DOUBLE PARAPROXIMITY SPACES

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Abstract: We introduce the concept of a double completely normal topological space or DT_5 – space and double Paraproximity space showing that every double space induces a double completely normal topological space and vice verse.

[A. Kandi l, O. Tantawy, K. Barakat, and N. Abdelnaby **Double Paraproximity Spaces]** Life Science Journal,. 2011; 8(4):800-804] (ISSN: 1097-8135). http://www.lifesciencesite.com.

Keywords: topological space; double Paraproximity space; mathematics.

1. Introduction

The mathematical idea of double sets was firstly introduced by Kere [9] and studied in many articles by A. Kandil and others "[1], [4], [6], [7], [8]". They introduced and studied many topics in the double topology. The Paraproximity structure subject has been introduced by *E*. Hayashi in 1964 [3]. Recently, Kandil and others introduced the fuzzy Paraproximity structure [5].

In this paper we shall introduce the separation axiom DT_5 (double completely normal) on double topological spaces and study some of its properties. Also, we shall introduce the notion of DP-proximity in the case of double topological space showing that every DP-Proximity on X generates a double completely normal (DT_5) topology.

2. **Preliminaries**

Throughout this section we mention the concepts and notations which we shall use in this paper.

2-1 Double set

Definition 2.1.1. [8] Let X be a non empty set.

1. A double set $\underline{A}$ is an ordered pair $\underline{A}=(A_1,A_2)\in P(X)\times P(X)$ such that $A_1\subseteq A_2$.
2. $D(X)=\{(A_1,A_2)\in P(X)\times P(X): A_1\subseteq A_2\}$ is the family of all double sets on *X*.
3. Let $x\in X$, then the double sets $x_{1/2}=(\phi,\{x\})$ and $x_1=(\{x\},\{x\})$ are said to be double points of X. $D(X)_p=\{x_t : x\in X, t=\{1/2,1\}\}$ is the set of all double points on *X*.
4. Let $\eta_1,\eta_2\subseteq P(X)$. Then the double product of η_1 and η_2 is denoted by $\eta_1\hat{\times}\eta_2$ and is defined by: $\eta_1\hat{\times}\eta_2=\{(A_1,A_2)\in\eta_1\times\eta_2 : A_1\subseteq A_2\}$.
5. The double set $\underline{X}$ = *(X, X)* is called the universal double set.
6. The double set $\underline{\phi}=(\phi,\phi)$ is called the empty double set.

Definition 2.1.2. [8] Let $\underline{A}=(A_1,A_2)$, $\underline{B}=(B_1,B_2)\in D(X)$. Then:

i- $\underline{A}=\underline{B}\Leftrightarrow A_i=B_i, i=1,2$.

ii- $\underline{A}\subseteq\underline{B}\Leftrightarrow A_i\subseteq B_i, i=1,2$.

iii- If $\{\underline{A}_s : s\in S\}\subseteq D(X)$, then $\bigcup_{s\in S}\underline{A}_s=(\bigcup_{s\in S}\underline{A}_{1s},\bigcup_{s\in S}\underline{A}_{2s})$ and $\bigcap_{s\in S}\underline{A}_s=(\bigcap_{s\in S}\underline{A}_{1s},\bigcap_{s\in S}\underline{A}_{2s})$

iv- The complement of a double set $\underline{A}$ is $\underline{A}^c=(\underline{A}_2^c,\underline{A}_1^c)$

v- $\underline{A}-\underline{B}=\underline{A}\cap\underline{B}^c$.

Proposition 2.1.3. [8] $(D(X),\cup,\cap,^c)$ is a Morgan Algebra.

Definition 2.1.4. [8] For any two double sets $\underline{A}$ and $\underline{B}$, $\underline{A}$ is called quasi-coincident to $\underline{B}$, denoted by $\underline{A}\ Q\underline{B}$, if $A_1\cap B_2\neq\phi$ or $A_2\cap B_1\neq\phi$.

