Statistical Analysis of Factorial Experiments for Quality Engineering and Similar Cases under Inverse Gaussian Model

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Abstract: Factorial experiments with k-factors have varieties of applications in life testing and engineering reliability studies. In the last few decades, researchers became aware of the appropriateness of assuming an Inverse Gaussian Model instead of the Normal Model in order to analyze such experiments. However, previous researches provide no complete study for the case of two-factor, neither it provide any attempt to consider the more general case of K-factor experiments. In this article we reconsider the case of two-factor experiments and provide an explicit algebraically form of maximum likelihood estimators for all main effects and all sum of reciprocals. This new development enable us to provide a complete decomposition for the total sum of reciprocals and construct an ANOR table in a complete analogue to the ANOVA table under the Normal Model. This will provide a complete statistical analysis of such experiments. This work is extended to the case of three-factor experiment. First, we considered the additive model that containing all main effects and obtain explicit algebraically form of the maximum likelihood estimators for the main effects. This procedure can be generalized to any k-factor additive model. Then, we considered some sub models of the complete three factor model, in particular, sub models with one main effect and one interaction effect, and sub models with two interaction effects. For those sub models we were again able to obtain an explicit form of the maximum likelihood estimators for the main effect and for the interaction effects. These estimators allow us to provide a perfect decomposition for the total sum of reciprocals and again to construct an ANOR table in a complete analogue to the ANOVA table under the Normal Model for the case of three-factor experiment. Applications of the procedures are illustrated with data set of strength measurements of an insulating material and a data set on Effect of Humidity and Several Surface Coatings on the Fatigue Life of 2024-T351 Aluminum Alloy.

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1. Introduction

For the case of balanced two-factor experiments, we derived an explicit form of maximum likelihood (ML) estimators and we were able to prove that they are indeed the ML estimators; the forms of those estimators have a great similarity with their analogues in the normal case. Regarding the analysis of reciprocals, we discovered that the reminder in the decomposition of the total sum of reciprocals founded by Fries and Bhattacharyya (1983) is irrelevant and we were able to establish a perfect decomposition of the total sum of reciprocals. This has a great impact on improving the approximate F tests for main effects and interaction. Factorial experiments are widely used in different fields of science to study main effects and interactions among several variables. The usual assumptions that such experiments relay on are three main assumptions: 1) each variable follow a normal distribution with specific mean, 2) all distributions has the same standard deviation (this is known in the literature as homoscedasticity assumption), 3) the observation selected from those normal distributions are independent. The assumptions of equal standard deviation and independency are realistic and crucial for analyzing data from any kinds of experiments; however the assumption of normality is not realistic in many cases, especially for the case of positively skewed data. Such data is often results from life testing and engineering reliability studies. Many authors studied the analysis of factorial experiments with two factors under the assumption of non-normal distributions such as the exponential, gamma and Weibull distributions (see for example, Zelen (1959, 1960), Lawless and Singhal (1980)). However, those distributions as a basis for analyzing positively skewed data have their own drawbacks. The lack of memory in the case of the exponential model is debatable in many practical situations and its scope does not fit with all data. The Weibull and gamma models offer wider scope but using them in analyzing factorial experiments confronts serious complexities.

The Inverse Gaussian distribution has a history dating back to 1915; it is the result of driving the density functions of the first passage time of Brownian motion with positive drift. In the last 30 years or so, Inverse Gaussian distribution won the attention of many statistician as a useful model for developing statistical methods for data supposed to have arisen from this distribution (see for example, Chhikara and Folks (1989), Seshadri (1993 and 1999).

Inverse Gaussian distribution is a family of distributions that share striking similarities with the Gaussian family. For examples: statistical inference for one- and two-sample under the normal model is developed under the Inverse Gaussian mode and the analysis of variance under the normal model developed under the Inverse Gaussian model and the result is the "hierarchical analysis of variance" which is known in the literature as "analysis of reciprocals". The pioneer work in this subject is given by Tweedie (1957).

Regarding the factorial experiment, some attempts have been done but relay on some artificial assumptions and has some kind of drawback; see for example Shuster and Miura (1972).

A remarkable development of the analysis of factorial experiments in case of balanced two factors has been accomplished by the work of Fries and Bhattacharyya (1983). Their assumptions entail a linear model for the reciprocal mean of the Inverse Gaussian distribution with constant precision parameter for all levels of the factors. The constancy of the precision parameter is parallel to the homoscedasticity assumption in the usual normal theory. They applied the maximum likelihood method to estimate the model's parameters, and provide a closed form expression for their solution of the normal equations, and established the limiting normality of those solutions. They break down the total sum of reciprocal to several components representing the main effects, interaction and the errors or residuals and hence constructed the analysis of reciprocals table. However, their work has its own drawbacks which are summarizing in the following: their solution to the normal equation has no explicit form with unknown structure as the case of the normal model; it is not easy to calculate the solution of the normal equation and hence the sums of reciprocals; they were unable to prove that the solution is indeed the maximum likelihood estimators; their decomposition of the total sum of reciprocals has a remainder that they called a non-orthogonality component. Since their work in 1983, neither further works in attempts to overcome those drawbacks were accomplished; nor has a new development in the analysis of factorial experiments under the Inverse Gaussian model been developed.

The objective of this article is to overcome and solve the drawbacks in the analysis of factorial experiments with the Inverse Gaussian model.

In this article we consider the same model of Fries and Bhattacharyya (1983). We provide explicit algebraic formulas for all MLE's of model's parameters. Then; explicit algebraic forms of all sums of squares are obtained.

We organized the work as follows. In section 2, the case of one factor experiment under an inverse Gaussian model assuming a reciprocal linear model is considered for the purpose of comparison with models having more than one factor. Section 3 gives a full description of the models with two factors and the normal equations that need to be solved, while section 4 describes the method of estimation of the model's parameters as given in the literature. The new contributions of this research regarding the case of two factors are given in sections 5 and 6 where we introduce explicit algebraic formulas for the ML estimators, perfect decomposition for the total sum of reciprocals and a construction of ANOR table as well as numerical application for illustration. Further contributions are given in the reminder sections (7 to 10), where a generalization to the case of three factors is given companion with numerical application. This work is applicable to any k-factor experiments. In all cases, the ML estimator for the effect of certain level of any factor can be seen to have the same form as in the case of one factor experiments added to it a term that can be interpreted as an adjustment or as a non-orthogonality component.

The Inverse family of distributions, denoted as $IG(\theta,\sigma)$ has probability density function given by

$$f(y;\theta,\sigma)=(2\pi\sigma)^{-1/2}\,y^{-3/2}\exp\left(-(2\sigma y)^{-1}\left(y\theta^{-1}-1\right)^2\right)\quad ;\, y>0,\,\theta>0,\,\sigma>0. \qquad (1)$$

This probability function belongs to the exponential family. The mean and variance of this distribution are θ and $\theta^3\sigma$ respectively.

2. The Case of One Factor Experiment

Consider a one factor life test with a levels of the factor. At each level, n items are tested and their failure times y_{ij}, $i=1,\ldots,a$, and $j=1,\ldots,n$ recorded. The observations are assumed to be independent with y_{ij} distributed as $IG(\theta_i,\sigma)$. Since the mean is inversely proportional to the drift, the usual parameterization suggests the model

$$\theta_i^{-1}=\mu+\alpha_i\,,\ \sum_{i=1}^{a}\alpha_i=0 \qquad (2)$$

where μ and α_i's represents the grand mean and the main factor effects respectively. For the IG distribution we must have $\theta_i \geq 0$ for all i and $\sigma > 0$. Thus the parameters μ , $\alpha' = (\alpha_1, ..., \alpha_a)$ and σ lie in the set

$$\Omega = \left\{ (\mu, \alpha', \sigma) : \sum_i \alpha_i = 0;\ \mu + \alpha_i > 0, i = 1, ..., a;\ \sigma > 0 \right\} \tag{3}$$

We introduced the basic notation for the totals and the means that will be used throughout the paper:

$$y_{i.} = \sum_j y_{ij} = n\bar{y}_{i.}, y_{..} = \sum_i \sum_j y_{ij} = na\bar{y}_{..},$$

$$SR = \sum_i \sum_j y_{ij}^{-1} \tag{4}$$

Referring to (1) and (2), the log-likelihood function has the form

$$l = const. - (1/2)an \log \sigma$$
$$-(2\sigma)^{-1} \sum \sum y_{ij}^{-1} \left[y_{ij}(\mu + \alpha_i) - 1 \right]^2 \tag{5}$$

Expanding the squared term, we find that the set $(\bar{y}_{1.}, ..., \bar{y}_{a.}, SR)$ represent a set of $(a+1)$-dimensional sufficient statistics, with the parameter space Ω of dimension $(a+1)$ as well.

Equating to zero the first partial derivatives of (5) with respect to μ and α_i, we obtain

$$\hat{\mu}\, y_{i.} + \sum_i \hat{\alpha}_i\, y_{i.} = n\, a$$
$$\hat{\mu}\, y_{i.} + \quad \hat{\alpha}_i\, y_{i.} = n, \qquad a \leq i \leq 1 \tag{6}$$

The derivative with respect to σ leads to

$$\hat{\sigma} = \frac{1}{an} \sum \sum y_{ij}^{-1} \left[y_{ij} (\hat{\mu} + \hat{\alpha}_i) - 1 \right]^2 \tag{7}$$

The system (6) of equations has the following unique solution

$$\hat{\mu} = \frac{1}{a} \sum_i \frac{1}{\bar{y}_{i.}}$$
$$\hat{\alpha}_i = \frac{1}{\bar{y}_{i.}} - \frac{1}{a} \sum_i \frac{1}{\bar{y}_{i.}}, \ i = 1, ..., a \tag{8}$$

while

$$\hat{\sigma} = \frac{1}{an} \left[R - \frac{a\,n}{a} \sum_i \frac{1}{\bar{y}_{i.}} \right] = \frac{1}{an} [R - an\hat{\mu}] \tag{9}$$

To test the hypothesis of no main effects, i.e. to test $H_0 : \alpha_1 = \alpha_2 = ... = \alpha_a = 0$, against $H_1 : \alpha_1 \neq \alpha_2 \neq ... \neq \alpha_a$, the LR test statistic is given by

$$\Lambda = 2\left[l_{\max(\Omega_1)} - l_{\max(\Omega_0)} \right] = a\,n \log\left(\frac{\hat{\sigma}_0}{\hat{\sigma}_1} \right) \tag{10}$$

The last expression in (10) obtains from the general result that the maximized log-likelihood, under each model $\Omega_i, i = 0,1$, has the value $-\left(\frac{1}{2}\right) a\,n (\log \hat{\sigma}_i + 1)$, ignoring the constant term. Expression for $\hat{\sigma}_1$ is given by (9) while $\hat{\sigma}_0$ is given by

$$\hat{\sigma}_0 = \frac{1}{an} \left[SR - a\,n \frac{1}{\bar{y}_{..}} \right] \tag{11}$$

The rejection region consists of the large values of the statistic in (10).

