On The (Co)Homology With Inner Symmetry Of Schemes

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Abstract. The cyclic (co)homology group is the first type of what called (co)homology with inner symmetry. The dihedral, reflexive, symmetry, bisymmetry and octahedral (co)homology are other types of (co)homology with inner symmetry [4].The cyclic (co)homology initial and studied by Connes [2] and others. The cyclic homology for schemes are studied in [11]. In this article we concerned with the dihedral and reflexive (co)homology of schemes with an involution over a commutative ring *k* by sheafifying the (co)dihedral and reflexive complex. We give some properties of theses (co)homologies.

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1. Introduction

Firstly, we recall some definitions and basic properties of schemes see[1]. (For detail see [5],[10]).

Definition 1.1. Let R be a ring with unit. We shall denote by *spec(*R*),* the set of all prime ideals over *R*. *F*or any ideal *P* in R, we denote by $V(P)$ the set of all prime ideal in *spec(*R*)* containing P.these sets define a topology, which called a Zariski topology.

Definition 1.2. For given topological space *X*, a ringed space is a pair (X,θ_X) where θ_x is the structure of sheaf on X. The space (X,θ_X) is called locally ringed space if the stalks $\theta_{x|X}$ are locally rings for any $x \in X$.

Definition 1.3. A locally ringed space (X,θ_X) is called affine space if $(X,\theta_X) \cong (spec(R),\theta_{spec(R)})$ and a scheme if it has an open covering $X = \bigcup_{i\in I} U_i$ such that $(X,\theta_{X|U_i})$ is an affine scheme for $i \in I$.

A pair $(spec(R),\theta_{spec(R)})$ is locally ringed space.

Definition 1.4. For any scheme *X*, the structure of sheaf θ_X is defined to be the ring of all regular functions denoted by:

$$\theta_X(U) = \{f \mid f : U \to U_{p\in U} A_p,\ U \in X\},$$

where A_p is the local ring on *X* at *p*.

Definition 1.5. An involution * on the sheaf θ_x $(* : \theta_X \to \theta_X)$ is defined as an automorphism of order two by considering the inverse regular function, that is, $(*)^2 = id$, $f^* = f^{-1}$ and $(fg)^* = g^{-1}f^{-1}$. The scheme X with this property on sheaf θ_X is called a scheme with an involution.

Secondly, we recall the concept of dihedral category, object and modules that will be useful in the sequel (see [3],[6]).

Definition 1.6. The dihedral category ΔD is a category with objects order set $[n],\ n \in N$ and the following family of morphisms:

$$\delta_n^i : [n-1] \to [n],\ \sigma_n^j : [n+1] \to [n],$$
$$\tau_n : [n] \to [n],\ \rho_n : [n] \to [n],$$

such that the following framework are hold:

$$\left.\begin{array}{lll} \delta_{n+1}^j\delta_n^i = \delta_{n+1}^i\delta_n^{j-1}, & \text{if} & i \prec j \\ \sigma_n^j\sigma_{n+1}^i = \sigma_n^i\sigma_{n+1}^{j+1}, & \text{if} & i \le j \\ \sigma_n^j\delta_{n+1}^i = \begin{cases} \delta_{n-1}^i\sigma_{n-2}^{j-1}, & if\ \ i<j, \\ Id_{[n]}, & if\ \ i=j,j+1, \\ \delta_{n+1}^{i-1}\sigma_n^j, & if\ \ i>j+1, \end{cases} & & \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{ll} \tau_n\delta_n^i = \delta_n^{i-1}\tau_{n-1}, & 1 \le i \le n \\ \tau_n\sigma_n^j = \sigma_n^{j-1}\tau_{n+1}, & 1 \le j \le n \\ \tau_n^{n+1} = Id_{[n]} & \\ \rho_n\delta_n^i = \delta_n^{i-1}\rho_{n-1}, & 0 \le i \le n \\ \rho_n\sigma_n^j = \sigma_n^{j-1}\rho_{n+1}, & 0 \le j \le n \\ \rho_n^2 = Id_{[n]} & \\ \tau_n\rho_n = \rho_n\tau_n^{-1} & \end{array}\right\} \quad (2)$$

Definition 1.6. The category generated by the family of morphisms d_n^i, S_n^j and r_n is called a reflexive category and is denoted by *ΔR*.

Definition 1.7. Let ζ be an arbitrary category. A dihedral object of the category ζ is a functor $F : \Delta D^{op} \to \zeta$, such that $F(n) = X_n$, $F(\delta_n^i) = d_n^i$,

$F(\sigma_n^j)=S_n^j$, $F(\tau_n)=t_n$ and $F(\rho_n)=r_n$ (ΔD^{op} is the inverse of ΔD). If ζ is a category of modules, then the dihedral object is called a dihedral module.