$\underline{A}$ is not quasi – coincident to $\underline{B}$, denoted by $\underline{A}\ \overline{Q}\ \underline{B}$, if $A_1\cap B_2=\phi$ and $A_2\cap B_1=\phi$.

Theorem 2.1.5. [8] Let $\underline{A},\underline{B},\underline{C}\in D(X)$ and $x_t\in D(X)_p$. Then :

1- $\underline{A}Q\underline{B}\Rightarrow\underline{A}\cap\underline{B}\neq\underline{\phi}$

2- $\underline{A} Q \underline{B} \Leftrightarrow \exists x_t \in \underline{A}$ such that $x_t \; Q \underline{B}$

3- $\underline{A} \; \overline{Q} \; \underline{B} \Leftrightarrow \underline{A} \subseteq \underline{B}^c$, and $x_t \; \overline{Q} \; \underline{A} \Leftrightarrow$ $x_t \in \underline{A}^c$.

4- $\underline{A} \; \overline{Q} \; \underline{A}^c$.

5- $\underline{A} \subseteq \underline{B} \Leftrightarrow x_t \; Q \underline{A} \Rightarrow x_t \; Q \underline{B}$.

6- $\underline{A} \; \overline{Q} \; \underline{B}, \underline{B} \subseteq \underline{C} \Rightarrow \underline{A} \; \overline{Q} \; \underline{C}$.

7- $x_t \; \overline{Q} \; (\underline{A} \cup \underline{B}) \Leftrightarrow x_t \; \overline{Q} \; \underline{A}$ and $x_t \; \overline{Q} \; \underline{B}$.

8- $x_t \; \overline{Q} \; (\underline{A} \cap \underline{B}) \Leftrightarrow x_t \; \overline{Q} \; \underline{A}$ or $x_t \; \overline{Q} \; \underline{B}$.

2.2 Double Topological Space

Definition 2.2.1. [8] Let X be a non-empty set. Then:

1- $\tau \subseteq D(X)$ is called a double toplogy on X if the following axioms are satisfied:

DT_1 $\quad \phi, X \in \tau$.

DT_2 $\quad$ if $\underline{A}, \underline{B} \in \tau$, then $\underline{A} \cap \underline{B} \in \tau$.

DT_3 $\quad$ if $\{\underline{A}_s : s \in S\} \subseteq \tau$, then $\bigcup_{s \in S} \underline{A}_s \in \tau$.

The pair (X, τ) is called a double topological space and member of τ are called open double set.

2- $\underline{F} \in D(X)$ is called a closed double set, if $\underline{F}^c \in \tau$, and the family of all closed double sets is denoted by $\tau^c = \{\underline{F} : \underline{F}^c \in \tau\}$.

3- A double set $\underline{O}_{x_t}$ is called a double neighborhood of the double point x_t if $x_t \in \underline{O}_{x_t} \in \tau$. The family of all double neighborhoods of xt will be denoted by $\underline{N}(x_t)$.

4- If $\underline{A} \in D(X)$, Then:

(i) The closure of $\underline{A}$ is denoted by $\bar{\underline{A}}$ or cL$(\underline{A})$ and is defined by $\bar{\underline{A}} = \bigcap\{\underline{F} : \underline{A} \subseteq \underline{F} \in \tau^c\}$.

(ii) The interior of $\underline{A}$ is denoted by $\underline{A}^o$ or int$(\underline{A})$ and $\underline{A}^o = \bigcup\{\underline{V} : \underline{V} \in \tau, \underline{V} \subseteq \underline{A}\}$.

(iii) The derived double set of $\underline{A}$ is denoted by $\underline{A}^d$ and is given by: $x_t \; Q \underline{A}^d \Leftrightarrow x_t \; Q \; (\underline{A} - \{x_t\})$.

Definition 2.2.2. [6] A double topological space (X, τ) is called :

1- $D - R_0$ iff $x_t \; \overline{Q} \; \overline{y_r}$ implies $y_r \; \overline{Q} \; \overline{x_t}$.