We note that,

$$\Lambda = an \log\left(1 + \frac{\hat{\sigma}_0 - \hat{\sigma}_1}{\hat{\sigma}_1} \right) \tag{12}$$

which is strictly increasing function of $R_{01} = \dfrac{\hat{\sigma}_0 - \hat{\sigma}_1}{\hat{\sigma}_1}$. Consequently, the LR test can equivalently be based on R_{01} with large values in the rejection region.

Let $Q_1^* = \hat{\sigma}_1$ and,

$$Q_2^* = \hat{\sigma}_0 - \hat{\sigma}_1 = \sum_i \sum_j \left[\frac{1}{y_{ij}} - \frac{1}{\bar{y}_{..}} \right] - \sum_i \sum_j \left[\frac{1}{y_{ij}} - \frac{1}{\bar{y}_{i.}} \right]$$
$$= \sum_i \left[\frac{n}{\bar{y}_{i.}} - \frac{n}{\bar{y}_{..}} \right] \tag{13}$$

Hence,

$$R_{01} = \frac{\hat{\sigma}_0 - \hat{\sigma}_1}{\hat{\sigma}_1} = \frac{Q_2^*}{Q_1^*} \tag{14}$$

The statistic $Q_1^* = \sum_i \sum_j \left[\dfrac{1}{Y_{ij}} - \dfrac{1}{\bar{Y}_{i.}} \right]$ divided by σ has χ^2 distribution with $a(n-1)$. While, under the assumption of no main effects, the statistics $Q_2^* = \sum_i \left[\dfrac{n}{\bar{Y}_{i.}} - \dfrac{n}{\bar{Y}_{..}} \right]$ divided by σ has χ^2 distribution with $a-1$ degrees of freedom. The two statistics are independent (see Seshadri (1983) and Datta (2005) for properties of inverse Gaussian distribution); hence one can use the *F* test based on the statistic

$$T_{01} = \frac{a(n-1)Q_2^*}{(a-1)Q_1^*} \tag{15}$$

with $(a-1)$ and $a(n-1)$ degrees of freedom.

3. The Model: Balanced Two-factor Experiment With no Interaction

In this section, we consider the same model as Fries and Bhattacharyya (1983). A full description of the models having two factors is given along with the normal equations that need to be solved.

For the balanced two-factor life test, assume a levels of factor A and b levels of factor B. At each cell (i, j), n items are tested and failure times $Y_{ijk}, k = 1, 2, \ldots, n$ are recorded. The observations are assumed to be independent with $Y_{ijk} \sim IG(\theta_{ij}^{-1}, \sigma)$; $i = 1, 2, \ldots, a$, $j = 1, 2, \ldots, b$, and $k = 1, \ldots, n$. We focus on the additive or no-interaction model; hence the means have the structure

$$\theta_{ij} = (\mu + \alpha_i + \beta_j)^{-1}, \; i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b, \quad \& \sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = 0 \tag{16}$$

where μ, α_i's , and β_j's represent the grand mean, the main effects of factor A, and the main effects of factor B, respectively. We must have $\theta_{ij} > 0$ for all i, j and $\sigma > 0$. Thus the parameters $\mu, \alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_a)$, $\beta' = (\beta_1, \beta_2, \ldots, \beta_a)$ and σ lie in the set

$$\Omega = \{(\mu, \alpha', \beta', \sigma) : \sum_i \alpha_i = \sum_j \beta_j = 0; \; \mu + \alpha_i + \beta_j > 0, \quad i = 1, \ldots, a; \; \& \; j = 1, \ldots, b; \sigma > 0\} \tag{17}$$

The basic notation for the totals and the means are extended to the two-factors experiments as follows

$$\begin{aligned} y_{ij} &= \sum_k y_{ijk} = n\bar{y}_{ij} \\ y_{i.} &= \sum_j \sum_k y_{ijk} = nb\bar{y}_{i.} \\ y_{.j} &= \sum_i \sum_k y_{ijk} = na\bar{y}_{.j.} \\ y_{..} &= \sum_i \sum_j \sum_k y_{ijk} = nab\bar{y}_{...} \\ SR &= \sum_i \sum_j \sum_k y_{ijk}^{-1} \end{aligned} \tag{18}$$

The log-likelihood function has the form

$$l = const. - (1/2) an \log \sigma - (2\sigma)^{-1} \sum_i \sum_j \sum_k y_{ijk}^{-1} [y_{ijk}(\mu + \alpha_i + \beta_j) - 1]^2 \tag{19}$$

As it is previously done by Fries and Bhattacharyya, equating to zero the first partial derivatives of (19) with respect to μ, α_i, and β_j , to obtain the following normal equations:

$$\begin{aligned} &\hat{\mu} y_{..} + \sum_i \hat{\alpha}_i y_{i.} + \sum_j \hat{\beta}_j y_{.j} = nab \\ &\hat{\mu} y_{i.} + \hat{\alpha}_i y_{i.} + \sum_j \hat{\beta}_{.j.} y_{ij} = bn, \qquad 1 \le i \le a \\ &\hat{\mu} y_{.j} + \sum_i \hat{\alpha}_i y_{ij} + \hat{\beta}_{.j.} y_{.j} = an, \qquad 1 \le j \le b \end{aligned} \tag{20}$$

and the derivative with respect to σ leads to

$$\hat{\sigma} = \frac{1}{abn} \sum_i \sum_j \sum_k y_{ijk}^{-1} \left[y_{ijk} \left(\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j \right) - 1 \right]^2 \tag{21}$$

4. Estimation of the Model's parameters

This section demonstrates the method given by Fries and Bhattacharyya (1983) to solve the normal equations given by (20). While they provided an excellent way to solve those equations, they had not been able to prove that their solution is the maximum likelihood estimator generally. In the following sections, we will provide alternative way to solve those normal equations, and then express the solution in an explicit algebraic form of the reciprocals of level means. Moreover, we have been able to prove that this solution is the maximum likelihood estimator generally.

The system of linear equation given by (20) is linear in the parameters, however, the summation of the a equations associated with the α_i's yields the first equation; also, the summation of the b equations associated with the β_j's yields the first equation.

Fries and Bhattacharyya (1983) used the conditions $\sum \alpha_i = \sum \beta_j = 0$ to delete the last components of the vectors α and β, and define the new parameter

$$\phi = (\mu, \alpha_1, \alpha_2, \cdots, \alpha_{a-1}, \beta_1, \beta_2, \cdots, \beta_{b-1}) \tag{22}$$

Then they observed that, for every (i, j) there exists an $(a + b - 1)$ vector x_{ij} consisting of -1's, 0's and I's such that

$$\mu + \alpha_i + \beta_j = \phi' x_{ij} \quad 1 \le i \le a, 1 \le j \le b. \tag{23}$$

They define the $ab \times (a + b - 1)$ and $(a + b - 1) \times (a + b - 1)$ matrices X and M as

$$\begin{aligned} X' &= (x_{11}, x_{12}, \cdots, x_{ab}), \\ M &= X' D X, \end{aligned} \tag{24}$$

where $D = diag(\bar{y}_{11.}, \bar{y}_{12.}, \cdots, \bar{y}_{ab.})$.

Then they used the aforementioned notation to rewrite the log-likelihood function and obtain a new set of normal equations, and showed that these normal equations has the following unique solutions, that maximize the likelihood

$$\begin{aligned} \hat{\phi} &= M^{-1} X' J, \\ \hat{\sigma} &= (abn)^{-1} \left[SR - nJ'XM^{-1}X'J \right] \end{aligned} \tag{25}$$

where J is the vector of one's.

No proof is provided by Fries and Bhattacharyya that the above solution to be the maximum likelihood estimators, however, they provided a theorem stated that $\hat{\phi}$ and $\hat{\sigma}$ serve the primary goal of maximum likelihood estimation, by proving that they are asymptotically independent, strongly consistent, and the limiting distributions of $n^{1/2}\left(\hat{\phi}-\phi\right)$ and $n^{1/2}\left(\hat{\sigma}-\sigma\right)$ are normal.

From the above demonstration, we cannot call these solutions explicit; since they do not show the structures of these solutions as are the likelihood estimators under the usual normal model. It may better to define them as a closed form solutions.

In the next section, we will provide alternative way to solve the set of normal equation (20) that give an explicit algebraic form for μ and all α_i's and β_j's. Moreover, we proved that these estimators are indeed the maximum likelihood estimators. The proposed method looks deeply into the details structure of the ML estimators.

5. Explicit Algebraic Solutions of the Model's ML Estimators

In this subsection we will obtain an explicit form for the maximum likelihood estimators. The form of these estimators contains two parts; the first part can be viewed as a natural generalization for the solution obtained before in the case of one factor experiment, and the second part can be described as a nonorthogonality component. To obtain these solutions we proceed as follows.

