+b^2)^n} x_4.$

Again, let $Q: H^2 \to H^2$ be a folding from H^2 into itself such that $Q(H^2) = (a\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}t\right), -a\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}t\right), b\cos\left(\frac{1}{\sqrt{a^2r^2+b^2}}t\right), -b\sin\left(\frac{1}{\sqrt{a^2r^2+b^2}}t\right))$, $n \in \mathbb{N}$. The system of linear ordinary differential equations of $Q(H^2)$ is:

$\dot{x_1} = \frac{r}{\sqrt{a^2r^2+b^2}} x_2 ,$ $\dot{x_2} = -\frac{r}{\sqrt{a^2r^2+b^2}} x_1 ,$

$\dot{x_3} = \frac{1}{\sqrt{a^2r^2+b^2}} x_4 ,$ $\dot{x_4} = -\frac{1}{\sqrt{a^2r^2+b^2}} x_3,$

$\ddot{x_1} = -\frac{r^2}{a^2r^2+b^2} x_1,$ $\ddot{x_2} = -\frac{r^2}{a^2r^2+b^2} x_2 ,$

$\ddot{x_3} = -\frac{1}{a^2r^2+b^2} x_3,$ $\ddot{x_4} = -\frac{1}{a^2r^2+b^2} x_4, \ldots,$

$x_1{}^{(2n-1)} = (-1)^{n+1} \frac{r^{2n-1}}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_2 ,$

$x_2{}^{(2n-1)} = (-1)^n \frac{r^{2n-1}}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_1 ,$

$x_3{}^{(2n-1)} = (-1)^{n+1} \frac{1}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_4,$

$x_4{}^{(2n-1)} = (-1)^n \frac{1}{(a^2r^2+b^2)^{\frac{2n-1}{2}}} x_3.$

$x_1{}^{(2n)} = (-1)^n \frac{r^{2n}}{(a^2r^2+b^2)^n} x_1,$

$x_2{}^{(2n)} = (-1)^n \frac{r^{2n}}{(a^2r^2+b^2)^n} x_2 ,$

$x_3{}^{(2n)} = (-1)^n \frac{1}{(a^2r^2+b^2)^n} x_3,$

$x_4{}^{(2n)} = (-1)^n \frac{1}{(a^2r^2+b^2)^n} x_4.$

Thus, the above result can be formulated in the following theorem:

Theorem 11. Let $Q(H^2)$ be a folding from H^2 into itself where $Q(H^2) = (a\cos\left(\frac{(-1)^m r}{\sqrt{a^2r^2+b^2}}t\right), a\sin\left(\frac{(-1)^m r}{\sqrt{a^2r^2+b^2}}t\right), b\cos\left(\frac{(-1)^m}{\sqrt{a^2r^2+b^2}}t\right), b\sin\left(\frac{(-1)^m}{\sqrt{a^2r^2+b^2}}t\right))$, $m, q \in \mathbb{N}$, and m is odd or even, then $({x_1}^{(2q)}, {x_2}^{(2q)}, {x_3}^{(2q)}, {x_4}^{(2q)})$ is same as $({x_1}^{(2q)}, {x_2}^{(2q)}, {x_3}^{(2q)}, {x_4}^{(2q)})$.

Now, let $\alpha(t) = (at, \sinh t, \cosh t)$, $a \neq 0$, be the curve in Minkowski space E_1^3 and $\acute{\alpha}(t) = (a, \cosh t, \sinh t)$. Consider the following types of folding on $\alpha(t)$. Let $f_1 : \alpha(t) \to \alpha(t)$ be a folding from $\alpha(t)$ into itself such that $f_1(\alpha(t)) = (at^*, \sinh t^*, \cosh t^*)$, $c_1 \leq t^* < \infty$, $c_1 \in \mathbb{R}$. Then the point $(ac_1, \sinh c_1, \cosh c_1)$ is a singular point. Also, $f_2 : \alpha(t) \to \alpha(t)$ were $f_2(\alpha(t)) = (at^*, \sinh t^*, \cosh t^*)$, $c_1 \leq t^* \leq c_2$, $c_1, c_2 \in \mathbb{R}$, then f_2 is a kind of a folding from $\alpha(t)$ into itself with two singular points $(ac_1, \sinh c_1, \cosh c_1)$ and $(ac_2, \sinh c_2, \cosh c_2)$. Again, if $f_3 : \alpha(t) \to \alpha(t)$ be defined as $f_3((at, \sinh t, \cosh t)) = (at, \sinh t, \cosh t)$ then f_3 is a folding from $\alpha(t)$ into itself and f_3 give1-manifold without singular point. Then by changing in the value of $t \in (-\infty, \infty)$, we get types of folding can be represented by this differential equations:

$\dot{x} = a$, $\dot{y} = z$, $\dot{z} = y$,
$\ddot{x} = 0$, $\ddot{y} = y$, $\ddot{z} = z$.