2- $D - R_1$ iff $x_t \; \overline{Q} \; \overline{y_r}$ implies there exist $\underline{O}_{x_t}$, $\underline{O}_{y_r}$ such that $\underline{O}_{x_t} \; \overline{Q} \; \underline{O}_{y_r}$.

3- $D - R_2$ iff $x_t \; \overline{Q} \; \underline{F}$, $\underline{F}^c \in \tau$ implies that there exist $\underline{O}_{x_t}$, $\underline{O}_{F}$ such that $\underline{O}_{x_t} \; \overline{Q} \; \underline{O}_F$.

4- $D - R_3$ iff $\underline{F} \; \overline{Q} \; \underline{G}$ and $\underline{F}, \underline{G} \in \tau^c$ implies that there exist $\underline{O}_F$, $\underline{O}_G$ such that $\underline{O}_F \; \overline{Q} \; \underline{O}_G$.

5- D-T_0 iff $x_t \; \overline{Q} \; y_r$ implies $y_r \; \overline{Q} \; \bar{x}_t$ or $x_t \; \overline{Q} \; \bar{y}_r$.

6- D-T_1 iff $x_t \; \overline{Q} \; y_r$ implies $y_r \; \overline{Q} \; \bar{x}_t$ and $x_t \; \overline{Q} \; \bar{y}_r$.

7- D- T_2 iff $x_t \; \overline{Q} \; y_r$ implies that there exist $\underline{O}_{x_t}$, $\underline{O}_{y_r}$ such that $\underline{O}_{x_t} \; \overline{Q} \; \underline{O}_{y_r}$

8- $D - T_3$ iff it is $D - R_2$ and D-T_1.

9- D-T_4 iff it is D-R_3 and D-T_1 .

Theorem2.2.3. [6] The interrelation between the pervious axioms given in the following diagram:

$$\begin{array}{ccccccccc} DT_4 & \Rightarrow & DT_3 & \Rightarrow & DT_2 & \Rightarrow & DT_1 & \Rightarrow & DT_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ DR_3 & \Rightarrow & DR_2 & \Rightarrow & DR_1 & \Rightarrow & DR_0 & \Rightarrow & \text{any double topology} \end{array}$$

2.3 Completely normal spaces and paraproximity spaces

Definition2.3.1. [king] An ordinary toplogical space (X, τ) is called completely normal space iff for every pair of separated sets A, B $\subset X$, there exist open sets U, $V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$.

Definition *2.3.2.* [king] A mapping $\delta : P(X) \times P(X) \to \{0, 1\}$ is called a paraproximity on a set X if the following axioms are satisfied:

1. $\delta(A, \phi) = 1, \; \forall A \subseteq X$.

2. $\delta(A, B \cup C) = \delta(A, B) \, . \, \delta(A, C), \; \forall A, B, C \subseteq X$.

3. For an arbitrary index set $\wedge$, δ $(\bigcup_{\lambda\in\wedge} A_\lambda$, $B) = 0 \Leftrightarrow \delta$ $(A_\mu, B) = 0$, for some $\mu \in \wedge$.

4. for any two points x, y of X, δ $(x, y) = 0 \Leftrightarrow x = y$.

5. δ $(A, B) = \delta$ $(B, A) = 1 \Rightarrow \exists\, U, V \subseteq X$ with $U \cap V = \phi$, satisfying $\delta(A, U^c) = \delta$ $(B, V^c) = 1$ and δ $(U, U^c) = \delta$ $(V, V^c) = 1$.

Theorem 2.3.3. [king] Let (X, δ) be a paraproximity space. Then the collection $\tau_\delta = \{V \subseteq X: \delta$ $(V, V^c) = 1\}$ is a completely normal topology on X.

Theorem 2.3.4. [king] Let (X, τ) be a completely normal ordinary topological space. Then the relation δ given by: δ $(A, B) = 0 \Leftrightarrow A \cap \overline{B} \neq \phi$, is a paraproximity on X, for which $\tau_\delta = \tau$

3- Double complete normal spaces

Definition 3.1. Let (X, τ) be a double topological space, and let $\underline{A}$, $\underline{B} \in D(x)$. $\underline{A}$ and $\underline{B}$ are called double separated sets if $\underline{A}\ \overline{Q}\ \underline{\overline{B}}$ and $\underline{B}\ \overline{Q}\ \underline{\overline{A}}$.