First, we rewrite the normal equations (20) in the new form

$$\hat{\mu}=\frac{1}{\bar{y}_{..}}-\frac{1}{a}\sum_i\frac{\bar{y}_{i.}}{\bar{y}_{..}}\hat{\alpha}_i-\frac{1}{b}\sum_j\frac{\bar{y}_{.j.}}{\bar{y}_{..}}\hat{\beta}_j \qquad (26)$$

$$\hat{\alpha}_i=\frac{1}{\bar{y}_{i\,.}}-\hat{\mu}-\frac{1}{b}\sum_j\frac{\bar{y}_{.j.}}{\bar{y}_{..}}\hat{\beta}_j \qquad (27)$$

$$, i=1,2,\dots,a$$

$$\hat{\beta}_j=\frac{1}{\bar{y}_{.j.}}-\hat{\mu}-\frac{1}{a}\sum_i\frac{\bar{y}_{i.}}{\bar{y}_{..}}\hat{\alpha}_i \qquad (28)$$

$$, j=1,2,\dots,b$$

Second, we get rid of $\hat{\mu}$ that appears in (27) and (28) this can be accomplished by applying the constraint $\sum_i\alpha_i=0$ on the $\hat{\alpha}_i$'s equations and solve for $\hat{\mu}$, to obtain another form of it, denoted it as $\hat{\mu}_\alpha$ (referring to that it is obtained from the $\hat{\alpha}_i$'s equations). This $\hat{\mu}_\alpha$ is given by

$$\hat{\mu}_\alpha=\frac{1}{a}\sum_i\frac{1}{\bar{y}_{i.}}-\frac{1}{ab}\sum_i\sum_j\frac{\bar{y}_{ij}}{\bar{y}_{i.}}\hat{\beta}_j \qquad (29)$$

Substitute this value of $\hat{\mu}$ as given by (29) back into equation (27) to obtain a new form of the $\hat{\alpha}_i$'s equations that free of $\hat{\mu}$ as:

$$\hat{\alpha}_i=\left(\frac{1}{\bar{y}_{i..}}-\frac{1}{a}\sum_i\frac{1}{\bar{y}_{i..}}\right)-\left(\frac{1}{b}\sum_j\frac{\bar{y}_{ij.}}{\bar{y}_{i..}}\hat{\beta}_j-\frac{1}{ab}\sum_i\sum_j\frac{\bar{y}_{ij.}}{\bar{y}_{i..}}\hat{\beta}_j\right) \qquad (30)$$

$$, i=1,2,\dots,a$$

Similarly, applying the constraint $\sum_j\beta_j=0$ on the $\hat{\beta}_j$'s equations and solve for $\hat{\mu}$ to obtain

$$\hat{\mu}_\beta=\frac{1}{b}\sum_j\frac{1}{\bar{y}_{.j.}}-\frac{1}{ab}\sum_i\sum_j\frac{\bar{y}_{ij.}}{\bar{y}_{.j.}}\hat{\alpha}_i\,, \qquad (31)$$

and hence obtain

$$\hat{\beta}_j=\left(\frac{1}{\bar{y}_{.j.}}-\frac{1}{b}\sum_j\frac{1}{\bar{y}_{.j.}}\right)-\left(\frac{1}{a}\sum_i\frac{\bar{y}_{ij.}}{\bar{y}_{.j.}}\hat{\alpha}_i-\frac{1}{ab}\sum_i\sum_j\frac{\bar{y}_{ij.}}{\bar{y}_{.j.}}\hat{\alpha}_i\right) \qquad (32)$$

$$, j=1,2,\dots,b$$

Our goal now is to solve equations (30) & (32) simultaneously for $\hat{\alpha}_i$'s and $\hat{\beta}_j$'s, then to use the result to find $\hat{\mu}$ that satisfies the three forms of $\hat{\mu}$ namely equation (26), (29), and (31).

Interestingly, Equations (30) & (32) show that both $\hat{\alpha}_i$'s and $\hat{\beta}_j$'s consist of two terms. The first term is analogue to that of the one-factor case discussed in section 2. The second part is a linear function of the other factor's main effects. The sum of each part is zero.

We use an algebraic iteration method to solve these equations. The initial values for $\hat{\alpha}_i$'s are $\hat{\alpha}_i^{(0)}=0, i=1,2,\dots,a$ while, the initial values for $\hat{\beta}_j$'s are $\hat{\beta}_j^{(0)}=0,\ j=1,2,\dots,b$.

Substitute these values into equations (30) and (32) yield the values of the first iteration as

$$\hat{\alpha}_i^{(1)}=\frac{1}{\bar{y}_{i..}}-\frac{1}{a}\sum_i\frac{1}{\bar{y}_{i..}}\,, i=1,2,\dots,a\,, \qquad (33)$$

$$\hat{\beta}_j^{(1)}=\frac{1}{\bar{y}_{.j.}}-\frac{1}{b}\sum_j\frac{1}{\bar{y}_{.j.}}\,,\ j=1,2,\dots,b \qquad (34)$$

Substitute these new values into equations (30) and (32) yield the values of the second iteration as

$$\hat{\alpha}_i^{(2)} = \left(\frac{1}{\bar{y}_{i..}} - \frac{1}{a}\sum_i \frac{1}{\bar{y}_{i..}}\right) - \left(w_{i.}^{(1)} - \frac{1}{a}\sum_i w_{i.}^{(1)}\right),\ \forall\, i \quad (35)$$

$$\hat{\beta}_j^{(2)} = \left(\frac{1}{\bar{y}_{.j.}} - \frac{1}{b}\sum_j \frac{1}{\bar{y}_{.j.}}\right) - \left(w_{.j}^{(1)} - \frac{1}{b}\sum_j w_{.j}^{(1)}\right),\ \forall\, j \quad (36)$$

where

$$w_{i.}^{(1)} = \frac{1}{\bar{y}_{i..}} \frac{1}{b} \sum_j \bar{y}_{ij.} w_{.j}^{(0)}$$

$$w_{.j}^{(1)} = \frac{1}{\bar{y}_{.j.}} \frac{1}{a} \sum_i \bar{y}_{ij.} w_{i.}^{(0)}, \quad (37)$$

$$w_{i.}^{(0)} = \frac{1}{\bar{y}_{i..}} \text{ and } w_{.j}^{(0)} = \frac{1}{\bar{y}_{.j.}}$$

At the R^{th} iteration, the formulas of $\hat{\alpha}_i^{(R)}$ and $\hat{\beta}_j^{(R)}$ are

$$\hat{\alpha}_i^{(R)} = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{i.}^{(r-1)} - \frac{1}{a}\sum_i w_{i.}^{(r-1)}\right) \quad (38)$$
$$,\ i = 1, 2, \ldots, a$$

and

$$\hat{\beta}_j^{(R)} = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{.j}^{(r-1)} - \frac{1}{b}\sum_j w_{.j}^{(r-1)}\right) \quad (39)$$
$$,\ j = 1, 2, \ldots, b$$

with $R = 1, 2, 3, \ldots$ Since both $\hat{\alpha}_i\text{'s}$ and $\hat{\beta}_j\text{'s}$ are exist, then each of the infinite series in the second terms of (38) and (39) converge to some limit. However, being these series having alternate signs, then each of them must converge to zero. Hence, there exist some odd integer R such that

$$w_{i.}^{(R)} - \frac{1}{a}\sum_i w_i^{(R)} = 0,\ \forall\, i \quad (40)$$

and

$$w_{.j}^{(R)} - \frac{1}{b}\sum_j w_j^{(R)} = 0,\ \forall\, j \quad (41)$$

Together with the relations

$$w_i^{(r)} = \frac{1}{\bar{y}_{i..}} \frac{1}{b} \sum_j \bar{y}_{ij.}\, w_j^{(r-1)}, \quad (42)$$

and

$$w_j^{(r)} = \frac{1}{\bar{y}_{.j.}} \frac{1}{a} \sum_i \bar{y}_{ij.}\, w_i^{(r-1)}, \quad (43)$$

we come to the conclusion that the integer R satisfies

$$w_i^{(R)} = w_j^{(R)} = M \text{ (say).}$$

To find the value M, we investigate the following general formulas of $\hat{\mu}^{(R)}$, $\hat{\mu}_{\alpha}^{(R)}$ and $\hat{\mu}_{\beta}^{(R)}$ at the R^{th} iteration

$$\hat{\mu}^{(R)} = \frac{1}{a}\sum_i \sum_{r=1}^{R} (-1)^{(r-1)} w_{i.}^{(r-1)} + \frac{1}{b}\sum_j \sum_{r=1}^{R} (-1)^{(r-1)} w_{.j}^{(r-1)} + (-1)^r \frac{1}{\bar{y}_{..}} \quad (44)$$

$$\hat{\mu}_{\alpha}^{(R)} = \frac{1}{a}\sum_i \sum_{r=1}^{R} (-1)^{(r-1)} w_{i.}^{(r-1)} + \frac{1}{b}\sum_j \sum_{r=1}^{R} (-1)^{(r-1)} w_{.j}^{(r-1)} + (-1)^R \frac{1}{a}\sum_i w_{i.}^{(R)} \quad (45)$$

and

$$\hat{\mu}_{\beta}^{(R)} = \frac{1}{a}\sum_i \sum_{r=1}^{R} (-1)^{(r-1)} w_{i.}^{(r-1)} + \frac{1}{b}\sum_j \sum_{r=1}^{R} (-1)^{(r-1)} w_{.j}^{(r-1)} + (-1)^R \frac{1}{b}\sum_j w_{.j}^{(R)} \quad (46)$$

The necessary and sufficient condition for the above three equations to be equal is

$$\frac{1}{a}\sum_i w_{i.}^{(R)} = \frac{1}{b}\sum_j w_{.j}^{(R)} = 1/\bar{y}_{..} \quad (47)$$

Hence, the constant M is equal to $1/\bar{y}_{..}$.

The integer R must be an odd number, for the iteration producer completes a full cycle of changing signs at $R-1$. Hence, we introduce the following theorem.

