More on p-Continuity and New Decompositions

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Abstract: The aim of this paper is to present the notion of relatively almost p-continuity and p^*-continuity and we obtain some decomposition theorems of p-continuity by proving that, A function $f: X \to Y$ is p-continuous if and only if it is almost p-continuous and relatively almost p-continuous. Also, A function $f: X \to Y$ is p-continuous if and only if it is almost p-continuous and p^*-continuous.

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1. Introduction

The decomposition of continuity is a classical problem of Real Analysis. Recently, El Naschie et al.,[6, 7, 8],have pointed out that topology plays a significant role in quantum physics, high energy physics and superstring theory. One of the most important subjects in studying topology is continuity which have been researched by many mathematicians. A weaker form of continuity associated with preopen sets was considered back in 1992, when Mashhour at al. [14] introduced the concept of precontinuity. A function $f: X \to Y$ is called precontinuous [14] if the inverse image of every open subeset of Y is preopen in X. Blumberg [2] proved that every function $f: \mathbb{R} \to \mathbb{R}$ is nearly continuous on a dense set of the real line $\mathbb{R}$. Since the begining of the 80s, the name precontinuity dominates in the literature although the term near continuity is also often used. The following result shows the importance of precontinuity. "Every linear function from one Banach space to another Banach space is precontinuous"[16]. Note also that in Functional Analysis, precontinuous function are important in the context with the well known closed graph and open mapping theorems.

In 1990, Abd El-Monsef et al.[1] introduced and studied a common strengthening of continuity and preirresoluteness called strongly M-precontinuous functions (written in short as p-continuous) by requiring the inverse image of each preopen set in the codomain to be open in the domain. The aim of this paper is to introduce the notion of relatively almost p-continuity and p^*-continuity and obtain decompositions of p-continuity.

2. Some preliminary topological concepts

Throughout this note, X and Y are always topological spaces. The closure and the interior of a set $A \subset X$ are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of X is said to be preopen [14] (=nearly open or locally dense [3]) if. $A \subset Int(Cl(A))$. The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing a subset A is called the preclosure [5] of A and is denoted by $pCl(A)$. The family of all preopen sets of X is denoted by $PO(X)$. For a point x in X, we set $PO(X,x) = \{U : x \in U \subset PO(X)\}$.

Definition 2.1 A function $f:(X,\tau) \to (Y,\sigma)$ is said to be

(1) strongly M-precontinuous [1] (or p-continuous) if for each $x \in X$ and each preopen set $V \subset Y$ containing $f(x)$, there exists an open set $U \subset X$ containing x such that $f(U) \subset V$.

(2) almost p-continuous [11] (or $p(\theta)$-continuous [4]) if for each $x \in X$ and each preopen

set $V \subset Y$ containing $f(x)$, there exists an open set $U \subset X$ containing x such that $f(U) \subset pCl(V)$.

Definition 2.2 A function $f: X \to Y$ is called relatively almost p-continuous at $x \in X$ if for each $V \in PO(Y, f(x))$, the set $f^{-1}(V)$ is open in the subspace $f^{-1}(pCl(V))$.

The function f is called relatively almost p-continuous if it has this property at each point x of X.

The pre-frontier or the preboundary [19] of a subset A of X, denoted by $pFr(A)$, is defined by $pFr(A) = pCl(A) \cap pCl(X \setminus A)$.

Definition 2.3 A function $f: X \to Y$ is called p^*-continuous if for each $V \in PO(Y)$, $f^{-1}(pFr(V))$ is closed in X.

Remark 2.4 By means of easy examples on finite topological spaces one can see that almost p-continuity and relatively almost p-continuity are independent of each other. The same is also true for almost p-continuity and p^*-continuity.

3. On p-continuity

For any space (X, τ) let τ_p be the smallest topology on X containing $PO(X, \tau)$. The topology τ^α [10] is $PO(X, \tau) \cap SO(X, \tau)$ where $A \in SO(X, \tau)$ if and only if A is semi-open [13] i.e. $A \subseteq Cl(Int(A))$. Thus, for any space (X, τ), $\tau \subseteq \tau^\alpha \subseteq PO(X, \tau) \subseteq \tau_p$. It is also known that $PO(X, \tau^\alpha) = PO(X, \tau)$ (Corollary 1 of [21]).