Note that the morphisms $\{d_n^i, S_n^j, t_n, r_n\}$ satisfy the relations (1),(2).

Definition 1.8 Let $\wp$ be an arbitrary category. A reflexive object of the category $\wp$ is a functor $F:\Delta R^{op}\to\wp$, such that $F(n)=X_n$, $F(\delta_n^i)=d_n^i$, $F(\sigma_n^j)=S_n^j$, and $F(\rho_n)=r_n$ (ΔR^{op} is the inverse of ΔR). If $\wp$ is a category of modules, then the reflexive object is called a reflexive module.

Definition 1.9. Let $M=\{M_n\}$ be a dihedral (reflexive) k-module. The dihedral (reflexive) homology groups of M is given by:

$$HD_n(M)=Tor_n^{k[\Delta D^{op}]}(K^D,M),\quad n\geq 0$$

$$(HR_n(M)=Tor_n^{k[\Delta R^{op}]}(K^D,M),\quad n\geq 0),$$

where K^D (K^R) is trivial dihedral (reflexive) k-module.

***2.* Reflexive and dihedral homology of schemes**

In this section we define the dihedral homology of schemes and study some of its properties.

Definition 2.1. [8] A sheaf of θ_X-modules is a sheaf $\Im$ on X, such that for each open set $u\in X$, the group $\Im(u)$ is a $\theta_X(u)-\text{mod}\,ule$ and for each inclusion of open sets $v\subseteq u$ the restriction homomorphism $\Im(u)\to\Im(v)$ is compatible with the module structures via the ring. The set of all sheaves of θ_x-module defines a category, called a category of sheaves of modules and denoted by $Mod(\theta_X)$.

Definition 2.2. The dihedral module of sheaves is a functor: $F:\Delta D^{op}\to Mod(\theta_X)$, such that:

$$F([n])=\theta_X^{\otimes(n+1)}$$

$$F(\delta_n^i)=d_n^i:\theta_X^{\otimes(n+1)}\to\theta_X^{\otimes(n)}$$

$$d^i{}_n(f_0\otimes f_1\otimes\cdots\otimes f_n)=$$
$$(f_0\otimes f_1\otimes\cdots\otimes f_if_{i+1}\otimes\cdots\otimes f_n),\ 0\leq i\leq 1$$

$$F(\sigma_n^j)=S_n^j:\theta_X^{\otimes(n-1)}\to\theta_X^{\otimes(n)}$$

$$S^i{}_n(f_0\otimes f_1\otimes\cdots\otimes f_n)=$$
$$(f_0\otimes f_1\otimes\cdots\otimes f_i\otimes id\otimes f_{i+1}\otimes\ldots\otimes f_n),\ 0\leq i\leq 1$$

$$F(\tau_n)=t_n:\theta_X^{\otimes(n)}\to\theta_X^{\otimes(n)}$$

$$t_n(f_0\otimes f_1\otimes\cdots\otimes f_n)=(f_n\otimes f_0\otimes\cdots\otimes f_{n-1}),$$

$$F(\rho_n)=r_n:\theta_X^{\otimes(n)}\to\theta_X^{\otimes(n)},$$

$$r_n(f_0\otimes f_1\otimes\cdots\otimes f_n)=\alpha(f_0^{-1}\otimes f_n^{-1}\otimes\cdots\otimes f_1^{-1}),\ \alpha=\pm1,$$

with the following:

$$b_n=\sum_{i=0}^{n}(-1)^i d_i,\quad b_n`=\sum_{i=0}^{n-1}(-1)^i d_i,\quad T_n=(-1)^n t^n,$$

$$N=1+t+\ldots+t^{n-1},\quad R_n=(-1)^{\frac{n(n+1)}{2}}r_n.$$

We can construct the tricomplex of sheaves $({}^{\alpha}CD^n(\theta_X,\delta))$, $\alpha=\pm1$, (see [9]) where $\delta=\delta_1+\delta_2+\delta_3$, and:

$$\delta_1=\begin{cases} b_n \\ -b_n` \end{cases} :\theta_X^{\otimes(n)}\to\theta_X^{\otimes(n-1)}$$

$$\delta_2=\begin{cases} 1-T_n \\ N \end{cases} :\theta_X^{\otimes(n)}\to\theta_X^{\otimes(n)}$$

$$\delta_3=\begin{cases} 1\text{-R}_n \\ \text{-1-R}_nT_n \\ 1+R_nT_n \\ 1-R_n \end{cases} :\theta_X^{\otimes(n)}\to\theta_X^{\otimes(n)}$$

Clearly $(\delta_i)^2=0,\ i=1,2,3.$

In order to define the dihedral cohomology of schemes, we make a use of the hyperhomology of define in [6].

