The Bishop Frame of Bezier Curves

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Abstract: The purpose of this study is to discuss some differential geometric properties and Bishop frames of Bezier curves. Firstly, we investigate Frenet frame and contact theory of Bezier curves. Furthermore, we have given some necessary conditions and properties for the Bezier curve. Secondly we obtain the Bishop frame of Bezier curves in Euclidean 3-space.

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1. Introduction

The Bishop frames was initiated by L.Bishop (1975) in order to create some new frames which have different advantages from Serret-Frenet frames, [1]. A normal vector field N along a curve is relatively parallel if its derivative is tangential. If α is a curve, considered as a displacement vector function of a parameter t, then if N is relatively parallel, the curve with displacement vector $\alpha + N$ has velocity $(\alpha + N)' = (v + f)T$ where T is the unit tangent vector field of α, v is the speed of α an $N' = fT$. So, the segment between the two curves are perpendicular to both. Bishop frames can be constructed locally by appling the Gram-Schmidt process to T and two parallel field. L. Bishop introduced the derivative of the Bishop frames $\{T, N_1, N_2\}$ as following

$$T' = k_1 v N_1 + k_2 v N_2$$
$$N_1' = -k_1 v T$$
$$N_2' = -k_2 v T$$

here k_1 and k_2 are the curvatures of the Bishop frame. The matrix representation of the derivatives is

$$\begin{pmatrix} T' \\ N_{1'} \\ N_{2'} \end{pmatrix} = \begin{pmatrix} 0 & k_1 v & k_2 v \\ -k_1 v & 0 & 0 \\ -k_2 v & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ N_2 \end{pmatrix}$$

in the references [1]. A generalization of one parameter motion to Bishop motion in Euclidean 3 space was introduced in [2]. Some results of Bishop Frame in Minkowski 3-space was investigated in [3]. In the reference [4], the authors generalized for a spacelike curve with a spacelike principal normal which was studied by Bishop to Minkowski 3 space. Slant helix according to Bishop frame in Euclidean 3 space was studied in [5]. In [6], some characterizations of timelike curves according to Bishop Frame in Minkowski 3 space by using Laplacian operator. Serret-Frenet and parallel transport frame were produced with the help of the generalized quaternions in [7]. The parallel transport frame of a curve and the relation between the frame and Frenet frame of the curve in 4-dimensional Euclidean space in [8]. The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, [9]. In the late 1950s, hardware become available that allowed the machining of 3D shapes out of blocks of wood or steel. Bezier curves which represented by the formula

$$b^n(t) = \sum_{i=0}^{n-r} b_i^r(t) B_i^{n-r}(t)$$

were independently developed by P. de Casteljau at Citroen and by P. Bezier at Renault Company in France. The theory of Bezier curves plays a central role in CAGD. They are numerically the most stable among polynomial bases currently used in CAD systems, was shown by Farauki and Rajon. Thus Bezier curves are the ideal geometric standart for the representation of piecewise polynomial curves. Also, Bezier curves lend themselves easily to a geometric understanding of many CAGD phenomena, [10]. Bezier curve segments are defined only by the position vectors of polygon vertices. Bezier curve segments are expressed as a convex combination of the polygon vertex position vectors which define the curve, and possess a variation diminishing property. Consequently the curve shape can be approximately anticipated from the polygon shape. That is to say, Bezier curves and surfaces are in a form that is easy for a person to control, [11]. Computer aided geometric design (CAGD) concerns itself with the mathematical description of shape for use in computer graphics, manufacturing or analysis, approximation theory, data structures and computer algebra. CAGD is a young field. The first work in this field began in the mid 1960s. The term computer aided geometric design was coined in 1974 by R.E. Barnhill and R.F. Riesenfeld in connection with a conference at the University of Utah, [12].