Lemma 3.1 A function $f: (X, \tau) \to (Y, \sigma)$ is p-continuous if and only if $f: (X, \tau) \to (Y, \sigma_p)$ is continuous.

Proof. A basic open set in σ_p has the form $V = \cap\{B_k : k = 1, 2, \ldots, n\}$ where each $B_k \in PO(Y, \sigma)$. So if $f: (X, \tau) \to (Y, \sigma)$ is p-continuous, and V is a basic open set in σ_p, $f^{-1}(V) = \cap\{f^{-1}(B_k) : k = 1, 2, \ldots, n\} \in \tau$, so that $f: (X, \tau) \to (Y, \sigma_p)$ is continuous. The converse is clear since $PO(Y, \sigma) \subseteq \sigma_p$.

Remark 3.2 A function $f: (X, \tau) \to (Y, \sigma)$ is said to be weakly continuous [12] if for each $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq Cl(V)$.

We observe that the following relations hold:

p-continuity $\to$ almost p-continuity $\to$ weak continuity

Definition 3.3 The graph $G(f)$ of a function $f: (X, \tau) \to (Y, \sigma)$ is said to be strongly p-closed [10] if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist an open set $U \subseteq X$ containing x and a preopen set $V \subseteq Y$ containing y such that $(U \times pCl(V)) \cap G(f) = \phi$ (or,equivalently, $f(U) \cap pCl(V)) = \phi$).

Recall that a space (X, τ) is called pre-Urysohn [18] if for any two distinct points $x \neq y$ there exist preopen sets $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $pCl(U) \cap pCl(V)) = \phi$.

A space (X,τ) is called pc -compact [10] if every preclosed subset of (X,τ) is p -closed relative to (X,τ).

Theorem 3.4 If $f:(X,\tau)\to(Y,\sigma)$ is p -continuous and (Y,σ) is pre-Urysohn, then $G(f)$ is strongly p -closed.

Proof. Let $(x,y)\in(X\times Y)\setminus G(f)$, i.e $f(x)\neq y$. Since (Y,σ) is pre-Urysohn, there exist preopen sets $V,W\subseteq Y$ containing $f(x)$ and y, respectively, such that $pCl(V)\cap pCl(W)=\phi$. Since f is p -continuous, there exists an open set $U\subseteq X$ containing x such that $f(U)\subseteq V\subseteq pCl(V)$. Hence $f(U)\cap pCl(V)=\phi$, and so $G(f)$ is strongly p -closed.

Definition 3.5 A subset A of (X,τ) is called p -closed relative to (X,τ) [17] if every cover of A by preopen sets of (X,τ) has a finite subfamily whose preclosures cover A.

The following result follows directly from the above definitions and Theorem 3.4.

Corollary 3.6 Let $f:(X,\tau)\to(Y,\sigma)$ be a function where (Y,σ) is pre-Urysohn and pc -compact. Then the following properties are equivalent:

(a) f is p -continuous,

(b) f is almost p -continuous,

(c) $G(f)$ is strongly p-closed,

(d) $f^{-1}(K)$ is closed for each subset $K\subseteq Y$ which is p -closed relative to (Y,σ).

Recall that a space X is resolvable if there is a dense subset $D\subseteq X$ for which $X\setminus D$ is also dense in X.

Lemma 3.7 (Corollary 5 of [9]) If (X,τ) is resolvable, then $\tau_p=2^X$.

Proof. Let E_1 and E_2 be disjoint dense subset of X and let $x\in X$. Then $E_i\cup\{x\}$ where $i=1,2$ are dense and hence preopen. Then $\{x\}=(E_1\cup\{x\})\cap(E_2\cup\{x\})\in\tau_p$.

Theorem 3.8 Let either every open subset of Y be closed, or (Y,σ) be resolvable, then a function $f:(X,\tau)\to(Y,\sigma)$ is p -continuous if and only if $f:(X,\tau)\to(Y,2^Y)$ is continuous.

Proof. By Lemma 3.1 and Lemma 3.7 and the foregoing remarks, in either case, we have $\sigma_p=2^Y$.