Lemma 3.2. Let (X, τ) be a double topological space. Then:

(i) $\underline{A}$ and $\underline{B}$ are double separated and $\underline{A}_1 \subseteq \underline{A}$, $\underline{B}_1 \subseteq \underline{B} \Rightarrow \underline{A}_1$, $\underline{B}_1$ are double separated.

(ii) $\underline{A}$, $\underline{B} \in \tau^c$ and $\underline{A}\ \overline{Q}\ \underline{B} \Rightarrow \underline{A}$ and $\underline{B}$ are double separated.

(iii) $\underline{A}$, $\underline{B} \in \tau$ and $\underline{A}\ \overline{Q}\ \underline{B} \Rightarrow \underline{A}$ and $\underline{B}$ are double separated.

Proof: Clear.

Definition 3.3. A double topological space (X, τ) is called DR_4-space if for every two double separated sets $\underline{A}$, $\underline{B}$ in X, $\exists\, \underline{O}_A$, $\underline{O}_B$ such that $\underline{O}_A \overline{Q}\ \underline{O}_B$.

Definition 3.4. A double space (X, τ) is called double completely normal space (or DT_5-space) if it is DR_4 and DT_1.

Proposition 3.5. $(X, \tau) \in DT_5 \Rightarrow (X, \tau) \in DT_4$.

Proof: The result following from Definition 3.4 and Lemma 3.2.

Theorem 3.6 . Every closed double subspace of a DT_5-space is DT_4 .

Proof: Let $(X, \tau) \in DT_5$-space,Y be a double subspace of X, and $\underline{A}$, $\underline{B} \in \tau_Y^C$ such that $\underline{A}\ \overline{Q}\ \underline{B}$. Then $\underline{A}$, $\underline{B} \in \tau^C \wedge \underline{A}\ \overline{Q}\ \underline{B}$

Lemma3.2.5 $\Rightarrow$ $\underline{A}$, $\underline{B}$ are double separated in X

$X\in DT_5$ $\Rightarrow \exists\, \underline{O}_A$, $\underline{O}_B$ such that $\underline{O}_A \overline{Q}\ \underline{O}_B$ $\Rightarrow (\underline{Y} \cap \underline{O}_A \overline{Q}\ \underline{Y} \cap \underline{O}_B) \Rightarrow (Y, \tau_Y) \in DR_3 \wedge$ clearly $(Y, \tau_Y) \in DT_1 \Rightarrow (Y, \tau_Y) \in DT_4$.

4- Double Paraproximity Spaces

Definitions 4.1 Let $\underline{\delta}: D(X) \times D(X) \rightarrow \{0,1\}$ be a relation on $D(X)$ that satisfies the following axioms:

DH_1 $\underline{\delta}$ $(\phi, \underline{X}) = \underline{\delta}$ $(\underline{X}, \phi) = 1$.

DH_2 $\underline{\delta}$ $(\underline{A}, \underline{B} \cup \underline{C}) = \underline{\delta}$ $(\underline{A}, \underline{B})$. $\underline{\delta}$ $(\underline{A}, \underline{C})$, $\forall\, \underline{A}, \underline{B}, \underline{C} \in D(X)$.

DH_3 for an arbitrary index set $\wedge$, $\underline{\delta}$ $(\bigcup_{\lambda\in\wedge} \underline{A}_\lambda, \underline{B}) = 0 \Leftrightarrow \underline{\delta}(\underline{A}_\mu, \underline{B}) = 0$, for some $\mu \in \wedge$.

This $\underline{\delta}$ is called a double H- proximity on X, and the pair $(X, \underline{\delta})$ is called a doubleH-proximity space (or an DHP- space, for short).