Theorem 1

For the model described in section 2, the maximum likelihood estimators for the model parameters are:

$$\hat{\mu} = \frac{1}{a}\sum_i \sum_{r=1}^{R} (-1)^{r-1} w_{i.}^{(r-1)} + \frac{1}{b}\sum_j \sum_{r=1}^{R} (-1)^{r-1} w_{.j}^{(r-1)} - \frac{1}{\bar{y}_{..}} \quad (48)$$

$$\hat{\alpha}_i = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{i.}^{(r-1)} - \frac{1}{a}\sum_i w_{i.}^{(r-1)}\right) \quad (49)$$
$$,\ i = 1, 2, \ldots, a$$

and

$$\hat{\beta}_j = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{.j}^{(r-1)} - \frac{1}{b}\sum_j w_{.j}^{(r-1)}\right) \quad (50)$$
$$,\ j = 1, 2, \ldots, b$$

where

$$w_{i.}^{(r)} = \frac{1}{\bar{y}_{i.}} \frac{1}{b} \sum_j \bar{y}_{ij}\, w_{.j}^{(r-1)},\ r = 1, 3, \ldots, R-1$$

$$w_{.j}^{(r)} = \frac{1}{\bar{y}_{.j}} \frac{1}{a} \sum_i \bar{y}_{ij}\, w_{i.}^{(r-1)},\ r = 1, 3, \ldots, R-1$$

$w_{i.}^{(0)} = \dfrac{1}{\bar{y}_{i.}}$ and $w_{.j}^{(0)} = \dfrac{1}{\bar{y}_{.j}}$, finally R is an (odd) integer that satisfies

$$w_{i.}^{(R)} = w_{.j}^{(R)} = \frac{1}{\bar{y}_{..}} \forall\, i \;\&\; j \qquad (51)$$

The ML estimate of σ is

$$\hat{\sigma} = \frac{1}{abn}\left[R - abn\hat{\mu}\right] \qquad (52)$$

Proof

The equations (48) - (50) are the solution of the normal equation outlined above, those solution also satisfy the constrained imposed on the model, i. e. $1'\hat{\alpha} = 0$ and $1'\hat{\beta} = 0$. It remains to show that $\hat{\theta}_{ij}^{-1} > 0$. for this we observe that

$$\hat{\theta}_{ij}^{-1} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j$$

$$= \sum_{r=1}^{R}(-1)^{r-1} w_{i.}^{(r-1)} + \sum_{r=1}^{R}(-1)^{r-1} w_{.j}^{(r-1)} - \frac{1}{\bar{y}_{..}} \qquad (53)$$

Investigating the relationship between $w_{i.}^{(0)} = 1/\bar{y}_{i.}$, and $w_{i.}^{(R)} = 1/\bar{y}_{..}$, we observe two cases:

Case 1:

$$1/\bar{y}_{i.} = w_{i.}^{(0)} < w_{i.}^{(1)} < \cdots < w_{i.}^{(R-1)} < w_{i.}^{(R)} = 1/\bar{y}_{..}$$

In this case we replace $w_{i.}^{(r)}$ by $w_{i.}^{(r+1)}$ for all odds r this leads to show that

$$\sum_{r=1}^{R}(-1)^{r-1} w_{i.}^{(r-1)} > \frac{1}{\bar{y}_{i.}} \qquad (54)$$

Case 2:

$$1/\bar{y}_{i.} = w_{i.}^{(0)} > w_{i.}^{(1)} > \cdots > w_{i.}^{(R-1)} > w_{i.}^{(R)} = 1/\bar{y}_{..}$$

In this case we replace $w_{i.}^{(r)}$ by $w_{i.}^{(r-1)}$ for all evens r this leads to show that

$$\sum_{r=1}^{R}(-1)^{r-1} w_{i.}^{(r-1)} > \frac{1}{\bar{y}_{..}} \qquad (55)$$

A similar procedure is used for the quantity $\sum_{r=1}^{R}(-1)^{r-1} w_{.j}^{(r-1)}$. Hence, for all possible four cases we end up by showing that $\hat{\theta}_{ij}^{-1} > 0$.

Hence $\hat{\theta}_{ij}$ are obtained within the parameter space Ω and the estimators $\hat{\alpha}_i\, s$ and $\hat{\beta}_j\, s$ are the ML estimators.

The expression of $\hat{\sigma}$ is obtained in a straightforward way.

We illustrate the above result with the following application which used before by Fries and Bhattacharyya (1983).

An Application for the case of two Factors:

Shuster and Miura (1972) analyzed a data set from Ostel (1963), which is in the form of a randomized 2×5 layout with 10 replicates per cell. The data consist of the impact strength, in foot-pounds, from tests on 5 lots of the same type of insulating material that are cut either lengthwise or crosswise. The use of an IG distribution is plausible since the impact strength is determined by building up stresses until failure occurs. The assumption of constant diffusion parameter is also appropriate since the same type of insulating material is being tested under a fixed specification of the failure criterion.

The cell means of this experiment are shown in table 1 where the rows represent the levels of factor A (Type of cut) and the columns represent the levels of factor B (Type of material)

Table 1: The cell means $\bar{y}_{ij}$ s

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	.919	.997	.690	.870	.551
$i=2$	.743	1.022	.624	.899	.526

The reciprocal of the grand mean is $1/\bar{y}_{..} = 1.275347532$

Tables 2 and 3 show the values of $w_{i.}^{(r)}, w_{.j}^{(r)}$, $\frac{1}{a}\sum_i w_{i.}^{(r)}$ and $\frac{1}{b}\sum_j w_{.j}^{(r)}$ for $r = 0,1,\ldots,7(=R)$.

Table 2. The values of $w_{i.}^{(r)}$ and $\hat{\alpha}$

	$i=1$	$i=2$	Average
$w^{(0)}_{i.}$	1.241619071	1.310959622	1.276289347
$w^{(1)}_{i.}$	1.279007112	1.271483577	1.275245344
$w^{(2)}_{i.}$	1.275272062	1.275427218	1.27534964
$w^{(3)}_{i.}$	1.275355721	1.275338886	1.275347304
$w^{(4)}_{i.}$	1.275347363	1.275347711	1.275347537
$w^{(5)}_{i.}$	1.275347551	1.275347513	1.275347532
$w^{(6)}_{i.}$	1.275347532	1.275347533	1.275347532
$w^{(7)}_{i.}$	1.275347532	1.275347532	1.275347532
$\hat{\alpha}_i$	-0.038518231	0.038518231	

Table 3: The values of $w_{.j}^{(r)}$ and $\hat{\beta}_j$ for j =1 to 3

	$j=1$	$j=2$	$j=3$
$w^{(0)}_{.j}$	1.203369434	0.990589401	1.522070015
$w^{(1)}_{.j}$	1.272617886	1.276718647	1.274547917
$w^{(2)}_{.j}$	1.275643702	1.275198765	1.275434291
$w^{(3)}_{.j}$	1.275341424	1.2753506	1.275345743
$w^{(4)}_{.j}$	1.275348195	1.275347199	1.275347726
$w^{(5)}_{.j}$	1.275347519	1.275347539	1.275347528
$w^{(6)}_{.j}$	1.275347534	1.275347531	1.275347533
$w^{(7)}_{.j}$	1.275347532	1.275347532	1.275347532
$\hat{\beta}_j$	-0.13443577	-0.3517717	0.18212057

Table 3 (Continue): the values of $w_{.j}^{(r)}$ and $\hat{\beta}_j$ for j = 4 to 5

	$j=4$	$j=5$	Average
$w^{(0)}_{.j}$	1.13058225	1.857010214	1.340724263
$w^{(1)}_{.j}$	1.276857712	1.275484559	1.275245344
$w^{(2)}_{.j}$	1.275183676	1.275332665	1.27535862
$w^{(3)}_{.j}$	1.275350911	1.275347839	1.275347304
$w^{(4)}_{.j}$	1.275347166	1.275347499	1.275347572
$w^{(5)}_{.j}$	1.27534754	1.275347533	1.275347532
$w^{(6)}_{.j}$	1.275347531	1.275347532	1.275347532
$w^{(7)}_{.j}$	1.275347532	1.275347532	1.275347532
$\hat{\beta}_j$	-0.21193335	0.516020172	

Finally the value of $\hat{\mu}$ is 1.341884136 the above estimates are identical to those previously obtained by Fries and Bhattacharyya (1983).

6. Statistical Inference and Analysis of Reciprocals

Being the estimators obtained in section 5 the ML estimators, enabling statistician to use the large sampling properties of the likelihood ratio tests for testing different hypotheses; such as additively or absence of factor effects and to construct confidence intervals for contrasts.

Fries and Bhattacharyya (1983) considered some variants of these testes by mixing the asymptotic and exact sampling distribution of the component statistics, and summarized the results in an analysis of reciprocals (ANOR) table. However, being the solution they provide for the normal equations not truly explicit, the structures of those estimators were hidden. This had a reflection on the analysis of the total sum of reciprocals by the appearance of a nonexistent component they had to call it a remainder and interpreted it as a nonorthogonality component.

In the following, using our explicit form of the maximum likelihood estimators given above, we provide an adjustment for the sums of reciprocals of the main effects. This adjustment results in a perfect break down of the total sum of reciprocals. This adjustment leads to a complete analogue between the ANOVA and ANOR Tables. Moreover, we were able to show that all different sums of reciprocals are positive. This adjustment is outlined in the ensuing discussion.

For testing the hypotheses of additively or absence of main factor effects, the relevant models (hypotheses) are

$\Omega_4 : \theta_{ij}^{-1}$ Unrestricted (general model)

$\Omega_3 : \theta_{ij}^{-1} = \mu + \alpha_i + \beta_j$; $1'\alpha = 1'\beta = 0$ (additive model)

$\Omega_2 : \theta_{ij}^{-1} = \mu + \alpha_i \quad 1'\alpha = 0$ (no *B* effects)

$\Omega_1 : \theta_{ij}^{-1} = \mu + \beta_j$; $1'\beta = 0$ (no *A* effects)

$\Omega_0 : \theta_{ij}^{-1} = \mu$ (no factor effects)

It each of the above model σ is unknown nuisance parameter, and $\theta_{ij}'s$ are constrained to be positive.

Let $\hat{\sigma}_s$ denote the ML estimator of σ and $l_{\max}(\Omega_s)$ denote the maximized log-likelihood under Ω_s, $s = 0,1,\ldots,4$. Expressions for $\hat{\sigma}_s's$ are given by

$$
\begin{aligned}
abn\,\hat{\sigma}_4 &= SR - \sum_i \sum_j y_{ij}^{-1} \\
abn\,\hat{\sigma}_3 &= SR - \sum_i \sum_j \hat{\theta}_{ij}^{-1} \\
abn\,\hat{\sigma}_2 &= SR - \sum_i \bar{y}_{i.}^{-1} \\
abn\,\hat{\sigma}_1 &= SR - \sum_j \bar{y}_{.j}^{-1} \\
abn\,\hat{\sigma}_0 &= SR - \bar{y}_{..}^{-1}
\end{aligned}
\tag{56}
$$

where *SR* is given in (18).