Corollary 3.9 If (Y,σ) is resolvable, the following are equivalent:

(a) $f:(X,\tau)\to(Y,\sigma)$ is p -continuous,

(b) $f^{-1}(V)$ is clopen (closed and open) for each $V\subseteq Y$,

(c) $f^{-1}(W)$ is clopen for each $W\subseteq Y$,

(d) $f^{-1}(W)$ is open for each $W\subseteq Y$,

(e) $f:(X,\tau)\to(Y,2^Y)$ is continuous.

Proof. It follows from Lemma 3.1 and Lemma 3.7.

Corollary 3.10 If (X,τ) is connected and (Y,σ) is resolvable, then $f:(X,\tau)\to(Y,\sigma)$ is p -continuous if and only if f is a constant function.

For example if $\mathbb{P}$ is the usual space of real numbers, every nonconstant function $f:\mathbb{P}\to\mathbb{P}$ is not p-continuous.

Corollary 3.11 If (X,τ) is dense-in-itself (has no isolated points) and (Y,σ) is a nonempty resolvable space then there is no p-continuous injection $f:(X,\tau)\to(Y,\sigma)$.

4. New decomposition of p-continuity

Theorem 4.1 A function $f:X\to Y$ is p-continuous if and only if it is almost p-continuous and relatively almost p-continuous.

Proof. We prove only the sufficient condition since the necessary condition is evident. Let $V\in PO(Y)$. Since f is relatively almost p-continuous, we have $f^{-1}(V)=f^{-1}(pCl(V))\cap G$ for some open set G of X. Suppose that $x\in f^{-1}(V)$. It follows that $f(x)\in V$ and $x\in G$. By the almost p-continuity of f, there exists an open set U in X containing x such that $f(U)\subseteq pCl(V)$. Therefore, we have $x\in U\cap G\subseteq f^{-1}(pCl(V))\cap G=f^{-1}(V)$. Since $U\cap G$ is open in X, x is an interior point of $f^{-1}(V)$ and hence $f^{-1}(V)$ is open in X. This shows that f is p-continuous.

A space X is said to be pre-regular [20] if for each preclosed set A and each point x in the complement of A, there exist disjoint $V\in PO(X)$ and $W\in PO(X)$ such that $x\in V$ and $A\subseteq W$. Pal and Bhattacharya [[20], Lemma 4.2] has shown that a space X is pre-regular if and only if for each point x of X and each $V\in PO(X,x)$, there exists $W\in PO(X,x)$ such that $x\in V\subseteq pCl(V)\subseteq W$.

Theorem 4.2 A function $f:X\to Y$ is p-continuous if and only if it is almost p-continuous and p^*-continuous.

Proof. Suppose that f is p-continuous. It is evident that f is almost p-continuous. For each $V\in PO(Y)$, we have

$$pFr(V)=pCl(V)\cap pCl(Y\setminus V)$$
$$=pCl(V)\cap(Y\setminus V)$$

. Since f is p-continuous, it follows that $f^{-1}(pFr(V))=f^{-1}(pCl(V))\cap f^{-1}(Y\setminus V)$ is closed in X. This shows that p-continuity implies also p^*-continuity. Conversely, for each $V\in PO(Y)$, we have

$$pCl(V)\cap(Y\setminus pFr(V))$$
$$=pCl(V)\cap[Y\setminus(pCl(V)\cap(Y\setminus V))]$$
$$=pCl(V)\cap[(Y\setminus pCl(V))\cup(Y\setminus(Y\setminus V))]$$
$$=pCl(V)\cap[(Y\setminus pCl(V)\cup(V)]$$
$$=pCl(V)\cap V=V$$

Thus we obtain

$$f^{-1}(V)=f^{-1}(pCl(V))\cap f^{-1}(Y\setminus pFr(V))$$
$$=f^{-1}(pCl(V)\cap[X\setminus f^{-1}(pFr(V))]$$

Since f is almost p-continuous and $pCl(V)$ is pre-regular in Y, $f^{-1}(pCl(V))$ is closed and open in X. Since f is p^*-continuous, $f^{-1}(pCl(V))$ is closed in X. It follows that $f^{-1}(V)$ is open in X and hence f is p-continuous.

Conclusion

In Section 3, we obtain some properties of p-continuous functions. Moreover, in Section 4, we show that a function $f:X\to Y$ is p-continuous if and only if it is almost p -continuous and relatively almost p -continuous. We also provide a decomposition of p -continuity.

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