Definition 4.2. An DHP- space $(X, \underline{\delta})$ is called separated if $\underline{\delta}$ satisfies the following

Axiom: DH_4 $\underline{\delta}$ $(x_t, y_r) = 0 \Leftrightarrow x_t\ Q\, y_r$, $\forall\, x_t, y_r \in D(X)_p$.

If $\underline{\delta}$ is a separated DH – proximity. Then $(X, \underline{\delta})$ is called a separated DHP- space (or an SDHP-space, for short).

Proposition 4.3. Let $(X, \underline{\delta})$ be an SDHP-space, then $\underline{\delta}$ $(\underline{A}, x_t) = 0 \Leftrightarrow \underline{A}\ Q\, x_t$.

Proof: $\underline{\delta}$ $(\underline{A}, x_t) = \underline{\delta}$ $(\bigcup_{y_r\in\underline{A}} y_r, x_t) = 0 \overset{DH_3}{\Leftrightarrow}$ $\underline{\delta}$ $(y_r, x_t) = 0$, for some $y_r \in \underline{A}$. $\overset{DH_4}{\Leftrightarrow}$ $y_r\ Q\, x_t$, for some $y_r \in \underline{A} \Leftrightarrow \underline{A}\ Q\, x_t$.

Definition 4.4. An SDHP- space $(X, \underline{\delta})$ is called a double paraproximity space (or a DPP-space, for short) if $\underline{\delta}$ satisfies the following axiom.

DH_5 $\underline{\delta}$ $(\underline{A}, \underline{B}) = \underline{\delta}$ $(\underline{B}, \underline{A}) = 1 \Rightarrow \exists\, \underline{C}, \underline{D} \in D(X)$ Such that $\underline{C}\ \overline{Q}\ \underline{D}$, and $\underline{\delta}$ $(\underline{A}, \underline{C}^c) = \underline{\delta}$ $(\underline{C}, \underline{C}^c) = \underline{\delta}$ $(\underline{B}, \underline{D}^c) = \underline{\delta}$ $(\underline{D}, \underline{D}^c) = 1$.

Lemma 4.5. Let $(X, \underline{\delta})$ be a DPP-space. Then:

i- If $\underline{\delta}$ $(\underline{A}, \underline{B}) = 1$, then $\underline{\delta}$ $(\underline{A}, \underline{C}) = 1$, for any $\underline{C} \subseteq \underline{B}$.

ii- If $\underline{\delta}$ $(\underline{A}, \underline{B}) = 1$, then $\underline{\delta}$ $(\underline{C}, \underline{B}) = 1$, for any $\underline{C} \subseteq \underline{A}$.

iii- If $\underline{\delta}$ $(\underline{A}, \underline{B}) = 1$, then $\underline{A}\ \overline{Q}\ \underline{B}$.

iv- $\underline{\delta}$ $(x_t^{\,c}, x_t) = 1$, for any double point $x_t \in D(X)_P$.

Proof : (i) Since $\underline{B} = \underline{B} \cup \underline{C}$ $(\underline{C} \subseteq \underline{B})$, and $\underline{\delta}$ $(\underline{A}, \underline{B}) = 1$. $\underline{\delta}$ $(\underline{A}, \underline{B}) = \underline{\delta}$ $(\underline{A}, \underline{B} \cup \underline{C}) = \underline{\delta}$ $(\underline{A}, \underline{B})$. $\underline{\delta}$ $(\underline{A}, \underline{C}) = 1$ by DH_2 .Then $\underline{\delta}$ $(\underline{A}, \underline{C}) = 1$.

(ii) Proof of (ii) is similar to proof of (i).

(iii) $\underline{\delta}$ $(\underline{A}, \underline{B}) = 1 \Rightarrow \underline{\delta}\,(\bigcup_{x_{tt} \in A} x_t\ ,\ \underline{B}) = 1 \overset{DH_3}{\Rightarrow} \underline{\delta}\,(x_t\ ,\ \underline{B}) = 1,\ \forall x_t \in \underline{A}. \overset{Lema\,4.5}{\Rightarrow} \forall x_t \in \underline{A},\ \underline{\delta}\,(x_t, y_r) = 1,\ \forall y_r \in \underline{B} \overset{DH_4}{\Leftrightarrow} x_t\ \overline{Q}\ y_r,\ \forall x_t \in \underline{A}, y_r \in \underline{B} \Leftrightarrow \underline{A}\ \overline{Q}\ \underline{B}$.