2. Material and Methods

Bezier curves consist of Bernstein polynomial and control points. Bernstein polynomial is defined with the following equation:

$$B_i^n(t)=\binom{n}{i}t^i(1-t)^{n-i}$$

and meanwhile the following condition is provided:

$$\binom{n}{i}=\begin{cases}\dfrac{n!}{i!(n-i)!} & 0\le i\le n\\ 0 & \text{else.}\end{cases}$$

Bernstein polynomial become $B_0^0(t)\equiv 1$ and $B_j^n(t)\equiv 1 \text{ for } j\notin\{0,.,n\}$ in special conditions. Moreover, the sum of the coefficients of Bernstein polynomial is $\sum_{j=0}^{n}B_j^n(t)\equiv 1$. Alternatively, Bernstein polynomial may be written with the following equation

$$B_i^n(t)=(1-t)B_i^{n-1}(t)+tB_{i-1}^{n-1}(t)$$

The derivative of a Berstein polynomial B_i^n is obtained as

$$\frac{d}{dt}B_i^n(t)=n\left[B_{i-1}^{n-1}(t)-B_i^{n-1}(t)\right]$$

We can write a Bezier curve in the form $b^n(t)=\sum_{i=0}^{n-r}b_i^r(t)B_i^{n-r}(t)$. Let determine the r.th degree derivative of a Bezier curve b^n

$$\frac{d^rb^n}{dt^r}(t)=\frac{n!}{(n-r)!}.\sum_{i=0}^{n-r}\Delta^rb_i.B_i^{n-r}(t)$$

and here Δ^rb_i difference equation is in the form of:

$$\Delta^rb_i=\sum_{i=0}^{r}\binom{r}{j}(-1)^{r-j}b_{i+j}$$

and in addition, $\Delta^rb_j=\Delta^{r-1}b_{j+1}-\Delta^{r-1}b_j$. The important special case of Equation(2.1) are given by $t=0$ and $t=1$

$$\frac{d^rb^n(t)}{dt^r}\Big|_{t=0}=\frac{n!}{(n-r)!}\Delta^rb_0 \quad \frac{d^rb^n(t)}{dt^r}\Big|_{t=1}=\frac{n!}{(n-r)!}\Delta^rb_{n-r}$$

Let us find the first, second and third level derivations at the start and end points by using the formulas above. The $t=0$ derivation formulas at the start point are found like this:

$$\frac{db^n(t)}{dt}\Big|_{t=0}=n.\Delta b_0 \quad \frac{d^2b^n(t)}{dt^2}\Big|_{t=0}$$
$$=n(n-1)[\Delta b_1-\Delta b_0]\frac{d^3b^n(t)}{dt^3}\Big|_{t=0}$$
$$=n(n-1)(n-2)[\Delta b_2-2\Delta b_1+\Delta b_0]$$

The $t=1$ derivation formulas at the end point are found like this:

$$\frac{db^n(t)}{dt}\Big|_{t=1}=n.\Delta b_{n-1} \quad \frac{d^2b^n(t)}{dt^2}\Big|_{t=1}$$
$$=n(n-1)[\Delta b_{n-1}-\Delta b_{n-2}]\frac{d^3b^n(t)}{dt^3}\Big|_{t=1}$$
$$=n(n-1)(n-2)[\Delta b_{n-1}-2\Delta b_{n-2}+\Delta b_{n-3}]$$

3. Results

Let us calculate the $\{T,N,B\}$ orthonormal Serret-Frenet frame at the start and end points of the $b^n(t)$ Bezier curve that is without unit speed. The following conditions are provided because of orthonormal frame system:

$$<T,T>=1,\ <N,N>=1,\ <B,B>=1$$
$$<T,N>=0,\ <T,B>=0,\ <N,B>=0$$

The $b^n(t)$ Bezier curve velocity vector is given with the formula

$$v=\|\frac{db^n(t)}{dt}\|,$$

and the length of the arc parameter is given with the equation

$$s=\int_{t_0}^{t_1}\|\frac{db^n(t)}{dt}\|dt.$$

Theorem 3.1. Let $b_i\in E^3$ be the control points, the $\{T,N,B\}|_{t=0}$ Serret-Frenet frame at the $t=0$ start point is given with the equations