Then, the LR statistic, for testing a null hypothesis Ω_s, nested within the full model Ω_4, is given by

$$
\begin{aligned}
\Lambda_{s4} &= 2\left(l_{\max(\Omega_4)} - l_{\max(\Omega_s)}\right) \\
&= abn\log\left(\hat{\sigma}_s / \hat{\sigma}_s\right)
\end{aligned}
\tag{57}
$$

According to this general form of LR statistics, Fries and Bhattacharyya (1983) introduced the following quantities to be known as the sums of reciprocals:

$$
\begin{aligned}
R_A &= abn(\hat{\sigma}_1 - \hat{\sigma}_3) = n\sum_i \sum_j \left(\hat{\theta}_{ij}^{-1} - \bar{y}_{.j}^{-1}\right) \\
R_B &= abn(\hat{\sigma}_2 - \hat{\sigma}_3) = n\sum_i \sum_j \left(\hat{\theta}_{ij}^{-1} - \bar{y}_{i.}^{-1}\right) \\
R_{AB} &= abn(\hat{\sigma}_3 - \hat{\sigma}_4) = n\sum_i \sum_j \left(\bar{y}_{ij}^{-1} - \hat{\theta}_{ij}^{-1}\right) \\
R_E &= abn\hat{\sigma}_4 = SR - n\sum_i \sum_j \bar{y}_{ij}^{-1}
\end{aligned}
\tag{58}
$$

All of the above statistics in (57) are nonnegative.

Now, recall the explicit formula of $\hat{\theta}_{ij}^{-1}$ given by (53), namely

$$
\hat{\theta}_{ij}^{-1} = \sum_{r=1}^{R} (-1)^{r-1} w_{i.}^{(r-1)} + \sum_{r=1}^{R} (-1)^{r-1} w_{.j}^{(r-1)} - \frac{1}{\bar{y}_{..}}
$$

We can see that R_A in (58) contains an irrelevant term, namely $\sum_{r=2} (-1)^{r-1} w_{.j}^{(r-1)}$ and it should be discarded from this sum of reciprocals. Similarly, R_B in (58) contains the irrelevant term $\sum_{r=2}(-1)^{r-1}w_{i.}^{(r-1)}$, which should be discarded from it. The appearance of those irrelevant terms is caused by the unawareness of the explicit structures of the estimators. A justification of disregarding those two components is given by noting that, if the model "s" is nested within model "t", then we can obtain the partial differentiation of model "s" from the partial differentiation of model "t" by dropping off the components which represent the missing variable in model "t".

Hence we introduce the following complete decomposition of the reciprocal $1/y_{ijk}$

$$\begin{aligned}\frac{1}{y_{ijk}} &= \frac{1}{\bar{y}_{..}}\\ &+\left(\sum_{r=1}^{R}(-1)^{r-1}w_{i.}^{(r-1)}-\frac{1}{\bar{y}_{..}}\right)\\ &+\left(\sum_{r=1}^{R}(-1)^{r-1}w_{.j}^{(r-1)}-\frac{1}{\bar{y}_{..}}\right)\\ &+\left(\frac{1}{\bar{y}_{ij}}-\sum_{r=1}^{R}(-1)^{r-1}w_{i.}^{(r-1)}-\sum_{r=1}^{R}(-1)^{r-1}w_{.j}^{(r-1)}+\frac{1}{\bar{y}_{..}}\right)\\ &+\left(\frac{1}{y_{ijk}}-\frac{1}{\bar{y}_{ij}}\right)\end{aligned} \tag{59}$$

As usual, the first term is the reciprocal of general mean, the second term represents factor A effect, the third term represents factor B effect, while the fourth term can be ascribed to the interaction effect, and finally, the last term is interpreted as a residual.

Hence, our adjusted sums of reciprocals are:

$$SR(A)=abn\left(\frac{1}{a}\sum_{i}\sum_{r=1}^{R}(-1)^{r-1}w_{i,}^{(r-1)}-\frac{1}{\bar{y}_{..}}\right)$$

$$SR(B)=abn\left(\frac{1}{b}\sum_{j}\sum_{r=2}^{R}(-1)^{r-1}w_{.j}^{(r-1)}-\frac{1}{\bar{y}_{..}}\right)$$

$$SR(AB)=abn\left(\frac{1}{ab}\sum_{i}\sum_{j}\frac{1}{\bar{y}_{ij}}-\frac{1}{a}\sum_{i}\sum_{r=1}^{R}(-1)^{r-1}w_{i.}^{(r-1)}-\frac{1}{b}\sum_{j}\sum_{r=1}^{R}(-1)^{r-1}w_{.j.}^{(r-1)}+\frac{1}{\bar{y}_{..}}\right)$$

$$SR(E)=SR-n\sum_{i}\sum_{j}\bar{y}_{ij}^{-1} \tag{60}$$

All of the above statistics in (60) are nonnegative.

We can easily verify that

$$SR=abn\,\bar{y}_{...}^{-1}+SR_A+SR_B+SR_{AB}+SR_E \tag{61}$$

The distribution of the adjusted sums of reciprocals given above do not affected by this adjustment, hence we can perform the approximate F tests that demonstrate previously.

Fries and Bhattacharyya (1983) shows that each of the above sums of reciprocals divided by σ has a chi-square distribution; the first three approximately with (a -1), (b -1) and (a -1)(b -1) degrees of freedom respectively, while the fourth exactly with $ab(n$-1) degrees of freedom. Hence, an approximate F tests are used to test the usual hypotheses about the main effects and interaction. This is demonstrated in the analysis of reciprocals (ANOR) table given below.

Table 5. Analysis of Reciprocals (ANOR) Table

Source	**SR**	$d.f.$	**MSR**
A	$SR(A)$	a -1	$MSR(A)$
B	$SR(B)$	b -1	$MSR(B)$
AB	$SR(AB)$	(a-1)(b-1)	$MSR(AB)$
Error	$SR(E)$	ab(n-1)	$MSR(E)$

Then, an approximate F test can be used for testing the significant of each effect by dividing the mean sum of reciprocal of that effect by the mean sum of reciprocal of the error. For example, to test the hypotheses $H_0:\theta_{ij}^{-1}=\mu+\alpha_i+\beta_j,\ \forall i,j,k$, (i. e. the additivity of the model), we use the test statistic

$$\frac{MSR(AB)}{MSR(E)}\sim F\left((a-1)(b-1),ab(n-1)\right)$$

An Application:

The following table demonstrates the result of applying the above producers to the data given in the previous application.

Table 6: Application of ANOR Table

Source	SR	$d.f.$	MSR	F	Approx. p-value
A	0.10463	1	0.10463	4.71	0.033
B	6.54903	4	1.63726	73.64	0.0000
AB	0.26310	4	0.06578	2.96	0.024
Error	2.00094	90	0.02223		

7. Estimation for the Balanced Three Factors Experiments: the Additive Model.

The three-factor life test consists of a levels of factor A, b levels of factor B and c levels of factor C. At each factor setting or cell (i,j,k), n items are tested and their failure times y_{ijkl} recorded $\forall i,j,k,l$. The observations are independent with y_{ijkl} distributed as IG. The usual parameterization of the model with main effects and two factors interaction is

$$\begin{aligned}\theta_{ijk}^{-1}&=\mu+\alpha_i+\beta_j+\gamma_k+\delta_{ij}\\ &+\eta_{ik}+\tau_{jk}\end{aligned} \tag{62}$$

with the usual constrains

$\sum_i \alpha_i, \sum_j \beta_j, \sum_k \gamma_k = 0, \sum_i \delta_{ij} = \sum_j \delta_{ij} = \sum_{i,j} \delta_{ij} = 0,$ $\sum_i \eta_{ik} = \sum_k \eta_{ik} = \sum_{i,k} \eta_{ik} = 0$, and we must also have $\theta_{ijk}^{-1} > 0$.

In this section, we consider the additive model, i.e. we will assume that all two factors interactions are zero. The next sections deal with some sub models.

The additive model is parameterized as

$$\theta_{ijk}^{-1} = \mu + \alpha_i + \beta_j + \gamma_k, \quad \sum \alpha_i = \sum \beta_j = \sum \gamma_k = 0 \tag{63}$$

The log-likelihood function has the form

$$l = const. - (1/2) an \log \sigma - (2\sigma)^{-1} \sum_{i,j,k,l} y_{ijkl}^{-1} \left[y_{ijkl} (\mu + \alpha_i + \beta_j + \gamma_k) - 1 \right]^2 \tag{64}$$

The basic notation for the totals and the means are extended to the three-factor experiments in a straightforward way.

Equating to zero the first partial derivatives of (64) with respect to μ, α_i, β_j and γ_k to obtain the following normal equations:

$$\begin{aligned} &\hat{\mu} y_{...} + \sum_i \hat{\alpha}_i y_{...} + \sum_j \hat{\beta}_j y_{...} + \sum_k \hat{\gamma}_k y_{...} = nabc \\ &\hat{\mu} y_{i..} + \hat{\alpha}_i y_{i..} + \sum_j \hat{\beta}_j y_{ij.} + \sum_k \hat{\gamma}_k y_{ik.} = nbc; \ \forall i \\ &\hat{\mu} y_{.j.} + \sum_i \hat{\alpha}_i y_{ij..} + \hat{\beta}_j y_{.j.} + \sum_k \hat{\gamma}_k y_{.jk} = nac; \ \forall j \\ &\hat{\mu} y_{..k} + \sum_i \hat{\alpha}_i y_{i.k} + \sum_j \hat{\beta}_j y_{.jk} + \hat{\gamma}_k y_{..k} = nab; \ \forall k \end{aligned} \tag{65}$$

and the derivative with respect to σ leads to

$$\hat{\sigma} = \frac{1}{abcn} \sum_{i,j,k,l} y_{ijkl}^{-1} \left[y_{ijkl} (\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_k) - 1 \right]^2 \tag{66}$$