So we can state the following theorem.

Theorem 12. All the folding of $\alpha(t)$ into itself by changing the value of $t \in (-\infty, \infty)$ have the same linear ordinary differential equations.

Moreover, if $g : \alpha(t) \to \alpha(t)$, $g(\alpha(t)) = (at, \sinh^n t, \cosh^n t)$, $n \neq 0$, g is a folding from $\alpha(t)$ into itself. The system of linear ordinary differential equations of $g(\alpha(t))$ is:

$\dot{x} = a$, $\dot{y} = n\, y^{\frac{n-1}{n}} z^{\frac{1}{n}}$, $\dot{z} = n\, z^{\frac{n-1}{n}} y^{\frac{1}{n}}$

$\ddot{x} = 0$, $\ddot{y} = n(n-1)\, y^{\frac{n-2}{n}} z^{\frac{2}{n}} + ny$, $\ddot{z} = n(n-1)\, z^{\frac{n-2}{n}} y^{\frac{2}{n}} + nz$

$\dddot{x} = 0$,

$\dddot{y} = n(n-1)(n-2)\, y^{\frac{n-3}{n}} z^{\frac{3}{n}} + 2n(n-1)\, y^{\frac{n-1}{n}} z^{\frac{1}{n}} + n^2 y^{\frac{n-1}{n}} z^{\frac{1}{n}}$,

$\dddot{z} = n(n-1)(n-2)\, z^{\frac{n-3}{n}} y^{\frac{3}{n}} + 2n(n-1)\, z^{\frac{n-1}{n}} y^{\frac{1}{n}} + n^2 z^{\frac{n-1}{n}} y^{\frac{1}{n}}$.

Then we get the following theorem.

Theorem 13. Under the defined folding and any folding homeomorphic to this type of folding the system of linear ordinary differential equations of $g(\alpha(t))$ is an infinite system.

Let $M : \alpha(t) \to \alpha(t)$ be a folding from $\alpha(t)$ into itself such that $M(\alpha(t)) = (at^n, \sinh t, \cosh t)$. This folding can be represented by the differential equations:

$\dot{x} = ant^{n-1}$, $\dot{y} = z$, $\dot{z} = y$
$\ddot{x} = an(n-1)t^{n-2}$, $\ddot{y} = y$, $\ddot{z} = z$
$\dddot{x} = an(n-1)(n-2)t^{n-3}$, $\dddot{y} = z$, $\dddot{z} = y$, ...,
$x^{(n)} = a(n!)$, $y^{(n)} = y^{(n-2)}$, $z^{(n)} = z^{(n-2)}$
$x^{(n+1)} = 0$, $y^{(n+1)} = y^{(n-1)}$, $z^{(n+1)} = z^{(n-1)}$

Generally, we will arrive the following theorem:

Theorem 14. The folding of $M(\alpha(t))$ and any folding homeomorphic to this type of folding produce the (n+1) - order system of ordinary differential equations.

Now, if $E : \alpha(t) \to \alpha(t)$ where $E((at, \sinh t, \cosh t)) = (ct, b\sinh t, b\cosh t)$ is a folding from $\alpha(t)$ into itself, then the system of linear ordinary differential equations of $E(\alpha(t))$ is:

$\dot{x} = c$, $\dot{y} = z$, $\dot{z} = y$
$\ddot{x} = 0$, $\ddot{y} = y$, $\ddot{z} = z$
$\dddot{x} = 0$, $\dddot{y} = z$, $\dddot{z} = y$

Also, let the folding F: $\alpha(t) \to \alpha(t)$, $w \in \mathbb{N}$, be given by $F((at, \sinh t, \cosh t)) = (at, \sinh(wt), \cosh(wt))$, $w \in \mathbb{N}$, from $\alpha(t)$ into itself without singular point. Then, $F(\alpha(t))$ has the system of linear ordinary differential equations is:

$\dot{x} = a$, $\dot{y} = wz$, $\dot{z} = wy$
$\ddot{x} = 0$, $\ddot{y} = w^2 y$, $\ddot{z} = w^2 z$
$\dddot{x} = 0$, $\dddot{y} = w^3 z$, $\dddot{z} = w^3 y$.

Also, If $K : \alpha(t) \to \alpha(t)$ such that $K((at, \sinh t, \cosh t)) = \left(at, \sinh\frac{t}{w}, \cosh\frac{t}{w}\right)$, $w \in \mathbb{N}$, is a folding from $\alpha(t)$ into itself. Then the system of differential equations of $K(\alpha(t))$ is:

$\dot{x} = a$, $\dot{y} = \frac{1}{w} z$, $\dot{z} = \frac{1}{w} y$
$\ddot{x} = 0$, $\ddot{y} = \frac{1}{w^2} y$, $\ddot{z} = \frac{1}{w^2} z$
$\dddot{x} = 0$, $\dddot{y} = \frac{1}{w^3} z$, $\dddot{z} = \frac{1}{w^3} y$.