$$T|_{t=0}=\frac{\Delta b_0}{\|\Delta b_0\|}$$
$$N|_{t=0}=\frac{\Delta b_1}{\|\Delta b_1\|}\csc\phi-\frac{\Delta b_0}{\|\Delta b_0\|}\cot\phi$$
$$B|_{t=0}=\frac{\Delta b_0\wedge\Delta b_1}{\|\Delta b_0\wedge\Delta b_1\|}.$$

Proof: When we calculate the $\{T,N,B\}|_{t=0}$ Serret-Frenet roof of the $b^n(t)$ Bezier curve at the $t=0$ point, the tangent vector is found with the following equation:

$$T|_{t=0}=\frac{\frac{db^n(t)}{dt}}{\|\frac{db^n(t)}{dt}\|}=\frac{\Delta b_0}{\|\Delta b_0\|}$$

The Binormal vector is

$$B|_{t=0}=\left.\frac{\frac{db^n(t)}{dt}\wedge\frac{d^2b^n(t)}{dt^2}}{\left\|\frac{db^n(t)}{dt}\wedge\frac{d^2b^n(t)}{dt^2}\right\|}\right|_{t=0}$$

$$=\frac{[n(b_1-b_0)]\wedge[n(n-1)\{(b_2-b_1)-(b_1-b_0)\}]}{\|n(b_1-b_0)\wedge[n(n-1)\{(b_2-b_1)-(b_1-b_0)\}]\|}$$

$$=\frac{[n.\Delta b_0]\wedge[n(n-1)\{(\Delta b_1-\Delta b_0\}]}{\|n.\Delta b_0\wedge[n(n-1)\{\Delta b_1-\Delta b_0\}]\|}$$

$$=\frac{\Delta b_0\wedge[\Delta b_1-\Delta b_0]}{\|\Delta b_0\wedge[\Delta b_1-\Delta b_0]\|}=\frac{\Delta b_0\wedge\Delta b_1}{\|\Delta b_0\wedge\Delta b_1\|}$$

and the normal vector is obtained with the equation

$$N|_{t=0}=B\wedge T$$

$$=\frac{\Delta b_0\wedge\Delta b_1}{\|\Delta b_0\wedge\Delta b_1\|}\wedge\frac{\Delta b_0}{\|\Delta b_0\|}$$

$$=\frac{(\Delta b_0\wedge\Delta b_1)\wedge\Delta b_0}{\|\Delta b_0\wedge\Delta b_1\|.\|\Delta b_0\|}$$

$$=\frac{<\Delta b_0,\wedge b_0>\Delta b_1-<\Delta b_1,\Delta b_0>\Delta b_0}{\|\Delta b_0\wedge\Delta b_1\|.\|\Delta b_0\|}$$

$$=\frac{\Delta b_1}{\|\Delta b_1\|}\csc\phi-\frac{\Delta b_0}{\|\Delta b_0\|}\cot\phi,$$

here, the angle between Δb_1 and Δb_0 is the ϕ.

Theorem 3.2. Let the $b_i\in E^3$ be the control points, the $\{T,N,B\}|_{t=1}$ Serret-Frenet frame at the $t=1$ start point is given with equations

$$T|_{t=1}=\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}$$

$$N|_{t=1}=\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\cot\psi-\frac{\Delta b_{n-2}}{\|\Delta b_{n-2}\|}\csc\psi$$

$$B|_{t=1}=-\frac{\Delta b_{n-1}\wedge\Delta b_{n-2}}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|}.$$