(iv) Since $x_t^c\ \overline{Q}\ x_t \Rightarrow y_r\ \overline{Q}\ x_t,\ \forall y_r \in x_t^c \overset{DH_4}{\Leftrightarrow} \underline{\delta}\ (y_r\ ,\ x_t) = 1,\ \forall y_r \in x_t^{\,c} \overset{DH_3}{\Leftrightarrow} \underline{\delta}\ (x_t^{\,c}, x_t) = 1$ (By proposition 4.3).

Theorm 4.6. Let $(X, \underline{\delta})$ be a DPP – Space. Then $\tau_{\underline{\delta}} = \{\underline{V} \in D(X): \underline{\delta}\ (\underline{V}, \underline{V}^c) = 1\}$ is a double topology on X, induced by $\underline{\delta}$.

Proof: DT_1 $\underline{\delta}$ $(\underline{X}, \underline{X}^c) = \underline{\delta}$ $(\underline{X}, \underline{\phi}) = \underline{\delta}\,(\underline{\phi}, \underline{\phi}^c) = \underline{\delta}\,(\underline{\phi}, \underline{X}) = 1 \Rightarrow \underline{X}, \underline{\phi} \in \tau_{\underline{\delta}}$.

DT_2 $\underline{U}, \underline{V} \in \tau_{\underline{\delta}} \Rightarrow \underline{\delta}$ $(\underline{U}, \underline{U}^c) = 1$ and $\underline{\delta}$ $(\underline{V}, \underline{V}^c) = 1$ and by (ii) in Lemma (4, 5), $\underline{\delta}$ $(\underline{U} \cap \underline{V}, \underline{U}^c) = 1$ and $\underline{\delta}$ $(\underline{U} \cap \underline{V}, \underline{V}^c) = 1$, Hence (by DH_2), $\underline{\delta}$ $(\underline{U} \cap \underline{V}, \underline{U}^c \cup \underline{V}^c) = \underline{\delta}$ $(\underline{U} \cap \underline{V}, (\underline{U} \cap \underline{V})^c) = 1$. Consequently $\underline{U} \cap \underline{V} \in \tau_{\underline{\delta}}$.

DT_3 Let $\underline{V}_i\ ,\ \in \tau_{\underline{\delta}}$, $\forall i \in I$, for some index set I. Then $\underline{\delta}$ $(\underline{V}_i, \underline{V}_i^{\,c}) = 1,\ \forall i \in I \overset{DH_3}{\Rightarrow}$

$\underline{\delta}\ (\bigcup_{i \in I} \underline{V}_i\ ,\ \underline{V}_i^{\ c}) = 1,\ \forall i \in I \overset{4.5\,(i)}{\Rightarrow}$

$\underline{\delta}\ (\bigcup_{i \in I} \underline{V}_i\ ,\ \bigcap_{i \in I} \underline{V}_i^{\ c}) = 1$

$\Rightarrow \underline{\delta}\ (\bigcup_{i \in I} \underline{V}_i, (\bigcup_{i \in I} \underline{V}_i)^c) = 1 \Rightarrow \bigcup_{i \in I} \underline{V}_i \in \tau_{\underline{\delta}}$.

Therefore, $\tau_{\underline{\delta}}$ is a double topology on X, generated by $\underline{\delta}$.

Corollary 4.7. $\underline{V} \in \tau_{\underline{\delta}} \Leftrightarrow \underline{\delta}$ $(x_t, V^c) = 1$, $\forall\ x_t \in \underline{V}$.

Theorem 4.8. Let $(X, \underline{\delta})$ be a DPP- space. Then $\underline{\delta}$ $(\underline{A}, \underline{B}) = 0 \Rightarrow A\ Q\ \overline{B}$.