Let $Z : \alpha(t) \to \alpha(t)$ be a folding from $\alpha(t)$ into itself such that $Z(\alpha(t)) = (at, -\sinh(wt), \cosh(wt))$, $w \in \mathbb{N}$. This folding can be represented by the ordinary differential equations:

$\dot{x} = a$, $\dot{y} = -wz$, $\dot{z} = -wy$
$\ddot{x} = 0$, $\ddot{y} = w^2 y$, $\ddot{z} = w^2 z$
$\dddot{x} = 0$, $\dddot{y} = -w^3 z$, $\dddot{z} = -w^3 y$.

Also, if $W : \alpha(t) \to \alpha(t)$ is a folding from $\alpha(t)$ into itself such that $W((at, \sinh t, \cosh t)) = \left(at, -\sinh\frac{t}{w}, \cosh\frac{t}{w}\right)$, $w \in \mathbb{N}$. The system of differential equations of $W(\alpha(t))$ is:

$\dot{x} = a$, $\dot{y} = \frac{-1}{w} z$, $\dot{z} = \frac{-1}{w} y$
$\ddot{x} = 0$, $\ddot{y} = \frac{1}{w^2} y$, $\ddot{z} = \frac{1}{w^2} z$
$\dddot{x} = 0$, $\dddot{y} = \frac{-1}{w^3} z$, $\dddot{z} = \frac{-1}{w^3} y$.

Thus, the above result can be formulated in the following theorem:

Theorem 15. Let $V(\alpha(t)) = (at, \sinh(ct), \cosh(ct))$, $c \in (\mathbb{Q}^+ \cup \mathbb{Q}^-) - \{0\}$ be a folding from

$\alpha\ (t)$ into itself and any folding homeomorphic to this type of folding . Then the corresponding system of $V(\ \alpha\ (t))$ is given by:

$\dot{x} = a$, $\dot{y} = cz$, $\dot{z} = cy$

$x^{(i)} = 0$, $y^{(i)} = c^i y$, $z^{(i)} = c^i z$,

were $i > 1$, i is even

$x^{(j)} = 0$, $y^{(j)} = c^j z$, $z^{(j)} = c^j y,$

were $j > 1$, j is odd

Now, let $L = p + tv$, $S^1 = p + r(cost,sint,0)$, $r \neq 0$ be the straight line and the circle in Minkowski space E_1^3 and $\mathcal{H}$ be a folding defined as $\mathcal{H}: L \to S^1$, where $\mathcal{H}(L) = p + (b\cos nt,\ b\sin nt,\ 0)$, then $\mathcal{H}(L)$ is a covering space of S^1 and the system of differential equations of $\mathcal{H}(L)$ is:

$\dot{x} = -ny$, $\dot{y} = nx$, $\dot{z} = 0$

$\ddot{x} = -n^2 x$, $\ddot{y} = -n^2 y$, $\ddot{z} = 0$

$\dddot{x} = -n^3 y$, $\dddot{y} = -n^3 x$, $\dddot{z} = 0$

Also, let $\mathfrak{I}: S^1 \to S^1$, be a type of folding of a circle into itself without singular point defined as $\mathcal{H}(L) = p + (r\cos nt\ ,r\sin nt,\ 0)$, then $\mathfrak{I}(\ S^1)$ is a covering space of S^1.

So we can state the following theorem.

Theorem 16. The system of linear ordinary differential equations of a covering space of the circle S^1 is an infinite.

Theorem 17. If $\mathcal{H}:\ L\ \to\ S^1$, where $\mathcal{H}(L) = (b\cos nt\ ,b\sin nt,\ 0), \forall\ \ i \in \mathbb{N}$ be a folding of $\mathcal{H}(L)$ into S^1, then the corresponding system of a covering space of S^1 is given by:

$x^{(2i-1)} = (-1)^i\ n^{2i-1}\ y,\ y^{(2i-1)} = (-1)^{1+i}\ n^{2i-1}\ x$, $z^{(2i-1)} = 0$

$x^{(2i)} = (-1)^i\ n^{2i}\ x$, $y^{(2i)} = (-1)^{2+i}\ n^{2i}\ y$, $z^{(2i)} = 0$

3. Conclusion

In this paper we achieved the approval of the important of the foldings and differential equations of some curves in Minkowski space. The relations between foldings and types of linear ordinary differential equations are introduced. Theorems which governs these relations are presented.

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