Proof: Let us find the $\{T,N,B\}|_{t=1}$ Serret-Frenet frame of the $b^n(t)$ Bezier curve at the $t=1$ end point. The tangent vector; and

$$T|_{t=1}=\left.\frac{\frac{db^n(t)}{dt}}{\left\|\frac{db^n(t)}{dt}\right\|}\right|_{t=1}=\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}$$

The Binormal vector is calculated by the equation

$$B|_{t=1}=\left.\frac{\frac{db^n(t)}{dt}\wedge\frac{d^2b^n(t)}{dt^2}}{\left\|\frac{db^n(t)}{dt}\wedge\frac{d^2b^n(t)}{dt^2}\right\|}\right|_{t=1}$$

$$=\frac{n.[b_n-b_{n-1}]\wedge n.(n-1)[(b_n-b_{n-1})-(b_{n-1}-b_{n-2})]}{\|n.(b_n-b_{n-1})\wedge n.(n-1)[(b_n-b_{n-1})-(b_{n-1}-b_{n-2})]\|}$$

$$=\frac{(b_n-b_{n-1}).[(b_n-b_{n-1})-(b_{n-1}-b_{n-2})]}{\|(b_n-b_{n-1}).[(b_n-b_{n-1})-(b_{n-1}-b_{n-2})]\|}$$

$$=\frac{\Delta b_{n-1}\wedge[\Delta b_{n-1}-\Delta b_{n-2}]}{\|\Delta b_{n-1}\wedge[\Delta b_{n-1}-\Delta b_{n-2}]\|}$$

$$=-\frac{\Delta b_{n-1}\wedge\Delta b_{n-2}}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|},$$

and the normal vector is

$$N|_{t=1}=B\wedge T$$

$$=\left.\frac{-\Delta b_{n-1}\wedge\Delta b_{n-2}}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|}\wedge\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\right|_{t=1}$$

$$=\frac{[\Delta b_{n-2}\wedge\Delta b_{n-1}]\wedge\Delta b_{n-1}}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|\|\Delta b_{n-1}\|}$$

$$=\frac{\langle\Delta b_{n-2},\Delta b_{n-1}\rangle\Delta b_{n-1}-\langle\Delta b_{n-1},\Delta b_{n-1}\rangle\Delta b_{n-2}}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|\|\Delta b_{n-1}\|}$$

$$=\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\cot\psi-\frac{\Delta b_{n-2}}{\|\Delta b_{n-2}\|}\csc\psi,$$

here, ψ is the angle between Δb_{n-2} and Δb_{n-1}.

Theorem 3.3. The curvature and torsion of a $b^n(t)$ Bezier curve whose control points are $b_0,\ b_1,\ b_n$ from $n.$ degree at the $t=0$ point

$$\kappa=\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}.\sin\phi\quad\tau=\frac{(n-2)}{n}\frac{(\Delta b_0\Delta b_1\ \Delta b_2)}{\|\Delta b_0\wedge\Delta b_1\|^2}.$$

Proof : The curve of the $b^n(t)$ Bezier curve with no unit speed at the $t=0$ start point is obtained with the following equations:

$$\kappa|_{t=0}=\left.\frac{\left\|\frac{db^n(t)}{dt}\wedge\frac{d^2b^n(t)}{dt^2}\right\|}{\left\|\frac{db^n(t)}{dt}\right\|^3}\right|_{t=0}$$

$$= \frac{\left\| n(b_1-b_0)\wedge n(n-1)\left[(b_2-b_1)-(b_1-b_0)\right]\right\|}{\left\| n(b_1-b_0)\right\|^3}$$

$$= \frac{nn(n-1)\left\|(b_1-b_0)\wedge\left[(b_2-b_1)-(b_1-b_0)\right]\right\|}{n^3.\left\|(b_1-b_0)\right\|^3}$$

$$= \frac{n-1}{n}\frac{\left\|(b_1-b_0)\wedge\left[(b_2-b_1)-(b_1-b_0)\right]\right\|}{\left\|(b_1-b_0)\right\|^3}$$