Handling the above set of normal equations in the exact same way as we done before for the case of two-factor experiments, we obtain the following set of equations for the main effects and the grand mean

$$\begin{aligned} \hat{\alpha}_i = &\left(\frac{1}{\bar{y}_{i..}} - \frac{1}{a} \sum_i \frac{1}{\bar{y}_{i..}} \right) \\ &- \left(\frac{1}{b} \sum_j \frac{\bar{y}_{ij.}}{\bar{y}_{i..}} \hat{\beta}_j - \frac{1}{ab} \sum_{i,j} \frac{\bar{y}_{ij.}}{\bar{y}_{i..}} \hat{\beta}_j \right) \\ &- \left(\frac{1}{c} \sum_k \frac{\bar{y}_{i.k.}}{\bar{y}_{i..}} \hat{\gamma}_k - \frac{1}{ab} \sum_{i,k} \frac{\bar{y}_{i.k}}{\bar{y}_{i..}} \hat{\gamma}_k \right), \\ &i = 1, 2, \dots, a \end{aligned} \tag{67}$$

$$\begin{aligned} \hat{\beta}_j = &\left(\frac{1}{\bar{y}_{.j.}} - \frac{1}{b} \sum_j \frac{1}{\bar{y}_{.j.}} \right) \\ &- \left(\frac{1}{a} \sum_i \frac{\bar{y}_{ij.}}{\bar{y}_{.j.}} \hat{\alpha}_i - \frac{1}{ab} \sum_{i,j} \frac{\bar{y}_{ij.}}{\bar{y}_{.j.}} \hat{\alpha}_i \right) \\ &- \left(\frac{1}{c} \sum_c \frac{\bar{y}_{.jk}}{\bar{y}_{.j.}} \hat{\gamma}_k - \frac{1}{bc} \sum_{i,k} \frac{\bar{y}_{i.k}}{\bar{y}_{.j.}} \hat{\gamma}_k \right), \\ &j = 1, 2, \dots, b \end{aligned} \tag{68}$$

and

$$\begin{aligned} \hat{\gamma}_k = &\left(\frac{1}{\bar{y}_{..k}} - \frac{1}{c} \sum_k \frac{1}{\bar{y}_{..k}} \right) \\ &- \left(\frac{1}{a} \sum_i \frac{\bar{y}_{i..k}}{\bar{y}_{..k}} \hat{\alpha}_i - \frac{1}{ac} \sum_{i,k} \frac{\bar{y}_{i.k}}{\bar{y}_{..k}} \hat{\alpha}_i \right) \\ &- \left(\frac{1}{b} \sum_j \frac{\bar{y}_{.jk}}{\bar{y}_{..k}} \hat{\beta}_j - \frac{1}{bc} \sum_{j,k} \frac{\bar{y}_{.jk}}{\bar{y}_{..k}} \hat{\beta}_j \right), \\ &k = 1, 2, \dots, c \end{aligned} \tag{69}$$

while, the equations related to the grand mean are

$$\hat{\mu} = \frac{1}{\bar{y}_{...}} - \frac{1}{a} \sum_i \frac{\bar{y}_{i..}}{\bar{y}_{...}} \hat{\alpha}_i - \frac{1}{b} \sum_j \frac{\bar{y}_{.j.}}{\bar{y}_{...}} \hat{\beta}_j - \frac{1}{c} \sum_k \frac{\bar{y}_{..k}}{\bar{y}_{...}} \hat{\gamma}_k \tag{70}$$

$$\hat{\mu}_\alpha = \frac{1}{a} \sum_i \frac{1}{\bar{y}_{i..}} - \frac{1}{ab} \sum_{i,j} \frac{\bar{y}_{ij.}}{\bar{y}_{i..}} \hat{\beta}_j - \frac{1}{ac} \sum_{i,k} \frac{\bar{y}_{i.k}}{\bar{y}_{i..}} \hat{\gamma}_k \tag{71}$$

$$\hat{\mu}_\beta = \frac{1}{b} \sum_j \frac{1}{\bar{y}_{.j.}} - \frac{1}{ab} \sum_{i,j} \frac{\bar{y}_{ij.}}{\bar{y}_{.j.}} \hat{\alpha}_i - \frac{1}{bc} \sum_{j,k} \frac{\bar{y}_{.jk}}{\bar{y}_{.j.}} \hat{\gamma}_k \tag{72}$$

and

$$\hat{\mu}_\gamma = \frac{1}{c} \sum_k \frac{1}{\bar{y}_{..k}} - \frac{1}{ac} \sum_{i,k} \frac{\bar{y}_{i.k}}{\bar{y}_{..k}} \hat{\alpha}_i - \frac{1}{bc} \sum_{j,k} \frac{\bar{y}_{.jk}}{\bar{y}_{..k}} \hat{\beta}_j \tag{73}$$

The main objective now is to find the values of $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ that satisfy the condition

$$\hat{\mu} = \hat{\mu}_\alpha = \hat{\mu}_\beta = \hat{\mu}_\gamma$$

This can be done by using the algebraic iteration method starting by the initial values $\hat{\alpha}_i^{(0)} = \hat{\beta}_j^{(0)} = \hat{\gamma}_k^{(0)} = 0, \forall i, j, k$. At the R^{th} iteration we obtain

$$\hat{\alpha}_i^{(R)} = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{i..}^{(r-1)} - \frac{1}{a} \sum_i w_{i..}^{(r-1)} \right) \tag{74}$$

$$\hat{\beta}_j^{(R)} = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{.j.}^{(r-1)} - \frac{1}{b} \sum_j w_{.j.}^{(r-1)} \right) \tag{75}$$

and

$$\hat{\gamma}_k^{(R)} = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{..k}^{(r-1)} - \frac{1}{c} \sum_k w_{..k}^{(r-1)} \right) \quad (76)$$

where

$$w_{i..}^{(r)} = \frac{1}{\bar{y}_{i..}} \left[\frac{1}{b} \sum_j \bar{y}_{ij.} w_{.j.}^{(r-1)} + \frac{1}{c} \sum_k \bar{y}_{i.k} w_{..k}^{(r-1)} \right] \quad (77)$$

$$w_{.j.}^{(r)} = \frac{1}{\bar{y}_{.j.}} \left[\frac{1}{a} \sum_i \bar{y}_{ij.} w_{i..}^{(r-1)} + \frac{1}{c} \sum_k \bar{y}_{.jk} w_{..k}^{(r-1)} \right] \quad (78)$$

and

$$w_{..k}^{(r)} = \frac{1}{\bar{y}_{..k}} \left[\frac{1}{a} \sum_i \bar{y}_{i.k} w_{i..}^{(r-1)} + \frac{1}{b} \sum_j \bar{y}_{.jk} w_{.j.}^{(r-1)} \right] \quad (79)$$

while

$$w_{i..}^{(0)} = \frac{1}{\bar{y}_{i..}};\; w_{.j.}^{(0)} = \frac{1}{\bar{y}_{.j.}} \text{ and } w_{..k}^{(0)} = \frac{1}{\bar{y}_{..k}};$$
$$i = 1,2,\ldots,a;\; j = 1,2,\ldots,b \;\&\; k = 1,2,\ldots,c \quad (80)$$

Similarly, the equations related to the grand mean are

$$\hat{\mu}^{(R)} = \frac{1}{a} \sum_i \sum_{r=1}^{R} (-1)^{r-1} w_{i..}^{(r-1)} + \frac{1}{b} \sum_j \sum_{r=2}^{R} (-1)^{r-1} w_{.j.}^{(r-1)} + \frac{1}{c} \sum_k \sum_{r=1}^{R} (-1)^{r-1} w_{..k}^{(r-1)} + (-1)^R \frac{2^R}{\bar{y}_{...}} \quad (81)$$

$$\hat{\mu}_\alpha^{(R)} = \frac{1}{a} \sum_i \frac{1}{\bar{y}_{i..}} + \frac{1}{b} \sum_j \frac{1}{y_{.j.}} + \frac{1}{c} \sum_k \frac{1}{\bar{y}_{..k}} + (-1)^R \frac{1}{a} \sum_i w_{i..}^{(R)} + \sum_{r=2}^{R} (-1)^{r-1} \left[\frac{1}{a} \sum_i w_{i..}^{(r-1)} + \frac{1}{b} \sum_j w_{.j.}^{(r-1)} + \frac{1}{c} \sum_k w_{..k}^{(r-1)} \right] \quad (82)$$

$$\hat{\mu}_\beta^{(R)} = \frac{1}{a} \sum_i \frac{1}{\bar{y}_{i..}} + \frac{1}{b} \sum_j \frac{1}{y_{.j.}} + \frac{1}{c} \sum_k \frac{1}{\bar{y}_{..k}} + (-1)^R \frac{1}{b} \sum_i w_{.j.}^{(R)} + \sum_{r=2}^{R} (-1)^{r-1} \left[\frac{1}{a} \sum_i w_{i..}^{(r-1)} + \frac{1}{b} \sum_j w_{.j.}^{(r-1)} + \frac{1}{c} \sum_k w_{..k}^{(r-1)} \right] \quad (83)$$

$$\hat{\mu}_\gamma^{(R)} = \frac{1}{a} \sum_i \frac{1}{\bar{y}_{i..}} + \frac{1}{b} \sum_j \frac{1}{y_{.j.}} + \frac{1}{c} \sum_k \frac{1}{\bar{y}_{..k}} + (-1)^R \frac{1}{c} \sum_i w_{..k}^{(R)} + \sum_{r=2}^{R} (-1)^{r-1} \left[\frac{1}{a} \sum_i w_{i..}^{(r-1)} + \frac{1}{b} \sum_j w_{.j.}^{(r-1)} + \frac{1}{c} \sum_k w_{..k}^{(r-1)} \right] \quad (84)$$

Investigating equations (76)–(79), we can see that the necessary and sufficient condition for $\hat{\mu} = \hat{\mu}_\alpha = \hat{\mu}_\beta = \hat{\mu}_\gamma$ is

$$\frac{1}{a} \sum_i w_{i..}^{(R)} = \frac{1}{b} \sum_j w_{.j.}^{(R)} = \frac{1}{c} \sum_k w_{..c}^{(R)} = \frac{2^R}{\bar{y}_{...}} \quad (85)$$

Now, investigating equation (72), we find that a necessary and sufficient for the existence of $\hat{\alpha}_i$'s is the existence of an odd integer R that satisfies

$$w_{i..}^{(R)} - \frac{1}{a} \sum_i w_{i..}^{(R)} = 0 \;\forall i \quad (86)$$

The requirement of R to be odd integer is obvious.

According to (80), a necessary and sufficient for the existence of $\hat{\alpha}_i$'s is

$$w_{i..}^{(R)} = \frac{2^R}{\bar{y}_{...}} = 0 \;\forall i \quad (87)$$

where R is and odd integer.

In a similar way, we can see that a necessary and sufficient conditions for the existence of $\hat{\beta}_j$'s and $\hat{\gamma}_k$'s are

$$w_{.j.}^{(R)} = \frac{2^R}{\bar{y}_{...}} = 0 \;\forall j \;\&\; w_{..k}^{(R)} = \frac{2^R}{\bar{y}_{...}} = 0 \;\forall k \quad (88)$$

Hence we have the following theorem.