Proof: Let $\underline{\delta}$ $(\underline{A}, \underline{B}) = 0$ and suppose $\underline{A}\ \overline{Q}\ \underline{\overline{B}}$

$\Rightarrow A \subseteq \overline{B}^{\,C}$, if we choose all open double sets $\underline{O}$, which contain the closed double set $\underline{\overline{B}}$, then $\bigcap_\lambda \underline{\overline{O}}_\lambda = \underline{\overline{B}} \Rightarrow (\underline{A} \subseteq \underline{\overline{B}}^{\,C})\ \underline{\overline{B}}^{\,C} =$

$(\bigcap_\lambda \underline{\overline{O}}_\lambda)^c = \bigcup_\lambda \underline{\overline{O}}_\lambda^{\,c}$ Since $\underline{\overline{O}}_\lambda$ is open $\forall \lambda$,

$\underline{\delta}\,(\underline{\overline{O}}_\lambda^{\,c}\ ,\ \underline{\overline{O}}_\lambda) = 1,\ \forall \lambda. \overset{DH_3}{\Rightarrow}$

$\underline{\delta}\ (\bigcup_\lambda \underline{\overline{O}}_\lambda^{\,c}, \underline{\overline{O}}_\lambda) = 1 \overset{4.5\,(i)}{\Rightarrow}$

$\underline{\delta}\ (\bigcup_\lambda \underline{\overline{O}}_\lambda^{\,c}\ ,\ \bigcap_\lambda \underline{\overline{O}}_\lambda) = 1 \overset{4.5\,(ii)}{\Rightarrow} \underline{\delta}\ (\underline{A}\ , \underline{B}) = 1$.

This contradicts our assumption that $\underline{\delta}\ (\underline{A}\ , \underline{B}) = 0$.

Corollary 4.9. Let $(X, \underline{\delta})$ be a DPP-space and let $x_t \in D(X)_p$, $\underline{A} \in D(X)$, Then:

(i) $\underline{\delta}\ (\underline{A}\ , x_t) = 0 \Leftrightarrow x_t \in \underline{A}$.

(ii) $\underline{\delta}\ (x_t, \underline{A}) = 0 \Rightarrow x_t \in \overline{A}$.

Theorem 4.10: Let $(X, \underline{\delta})$ be a DPP- space, Then: $(X, \tau_{\underline{\delta}}) \in DT_5$.

Proof: first, we prove that $\tau_{\underline{\delta}} \in DT_1$, forwhich we show that every double point of $D(X)_p$ is closed. Since $\underline{\delta}$ $(x_t^{\ c}, x_t) = 1$, $\forall\ x_t \in D(X)_p$ $\overset{4.6}{\Rightarrow} x_t^{\ c} \in \tau_{\underline{\delta}}\ \ \forall\ x_t \in D(X)_p \Rightarrow x_t \in \tau_{\underline{\delta}}^c$, $\forall\ x_t \in D(X)_p \Rightarrow (X, \tau_{\underline{\delta}}) \in DT_1$.

Now we show that for every separated double sets $\underline{A}, \underline{B}$ in X, $\exists\ \underline{O}_{\underline{A}}, \underline{O}_{\underline{B}}$ such that $\underline{O}_{\underline{A}}\ \overline{Q}\ \underline{O}_{\underline{B}}$. Since $\underline{\delta}$ $(\overline{A}^{\,c}, \overline{A}) = 1$, and $\underline{A}, \underline{B}$ are separated, then $\underline{B} \subseteq \overline{A}^{\,c}$. Consequantly, $\underline{\delta}$ $(\underline{B}, \overline{A}) = 1$ and $\underline{\delta}$ $(\underline{B}, \underline{A}) = 1$, (by Lemma 4.5 (ii), (i)). Similarly, we can show that $\underline{\delta}\,(\underline{A}, \underline{B}) = 1$. Now $\underline{\delta}$ $(\underline{A}, \underline{B}) = \underline{\delta}$ $(\underline{B}, \underline{A}) = 1$