$$= \frac{n-1}{n}\frac{\left\|(b_1-b_0)\wedge\left[(b_2-b_1)\right]\right\|}{\left\|(b_1-b_0)\right\|^3}$$

$$= \frac{n-1}{n}\frac{\left\|\Delta b_0\wedge\Delta b_1\right\|}{\left\|\Delta b_0\right\|^3}$$

$$= \frac{n-1}{n}\frac{\left\|\Delta b_1\right\|}{\left\|\Delta b_0\right\|^2}.\sin\phi$$

here, ϕ is the angle between Δb_1 and Δb_0 vectors. The torsion of Bezier curve at $t=0$ is obtained with the following equations:

$$\tau\,|_{t=0} = \frac{\left(\frac{db^n(t)}{dt}\frac{d^2b^n(t)}{dt^2}\frac{d^3b^n(t)}{dt^3}\right)}{\left\|\frac{db^n(t)}{dt}\Lambda\frac{d^2b^n(t)}{dt^2}\right\|^2}$$

$$= \frac{<\frac{db^n(t)}{dt},\frac{d^2b^n(t)}{dt^2}\wedge\frac{d^3b^n(t)}{dt^3}>}{\left\|\frac{db^n(t)}{dt}\Lambda\frac{d^2b^n(t)}{dt^2}\right\|^2}$$

$$= \frac{(n-2)}{n}.\frac{<\Delta b_0,(\Delta b_1\text{-}\Delta b_0)\wedge(\Delta b_2-2\Delta b_1+\Delta b_0)>}{\left\|\Delta b_0\Lambda[\Delta b_1\text{-}\Delta b_0]\right\|^2}$$

$$= \frac{(n-2)}{n}.\frac{<\Delta b_0,(\Delta b_1\Lambda\Delta b_2+\Delta b_0\wedge\Delta b_1-\Delta b_2>}{\left\|\Delta b_0\Lambda\Delta b_1\right\|^2}$$

$$= \frac{(n-2)}{n}.\frac{<\Delta b_0,(\Delta b_1\Lambda\Delta b_2>+<\Delta b_0,\Delta b_0\wedge(\Delta b_1-\Delta b_2)>}{\left\|\Delta b_0\Lambda\Delta b_1\right\|^2}$$

$$= \frac{(n-2)}{n}\frac{(\Delta b_0\Delta b_1\Delta b_2)}{\left\|\Delta b_0\wedge\Delta b_1\right\|^2}.$$

Theorem 3.4. The curve and torsion of a $b^n(t)$ Bezier curve whose control points are $b_0, b_1, .b_n$ from $n.$ degree at the $t=1$ point,

$$\kappa\,|_{t=1} = \frac{n-1}{n}\frac{\left\|\Delta b_{n-2}\right\|}{\left\|\Delta b_{n-1}\right\|^2}.\sin\psi$$

$$\tau\,|_{t=1} = \frac{n-2}{n}\frac{-(\Delta b_{n-1}\Delta b_{n-2}\Delta b_{n-3})}{\left\|\Delta b_{n-1}\wedge\Delta b_{n-2}\right\|^2}.$$

Proof: Let us calculate the curve of the $b^n(t)$ Bezier curve at $t=1$;

$$\kappa|_{t=1} = \left.\frac{\left\|\frac{db^n(t)}{dt}\wedge\frac{d^2b^n(t)}{dt^2}\right\|}{\left\|\frac{db^n(t)}{dt}\right\|^3}\right|_{t=1}$$

$$= \frac{n-1}{n}\frac{\left\|(b_n-b_{n-1})\wedge\left[(b_n-b_{n-1})-(b_{n-1}-b_{n-2})\right]\right\|}{\left\|b_n-b_{n-1}\right\|^3}$$

$$= \frac{n-1}{n}\frac{\left\|\Delta b_{n-1}\wedge\left[\Delta b_{n-1}-\Delta b_{n-2}\right]\right\|}{\left\|\Delta b_{n-1}\right\|^3}$$

$$= \frac{n-1}{n}\frac{\left\|\Delta b_{n-1}\wedge\Delta b_{n-2}\right\|}{\left\|\Delta b_{n-1}\right\|^3}$$

$$= \frac{n-1}{n}\frac{\left\|\Delta b_{n-2}\right\|}{\left\|\Delta b_{n-1}\right\|^2}.\sin\psi.$$