Theorem 2:

For a balanced three-factor factorial experiment under the inverse Gaussian model that described above with positive means $\theta_{ijk}^{-1} = \mu + \alpha_i + \beta_j + \gamma_k$, the maximum likelihood estimators for the grand mean and main effects are given by:

$$\hat{\mu} = \frac{1}{a} \sum_i \sum_{r=1}^{R} (-1)^{r-1} w_{i..}^{(r-1)} + \frac{1}{b} \sum_j \sum_{r=1}^{R} (-1)^{r-1} w_{.j.}^{(r-1)} + \frac{1}{c} \sum_k \sum_{r=1}^{R} (-1)^{r-1} w_{..k}^{(r-1)} - \frac{2^R}{\bar{y}_{...}} \quad (89)$$

$$\hat{\alpha}_i = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{i..}^{(r-1)} - \frac{1}{a} \sum_i w_{i..}^{(r-1)} \right) \quad (90)$$
$$i = 1,2,\ldots,a$$

$$\hat{\beta}_j = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{.j.}^{(r-1)} - \frac{1}{b} \sum_j w_{.j.}^{(r-1)} \right) \quad (91)$$
$$j = 1,2,\ldots,b$$

and

$$\hat{\gamma}_k = \sum_{r=1}^{R} (-1)^{r-1} \left(w_{..k}^{(r-1)} - \frac{1}{c} \sum_k w_{..k}^{(r-1)} \right) \quad (92)$$
$$k = 1,2,\ldots,c$$

where $w_{i,,}^{(r)}$, $w_{,j,}^{(r)}$ and $w_{,,k}^{(r)}$ are given by (76) – (79), and R is an odd integer that satisfies

$$\frac{1}{a}\sum_i w_{i\,..}^{(R)} = \frac{1}{b}\sum_j w_{.j.}^{(R)} = \frac{1}{c}\sum_k w_{..c}^{(R)} = \frac{2^R}{\bar{y}_{...}} \tag{93}$$

Moreover, using a similar argument as in Theorem 1, we can show that

$$\hat{\theta}_{i\,j\,k}^{\;-1} = \sum_{r=1}^{R}(-1)^{r-1} w_{i..}^{(r-1)} + \sum_{r=1}^{R}(-1)^{r-1} w_{.j.}^{(r-1)} + \sum_{r=1}^{R}(-1)^{r-1} w_{..k}^{(r-1)} - \frac{2^R}{\bar{y}_{...}} > 0 \tag{94}$$

The ML estimate of σ (as usual) is

$$\hat{\sigma} = \frac{1}{abcn}\left[R - abc\,n\hat{\mu}\right] \tag{95}$$

8. Estimation for the Balanced Three Factors Experiments: A sub Models with one main effect and one interaction effect.

This section deals with estimation for the following sub model of the model (61)

$$\theta_{i\,j\,k}^{\;-1} = \mu + \alpha_i + \tau_{j\,k} \tag{96}$$

Our method of finding the maximum likelihood estimators is continuing used for the sub model (87). In this case, we obtain the following theorem.

Theorem 3:

For the sub model (95), the maximum likelihood estimators of the parameters μ, α_i, and τ_{jk} are given as

$$\hat{\mu} = \frac{1}{a}\sum_i\sum_{r=1}^{R}(-1)^{r-1} v_{i..}^{(r-1)} + \frac{1}{bc}\sum_{j,k}\sum_{r=1}^{R}(-1)^{r-1} v_{.jk.}^{(r-1)} - \frac{1}{\bar{y}_{...}} \tag{97}$$

$$\hat{\alpha}_i = \sum_{r=1}^{R}(-1)^{r-1}\left(v_{i..}^{(r-1)} - \frac{1}{a}\sum_i v_{i..}^{(r-1)}\right) \tag{98}$$

and

$$\hat{\tau}_{j\,k} = \sum_{r=1}^{R}(-1)^{r-1}\left(v_{.\,j\,k}^{(r-1)} - \frac{1}{bc}\sum_{j,k} v_{.\,j\,k}^{(r-1)}\right) \tag{99}$$

where

$$v_{i..}^{(r)} = \frac{1}{\bar{y}_{i..}}\frac{1}{bc}\sum_{j,k}\bar{y}_{i\,j\,k}\, v_{.\,j\,k}^{(r-1)},$$

$$v_{.\,jk}^{(r)} = \frac{1}{\bar{y}_{.\,j\,k}}\frac{1}{a}\sum \bar{y}_{i\,j\,k}\, v_{i...}^{(i-1)}, \tag{100}$$

$$v_{i..}^{(0)} = w_{i..}^{(0)} = \frac{1}{\bar{y}_{i..}},\; v_{.\,jk}^{(0)} = \frac{1}{\bar{y}_{.\,j\,k}}$$

and R is an odd integer that satisfies

$$\frac{1}{a}\sum_i v_{i\,..}^{(R)} = \frac{1}{bc}\sum_{j,k} w_{.\,j\,k}^{(R)} = \frac{1}{\bar{y}_{...}} \tag{101}$$

Furthermore,

$$\hat{\theta}_{i\,j\,k}^{\;-1} = \sum_{r=1}^{R}(-1)^{r-1} v_{i..}^{(r-1)} + \sum_{r=2}^{R}(-1)^{r-1} v_{.\,jk.}^{(r-1)} - \frac{1}{\bar{y}_{...}} > 0 \tag{102}$$

while the ML estimator of σ is

$$\hat{\sigma} = \frac{1}{abcn}\left[R - abc\,n\hat{\mu}\right]$$

as usual.

In a similar way, the maximum likelihood estimators can be obtained for the sub models

$$\theta_{i\,j\,k}^{\;-1} = \mu + \beta_j + \eta_{i\,k}$$

and

$$\theta_{i\,j\,k}^{\;-1} = \mu + \gamma_k + \delta_{i\,j}$$

9. Estimation for the Balanced Three Factors Experiments: A sub Models with two Interaction Effects.

This section deals with estimation for the following sub model of the model (61)

$$\theta_{i\,j\,k}^{\;-1} = \mu + \delta_{i\,j} + \eta_{j\,k} \tag{103}$$

Following the same procedure as previously done, we obtain the following ML estimators for model (103)

$$\hat{\mu} = \frac{1}{ab}\sum_{i,j}\sum_{r=1}^{R}(-1)^{r-1} w_{i\,j.}^{(r-1)} + \frac{1}{ac}\sum_{i,k}\sum_{r=1}^{R}(-1)^{r-1} w_{i.k}^{(r-1)} - \frac{1}{\bar{y}_{...}} \tag{104}$$

$$\hat{\delta}_{i\,j} = \sum_{r=1}^{R}(-1)^{r-1}\left(w_{i\,j.}^{(r-1)} - \frac{1}{ab}\sum_{i,j} w_{i\,j.}^{(r-1)}\right),\quad i = 1,2,\ldots,a \;\; j = 1,2,\ldots,b \tag{105}$$

$$\hat{\eta}_{i\,k} = \sum_{r=1}^{R}(-1)^{r-1}\left(w_{i.k}^{(r-1)} - \frac{1}{ac}\sum_{i,k} w_{i.k}^{(r-1)}\right)\quad i = 1,2,\ldots,a \;\; k = 1,2,\ldots,c \tag{106}$$

where

$$w_{i\,j.}^{(r)} = \frac{1}{\bar{y}_{i\,j.}}\frac{1}{c}\sum_k \bar{y}_{ijk}\, w_{i\,j.}^{(r-1)},$$

$$w_{i.k}^{(r)} = \frac{1}{\bar{y}_{i.k}} \frac{1}{b} \sum_{j} \bar{y}_{ijk} w_{i.k}^{(r-1)}, \qquad (107)$$

$w_{ij.}^{(0)} = \frac{1}{\bar{y}_{ij.}}$, $w_{i.k}^{(0)} = \frac{1}{\bar{y}_{i.k}}$ and R is an odd integer that satisfies the condition

$$\frac{1}{ab} \sum_{i,j} w_{ij.}^{(R)} = \frac{1}{ac} \sum_{i,k} w_{i.k}^{(R)} = \frac{1}{\bar{y}_{...}} \qquad (108)$$

10. Balanced Three Factors Experiments: Analysis of Reciprocals

In this section we provide a complete decomposition for the reciprocals observation $1/y_{ijkl}$; and then an ANOR table that is in a complete analogous to the ANOV table.

As a beginning, we define the effects as:

The effect of the i^{th} level of factor A:

$$R_i(A) = \sum_{r=1}^{R} (-1)^{r-1} v_{i..}^{(r-1)} - \frac{1}{\bar{y}_{...}} \qquad (109)$$

The effect of the j^{th} level of factor B:

$$R_j(B) = \sum_{r=1}^{R} (-1)^{r-1} v_{.j.}^{(r-1)} - \frac{1}{\bar{y}_{...}} \qquad (110)$$

The effect of the k^{th} level of factor *C*:

$$R_k(C) = \sum_{r=1}^{R} (-1)^{r-1} v_{..k}^{(r-1)} - \frac{1}{\bar{y}_{...}} \qquad (111)$$

where $v_{i..}^{(r)}$ & $v_{.jk}^{(0)}$ are as defined by (100), while R is an odd integer satisfies (101), the quantities and $v_{.j.}^{(r)}$; $v_{i.k}^{(r)}$; $v_{..k}^{(r)}$ & $v_{ij.}^{(r)}$ are defined in a similar way with similar *R*'s values.

The sum of each of the above effects is nonnegative; this can be seen by two different ways; first, each sum is obtained via the form $abcn \log(\hat{\sigma}_s / \hat{\sigma}_t)$. Secondly, by using the same argument that used to show that $\theta_{ijk} > 0$. The same argument is to show that all coming sums of reciprocals are nonnegative.