DH_5 $\Rightarrow$ $\exists\, \underline{C}, \underline{D} \in D(X)$ Such that $\underline{C}\ \overline{Q}\ \underline{D}$ and $\underline{\delta}(\underline{A}, \underline{C}^c) = \underline{\delta}(\underline{C}, \underline{C}^c) = \underline{\delta}(\underline{B}, \underline{D}^c) = \underline{\delta}(\underline{D}, \underline{D}^c) = 1 \Rightarrow \underline{C}, \underline{D} \in \tau_{\underline{\delta}}$ (by 4.6), and $\underline{A} \subseteq \underline{C}$, $\underline{B} \subseteq \underline{D}$ (by 4.5 (iii)). Since $\underline{C}\,\overline{Q}\,\underline{D}$, then $(X, \tau_{\underline{\delta}}) \in DT_5$.

Theorem 4.11. Let (X, τ) be a double complete normal space. Then: $\underline{\delta} : D(X) \times D(X) \longrightarrow \{0, 1\}$, given by $\underline{\delta}(\underline{A}, \underline{B}) = 0 \Leftrightarrow \underline{A}\, Q\, \overline{\underline{B}}$, $\forall\ \underline{A}, \underline{B} \in D(X)$, is a SDH-proximity on X. Moreover, if $\underline{\delta}$ satisfies DH_5, then $\tau_{\underline{\delta}} = \tau$.

Proof: DH_1 $\underline{X}\ \overline{Q}\ \underline{\delta}\,\underline{\phi} \Rightarrow \underline{\delta}(\underline{X}, \underline{\phi}) = 1$, and $\underline{\phi}\ \overline{Q}\ \underline{X} \Rightarrow \underline{\delta}(\underline{\phi}, \underline{X}) = 1$.

DH_2 $\underline{\delta}(\underline{A}, \underline{B} \cup \underline{C}) = 0 \Rightarrow \underline{A}\ Q\ (\overline{\underline{B} \cup \underline{C}})$ $\Leftrightarrow \underline{A}\ Q\ (\overline{\underline{B}} \cup \overline{\underline{C}}) \overset{2.1.5}{\Leftrightarrow} \underline{A}\ Q\, \overline{\underline{B}}$ or $\underline{A}\ Q\, \overline{\underline{C}}$ $\Leftrightarrow \underline{\delta}(\underline{A}, \underline{B}) = 0$ or $\underline{\delta}(\underline{A}, \underline{C}) = 0$.

DH_3 $\underline{\delta}(\bigcup_{i\in I} \underline{A}_i, \underline{B}) = 0 \Leftrightarrow (\bigcup_{i\in I} \underline{A}_i)\ Q\, \overline{\underline{B}} \overset{2.1.5}{\Leftrightarrow}$ $\exists\ i_o \in I$ Such that $\underline{A}_{i_0}\ Q\, \overline{\underline{B}} \overset{2.1.5}{\Leftrightarrow}$ $\underline{\delta}(\underline{A}_{i_0}, \underline{B}) = 0$ for some $i_o \in I$.

DH_4 $\underline{\delta}(x_t, y_r) = 0 \Leftrightarrow x_t\ Q\ \overline{y}_r \overset{(X,\tau)\in DT_1}{\Leftrightarrow}$ $x_t\ Q\, y_r$, $\forall\ x_t, y_r \in D(X)_p$. Thus $\underline{\delta}$ is SDH-Proximity on X .Moreover, if $\underline{\delta}$ satisfies DH_5, then: $\tau_{\underline{\delta}} = \{\underline{V} \in D(X) : \underline{\delta}(\underline{V}, \underline{V}^c) = 1\}$ $= \{\underline{V} \in D(X) : \underline{V}\ \overline{Q}\ \overline{\underline{V}^c}\}$. $= \{\underline{V} \in D(X) : \underline{V} \subseteq \overline{\underline{V}^c}^{\,c} = \underline{V}^o\}$ $= \{\underline{V} \in D(X) : \underline{V} \in \tau\} = \tau$.

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