Let us find the torsion of the $b^n(t)$ Bezier curve at $t=1$ end point;

$$\tau\,|_{t=1} = \left.\frac{\left(\frac{db^n(t)}{dt}\frac{d^2b^n(t)}{dt^2}\frac{d^3b^n(t)}{dt^3}\right)}{\left\|\frac{db^n(t)}{dt}\wedge\frac{d^2b^n(t)}{dt^2}\right\|^2}\right|_{t=1}$$

$$= \frac{n-2}{n}\frac{\left\langle\Delta b_{n-1},\left[\Delta b_{n-1}-\Delta b_{n-2}\right]\wedge\left[\Delta b_{n-1}-2\Delta b_{n-2}+\Delta b_{n-3}\right]\right\rangle}{\left\|\Delta b_{n-1}\wedge\left[\Delta b_{n-1}-\Delta b_{n-2}\right]\right\|^2}$$

$$= -\frac{n-2}{n}\frac{(\Delta b_{n-1}\Delta b_{n-2}\Delta b_{n-3})}{\left\|\Delta b_{n-1}\wedge\Delta b_{n-2}\right\|^2}.$$

Let the speed at the start point be $v = n\,\|\,b_1-b_0\,\|$; the derivation formulas of the Serret-Frenet frame at $t=0$ are given with

$$\mathbf{T}' = (n-1)\frac{\|\,\Delta b_1\,\|}{\|\,\Delta b_0\,\|}sin\theta.\mathbf{N}$$

$$\mathbf{N}' = -(n-1)\frac{\|\,\Delta b_1\,\|}{\|\,\Delta b_0\,\|}sin\theta.\mathbf{T}-(n-2)\,\|\,\Delta b_0\,\|\frac{(\Delta b_0\Delta b_1\Delta b_2)}{\left\|\Delta b_0\wedge\Delta b_1\right\|^2}.\mathbf{B}$$

$$\mathbf{B}' = -(n-2)\,\|\,\Delta b_0\,\|\frac{(\Delta b_0\Delta b_1\Delta b_2)}{\left\|\Delta b_0\wedge\Delta b_1\right\|^2}.\mathbf{N}.$$

Let the speed at the end point be $v = n\,\|\,b_n-b_{n-1}\,\|$; the derivation formula of the Serret-Frenet frame at $t=1$ are obtained as follows:

$$\mathbf{T}' = (n-1)\frac{\|\,\Delta b_{n-2}\,\|}{\|\,\Delta b_0\,\|}sin\psi.\mathbf{N}$$

$$\mathbf{N}' = -(n-1)\frac{\|\,\Delta b_{n-2}\,\|}{\|\,\Delta b_{n-1}\,\|}sin\psi.\mathbf{T}-(n-2)\,\|\,\Delta b_{n-1}\,\|\frac{(\Delta b_{n-1}\Delta b_{n-2}\Delta b_{n-3})}{\left\|\Delta b_{n-1}\wedge\Delta b_{n-2}\right\|^2}.\mathbf{B}$$

$$\mathbf{B}' = -(n-2)\,\|\,\Delta b_{n-1}\,\|\frac{(\Delta b_{n-1}\Delta b_{n-2}\Delta b_{n-3})}{\left\|\Delta b_{n-1}\wedge\Delta b_{n-2}\right\|^2}.\mathbf{N}$$