The interaction effect between the i^{th} level of factor A and the j^{th} level of factor B is introduced as

$$R_{ij}(AB) = \sum_{r=1}^{R} (-1)^{r-1} \left(v_{ij.}^{(r-1)} - v_{i..}^{(r-1)} - v_{i..}^{(r-1)} + \frac{1}{\bar{y}_{...}} \right) \qquad (112)$$

The interaction effect between the i^{th} level of factor A and the k^{th} level of factor C

$$R_{ik}(AC) = \sum_{r=1}^{R} (-1)^{r-1} \left(v_{i.k}^{(r-1)} - v_{i..}^{(r-1)} - v_{..k}^{(r-1)} + \frac{1}{\bar{y}_{...}} \right) \qquad (113)$$

The interaction effect between the j^{th} level of factor B and the k^{th} level of factor C

$$R_{jk}(BC) = \sum_{r=1}^{R} (-1)^{r-1} \left(v_{.jk}^{(r-1)} - v_{.j.}^{(r-1)} - v_{..k}^{(r-1)} + \frac{1}{\bar{y}_{...}} \right) \qquad (114)$$

The interaction effect between the i^{th} level of factor A, the j^{th} level of factor B, and the k^{th} level of factor C

$$R_{ijk}(ABC) = \frac{1}{\bar{y}_{ijk}} - R_{ij}(AB) - R_{ik}(AC) - R_{jk}(BC) + R_i(A) + R_j(B) + R_k(C) - \frac{1}{\bar{y}_{...}} \qquad (115)$$

Finally, the error effect is defined as

$$R_{ijkl}(E) = \frac{1}{y_{ijkl}} - \frac{1}{\bar{y}_{ijk}} \qquad (116)$$

Hence, the following identity is fulfilled

$$\begin{aligned} 1/y_{ijkl} &= \frac{1}{\bar{y}_{...}} + R_i(A) + R_j(B) + R_k(C) \\ &+ R_{ij}(AB) + R_{ik}(AC) + R_{jk}(BC) \\ &+ R_{ijk}(ABC) + R_{ijkl}(E) \end{aligned} \qquad (117)$$

The sum of reciprocal for a certain effect is obtained by taking the sum of its component over $i, j, k,$ and l, for example:

$$\begin{aligned} SR(A) &= bcn \sum_{i} R_i(A), \\ SR(AB) &= cn \sum_{i,j} R_{ij}(AB), \\ SR(ABC) &= n \sum_{i,j,k} R_{ijk}(ABC) \\ &\vdots \\ SR(E) &= \sum_{i,j,k,l} R_{ijkl}(E) \end{aligned} \qquad (118)$$

Hence, a complete analogue of the ANOVA table is provided by Table 7.

Table 7. Analysis of Reciprocals (ANOR) Table

Source	SR	$d.f.$	MSR
A	*SR*(*A*)	*a* -1	*MSR*(*A*)
B	*SR*(*B*)	*b* -1	*MSR*(*B*)
C	*SR*(*C*)	*b* -1	*MSR*(*C*)
AB	*SR*(*AB*)	(*a*-1)(*b*-1)	*MSR*(*AB*)
AC	*SR*(*AC*)	(*a*-1)(*c*-1)	*MSR*(*AC*)
BC	*SR*(*BC*)	(*b*-1)(*c*-1)	*MSR*(*BC*)
ABC	*SR*(ABC)	(a-1)(*b*-1)(*c*-1)	*MSR*(*ABC*)
Error	*SR*(*E*)	*abc*(*n*-1)	*MSR*(*E*)

Then, an approximate *F* test can be used for testing the significant of each effect by dividing the mean sum of reciprocal of the effect by the mean sum of reciprocal of the error. For example, to test the hypotheses $H_0 : (\alpha\beta\gamma)_{ijk} = 0 \; \forall i, j, k$, we use the test statistic

$$\frac{MSR\,(ABC)}{MSR\,(E)} \sim F((a-1)(b-1)(c-1), abc(n-1)) \quad (119)$$

An Application:

Corrosion fatigue in metals has been defined as the simultaneous action of cyclic stress and chemical attack on a metal structure. In the study *Effect of Humidity and Several Surface Coatings on the Fatigue Life of 2024-T351 Aluminum Alloy* conducted by the Department of mechanical Engineering at the Virginia Polytechnic Institute and State University, a technique involving the application of a protective chromate coating was used to minimize corrosion fatigue damage in aluminum. Three factors were used in the investigation with 5 replicates for each treatment combination: coating, at 2 levels, and humidity and shear stress, both with 3 levels, Walpole (2007).

Assuming the Normal model, the ANOVA table is given in Table 7 below

Table 7. The ANOVA Table for the Fatigue Life data of 2024-T351 Aluminum Alloy assuming normal model

Source	Sum of Squares	d. f.
A (Stress)	4.280E8	2
B (Coating)	216776.544	1
C (Humidity)	19873750.400	2
A × B	700826.422	2
A ×C	58614763.400	4
B × C	31734677.956	2
A × B × C	36028716.778	4
Error	3.352E8	72

Table 7. Continue

Mean Square	F	p-value
2.140E8	45.967	.000*
216776.544	0.047	.830
9936875.200	2.134	.126
350413.211	0.075	.928
14653690.850	3.147	.019*
15867338.978	3.408	.039*
9007179.194	1.935	.114
4655745.389		

At the 5% level of significance, we declare that the main effect of factor A (stress), the interaction effect of factor A (stress) × C (Humidity); and the interaction effect of B (Coating) × C (Humidity), are all significant.

Now, assuming the inverse Gaussian model, we drive Table 8 below.

Table 8. The ANOR for the Fatigue Life data of 2024-T351 Aluminum Alloy assuming Inverse Gaussian model

Source	Sum of Reciprocal	Degrees of Freedom
A (Stress)	0.029798622	2
B (Coating)	8.4463E-06	1
C (Humidity)	6.2869E-05	2
A × B	0.000704776	2
A ×C	0.003677721	4
B × C	0.000976524	2
A × B × C	0.000731863	4
Error	0.025407367	72

Table 8. Continue

MSR	Approximate F	Approximate p-value
0.014899311	42.22202164	.000*
8.4463E-06	0.02393534	0.877
3.14345E-05	0.089079897	0.915
0.000352388	0.9986057	0.373
0.00091943	2.605503303	0.043*
0.000488262	1.383647951	0.257
0.000182966	0.518492885	0.72
0.00035288		

At the 5% level of significance, we declare that only the main effect of factor A (stress) and the interaction effect of factor A (stress) × C (Humidity) are significant.

At the 1% level of significance, we declare that only the main effect of factor A (stress) is significant.

It may be reasonable at this stage to look at the additive model that we previously discussed in section 7, namely

$$\theta_{ijk}^{-1} = \mu + \alpha_i + \beta_j + \gamma_k \,,$$

$$\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{k=1}^{c} \gamma_k = 0$$

For this model, we introduce the following sum of reciprocals

$$SR(A) = bcn \sum_i R_i(A)$$
$$= bcn \sum_i \sum_{r=1}^{R} (-1)^{r-1} \left(v_{i..}^{\,(r-1)} - 2^{r-1} \frac{1}{\bar{y}_{...}} \right) \quad (120)$$

$$SR(B) = acn \sum_j R_j(B)$$
$$= acn \sum_j \sum_{r=1}^{R} (-1)^{r-1} \left(v_{.j.}^{\,(r-1)} - 2^{r-1} \frac{1}{\bar{y}_{...}} \right) \quad (121)$$

$$SR(C) = abn \sum_k R_k(C)$$
$$= abn \sum_k \sum_{r=1}^{R} (-1)^{r-1} \left(v_{..k.}^{\,(r-1)} - 2^{r-1} \frac{1}{\bar{y}_{...}} \right) \quad (122)$$

The Error is as usual given as

$$SR(E) = \sum_{i,j,k,l} R_{ijkl}(E) \qquad (123)$$

Hence, we propose the table 9, that shows the Analysis of Reciprocals (ANOR) Table for the above additive model.

Table 9. ANOR table for the additive Model

Source	SR	$d.f.$	MSR
A	$SR(A)$	a -1	$MSR(A)$
B	$SR(B)$	b -1	$MSR(B)$
C	$SR(C)$	b -1	$MSR(C)$
Reminder	*By subtraction*		$MSR(R)$
Error	$SR(E)$	$abc(n$-1)	$MSR(E)$

The Reminder and its degree of freedom is obtain by subtraction. given by the difference between the total sum of reciprocals and all other sums of reciprocals.

Hence, for the data at hand, the ANOR table for the additive model is given by Table 10 below.

Table 10. The ANOR for the Fatigue Life data of 2024-T351 Aluminum Alloy assuming Inverse Gaussian and additive model

Source	Sum of Reciprocal	Degrees of Freedom
A (Stress)	0.0304971	2
B (Coating)	3.83193E-07	1
C (Humidity)	6.94481E-05	2
Reminder	0.00539389	12
Error	0.025407367	72
Total	0.092324745	89

Table 10. Continue.

MSR	Approximate F	Approximate p-value
0.01524855	43.212	.000*
3.83193E-07	0.001	0.974
3.47241E-05	0.098	0.906
0.000449491	1.274	0.253
0.00035288		

At any reasonable level of significance, we might declare that only the main effect of factor A (stress) is significant.

According to the above additive model, the estimated cell means are shown in Table 11 below.

Table 11. The Estimated Cell Means $\hat{\theta}_{ijk}$ for the Fatigue Life data of 2024-T351 Aluminum Alloy assuming Inverse Gaussian and additive model

		$j=1$	$j=2$	$j=3$
$i=1$	$k=1$	5372.476	2127.175	712.329
	$k=2$	5668.284	2172.056	717.292
	$k=3$	5339.023	2121.911	711.737
$i=2$	$k=1$	5387.076	2129.460	712.585
	$k=2$	5684.538	2174.438	717.552
	$k=3$	5353.441	2124.185	711.993

11. Results

The main objectives of this article are focus on the statistical analysis of factorial experiments assuming the Inverse Gaussian model. Heading to these objectives, we provide an explicit forms for the MLE's in each of two factor and three factor experiments. The generalization to k-factor factorial experiments is straightforward. This achievement allows the researchers to perform many types of statistical inference, for example construction approximate confidence intervals and testing Hypotheses via the likelihood ratio test. Another accomplished is given by the construction of the "Analysis of Reciprocal Table", analogue to the "Analysis of Variance Table" in the normal Theory. Several applications are provided to illustrate our procedures.

12. Discussions

This work covered and illuminates a great gap in the statistical analysis of factorial experiments under an inverse Gaussian model. Since the work of Fries and Bhattacharyya (1983), there is no known new work accomplished in this area of research.

Many investigations still needed in this area. For example: a simulation study is required to look deeply inside the properties of the ML estimators and their behavior with the variation of the main and interaction effects. Another subject that needs deep study is the ANOR tables and the performance of the approximate F tests and comparing it with the ANOVA tables. Above all, there is the problem of estimation for the full model containing all main effects and all types of interactions caused by confounding effects..

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