Since the $\{\mathbf{T},\mathbf{N}_1,\mathbf{N}_2\}$ Bishop frame is an orthonormal, it gives the equation

$$<T,T>=<N_1,N_1>=<N_2,N_2>=1$$
$$<T,N_1>=<T,N_2>=<N_1,N_2>=0.$$

Now, let us examine the Bishop frame at the start and end point of the $b^n(t)$ Bezier curve:

Theorem 3.5. Let the $b_i \in E^3$ be the control points, the curve and torsion of the Bishop angle at the $t=0$ start point of the $b^n(t)$ Bezier curve with no unit speed are given with the equations

$$k_1\,|_{t=0}=\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}.\sin\phi\cos\theta$$

$$k_2\,|_{t=0}=\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}.\sin\phi\sin\theta,$$

here, the θ angle is calculated as:

$$\theta=\frac{(n-2)}{n}\frac{(\Delta b_0\ \Delta b_1\ \Delta b_2)}{\|\Delta b_0\wedge\Delta b_1\|^2}t+c.$$

İspat: Since the $\tau=\theta'$ is used to calculate the angle in the Bishop frame, when we take the integral of both sides of this equation, we obtain the following:

$$\theta=\int\tau(t)\,|_{t=0}\,dt$$
$$=-\int\frac{n-2}{n}\frac{(\Delta b_0\ \Delta b_1\ \Delta b_2)}{\|\Delta b_0\wedge\Delta b_1\|^2}dt$$
$$=-\int\frac{n-2}{n}\frac{(\Delta b_0\ \Delta b_1\ \Delta b_2)}{\|\Delta b_0\wedge\Delta b_1\|^2}t+c$$

In addition, when the curvature formulas of the Bishop frame for the curves are considered; the equations

$$k_1\,|_{t=0}=\kappa.\cos\theta$$
$$=\frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}.\sin\psi\cos\theta$$
$$k_2\,|_{t=0}=\kappa.\cos\theta$$
$$=\frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}.\sin\psi\sin\theta$$

are obtained.

Theorem 3.6. The curvature and torsion of the Bishop frame at the $t=1$ start point of the $b^n(t)$ Bezier curve with no unit speed and with the $b_i \in E^3$ control points are

$$k_1\,|_{t=1}=\frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}.\sin\psi\cos\varphi$$

$$k_2\,|_{t=1}=\frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}.\sin\psi\sin\varphi,$$

where the φ angle is calculated as:

$$\varphi=-\frac{(n-2)}{n}\frac{(\Delta b_{n-1}\ \Delta b_{n-2}\ \Delta b_{n-3})}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|^2}t+c.$$

Proof: Since the $\tau=\varphi'$ formula is used in order to calculate the angle at the Bishop frame, we obtain the following angles when we take the integral of both sides of this equation:

$$\varphi=\int\tau(t)\,|_{t=1}\,dt$$
$$=-\int\frac{n-2}{n}\frac{(\Delta b_{n-1}\ \Delta b_{n-2}\ \Delta b_{n-3})}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|^2}dt$$
$$=-\int\frac{n-2}{n}\frac{(\Delta b_{n-1}\ \Delta b_{n-2}\ \Delta b_{n-3})}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|^2}t+c.$$

In addition, when the curvature formulas of the Bishop roof for the curves are considered; the following equations are obtained:

$$k_1\,|_{t=1}=\kappa.\cos\varphi=\frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}.\sin\psi\cos\varphi$$

$$k_2\,|_{t=1}=\kappa.\cos\varphi=\frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}.\sin\psi\sin\varphi$$

Theorem 3.7. The derivation formulas of the Bishop frame belonging to Bezier curve at the start point are given by following equations

$$\mathbf{T}'\,|_{t=0}=(n-1)\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\phi[\cos\theta\mathbf{N}_1+\sin\theta\mathbf{N}_2]$$

$$\mathbf{N}_1'\,|_{t=0}=(1-n)\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\phi\cos\theta\mathbf{T}$$

$$\mathbf{N}_2'\,|_{t=0}=(1-n)\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\phi\sin\theta\mathbf{T}.$$

Proof: When the k_1, k_2 curvatures of the Bezier curve with the unit speed in the Bishop frame are taken and the necessary abbreviations are made after writing the υ constant number, the derivation formulas of the Bishop frame belonging to Bezier curve at the start point are calculated by

$$\mathbf{T}'\,|_{t=0}=k_1\upsilon\mathbf{N}_1+k_2\upsilon\mathbf{N}_2$$
$$=\frac{(n-1)}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}\sin\phi\cos\theta n\|\Delta b_0\|\mathbf{N}_1$$
$$+\frac{(n-1)}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}\sin\phi\sin\theta n\|\Delta b_0\|\mathbf{N}_2$$
$$=(n-1)\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\phi[\cos\theta\mathbf{N}_1+\sin\theta\mathbf{N}_2]$$

$$\mathbf{N}_1'|_{t=0} = -k_1\upsilon\mathbf{T}$$
$$= -\frac{(n-1)}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}\sin\phi\cos\theta n\|\Delta b_0\|\mathbf{T}$$
$$= (1-n)\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\phi\cos\theta\mathbf{T}$$
$$\mathbf{N}_2'|_{t=0} = -k_2\upsilon\mathbf{T}$$
$$= -\frac{(n-1)}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}\sin\phi\sin\theta n\|\Delta b_0\|\mathbf{T}$$
$$= (1-n)\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\phi\sin\theta\mathbf{T}.$$

Theorem 3.8. The derivation formulas of the Bishop frame belonging to Bezier curve at the end point are given by following equations

$$\mathbf{T}'|_{t=1} = (n-1)\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|}\sin\psi[\cos\varphi\mathbf{N}_1 + \sin\varphi\mathbf{N}_2]$$
$$\mathbf{N}_1'|_{t=1} = (1-n)\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|}\sin\phi\cos\theta\mathbf{T}$$
$$\mathbf{N}_2'|_{t=1} = (1-n)\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|}\sin\phi\sin\theta\mathbf{T}$$

Proof Similarly, when the k_1, k_2 and the υ constant number in the matrix of the Bishop roof of the Bezier curve with no unit speed is taken and the necessary abbreviations are made, the derivation formulas of the Bishop roof of the Bezier curve at the start points are calculated with the flowing equations:

$$\mathbf{T}'|_{t=1} = k_1\upsilon\mathbf{N}_1 + k_2\upsilon\mathbf{N}_2$$
$$= \frac{(n-1)}{n}\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|^2}\sin\psi\cos\varphi n\|\Delta b_{(n-1)}\|\mathbf{N}_1$$
$$+ \frac{(n-1)}{n}\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|^2}\sin\psi\sin\varphi n\|\Delta b_{(n-1)}\|\mathbf{N}_2$$
$$= (n-1)\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|}\sin\psi[\cos\varphi\mathbf{N}_1 + \sin\varphi\mathbf{N}_2]$$
$$\mathbf{N}_1'|_{t=1} = -k_1\upsilon\mathbf{T}$$
$$= -\frac{(n-1)}{n}\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|^2}\sin\psi\cos\varphi n\|\Delta b_{(n-1)}\|\mathbf{T}$$
$$= (1-n)\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|}\sin\psi\cos\varphi\mathbf{T}$$
$$\mathbf{N}_2'|_{t=1} = -k_2\upsilon\mathbf{T}$$
$$= -\frac{(n-1)}{n}\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|^2}\sin\psi\sin\varphi n\|\Delta b_{(n-1)}\|\mathbf{T}$$
$$= (1-n)\frac{\|\Delta b_{(n-2)}\|}{\|\Delta b_{(n-1)}\|}\sin\psi\sin\varphi\mathbf{T}$$

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Competing Interests:

The authors declare that they have no competing interests.

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