Proposition 2.3: [11] Let X be a schemes over k. Then each HH_n is a quasi-coherent sheaf on X. Moreover, on each affine open $\cup=spec(A)$ of X we have natural isomorphism $HH_n\xrightarrow{\cong}H^0(\cup,HH_n)$. By means there is a cyclic homology theory HC_n of schemes over a commutative ring k, extending the usual cyclic homology HC_n of k-algebras.

Theorem 2.4:[5] For any affine schemes $X=spec(A)$ over k we have:

$$HC_n(A)\cong HC_n(X).$$

Definition 2.5. The dihedral homology of involuative scheme X over a commutative ring k is the hyperhomology of the total complex of the tricomplex of sheaves $({}_{\alpha}CD^n(\theta_X,\delta_i))$:

$${}^{\alpha}HD^*(X)=H^*(Tot({}^{\alpha}CD^n(\theta_X),X)),$$

Where

$$Tot({}^{\alpha}CD^n(\theta_X))=\frac{\theta_X^n}{Im(1-R)+Im(1-T)},$$

$$n=0,1,2,\ldots,\quad \alpha=\pm1.$$

Note that:

The reflexive homology of involuative scheme X over a commutative ring k is the hyperhomology of the total complex of the tricomplex of sheaves ${}_{\alpha}CR^n(\theta_X,\delta_i)$:

$${}^{\alpha}HR^*(X)=H^*(Tot({}^{\alpha}CR^n(\theta_x),X)),$$

Where

$$Tot({}^{\alpha}CR^{n}(\theta_X)) = \frac{\theta_X^n}{Im(1-R)}, \quad n = 0,1,2,..., \ \alpha = \pm 1.$$

In the following part we study some important properties of the dihedral homology of schemes.

3.The Mayer-Vietoris sequence for dihedral homology of schemes

In this part we get the Mayer-Vietoris sequence for dihedral homology of schemes. Firstly, we suppose the following lemma.

Lemma 3.1. The following sequence is exact:

$$0 \to {}_{\alpha}CD^{n}(\theta_X) \xrightarrow{J} {}_{\alpha}CD^{n}(\theta_{X_1}) \oplus {}_{\alpha}CD^{n}(\theta_{X_2}) \xrightarrow{I}$$

$${}_{\alpha}CD^{n}(\theta_{X_1} \cap \theta_{X_2}) \to 0, \qquad \alpha = \pm 1$$

Where $\theta_X = (\theta_{X_1} \cap \theta_{X_2})$, ${}_{\alpha}CD^{n}$ is the dihedral complex, and $J = J_1 - J_2$, $I = I_1 \oplus I_2$, are defined by:

$$I_1 : X_1 \cup X_2 \to X_1, \quad I_2 : X_1 \cup X_2 \to X_2,$$

$$J_1 : X_1 \to X_1 \cap X_2, \quad J_2 : X_2 \to X_1 \cap X_2.$$

Proof. Clearly, J is an epimorphism and I is a monomorphism and $J \circ I = 0$. Let $(\theta_{X_1} \cap \theta_{X_2})(u) \in CD^{n}(\theta_X)(u)$, then

$$(JoI)(\theta_{X_1 \cup X_2})(u) = J(\theta_{X_1}(u), \theta_{X_2}(u)) = 0$$

where

$$((\theta_{X_1}(u), \theta_{X_2}(u)) \in [CD^{n}(\theta_{X_1}) \oplus CD^{n}(\theta_{X_2})]$$

and $(\theta_{X_1} \cap \theta_{X_2})(u) \in {}_{\alpha}CD^{n}(\theta_{X_1} \cap \theta_{X_2})$.

Theorem 3.2. If $X = X_1 \cup X_2$ where X_1 and X_2 are open subsets of scheme X and the diagram:

$$\begin{array}{ccc} X & \to & X_1 \\ \downarrow & & \downarrow \\ X_2 & \to & X_1 \cap X_2 \end{array}$$

is commutative, then there exist the following long exact sequence:

$$... \to {}^{\alpha}HD_n(X_1 \cup X_2) \xrightarrow{J^*} HD_n(X_1) \oplus HD_n(X_2)$$

$$\xrightarrow{I^*} HD_n(X_1 \cap X_2) \xrightarrow{E} HD_{n-1}(X_1 \cup X_2) \to ....$$

where $I^* = (I_1^*, I_2^*)$, $J^* = (J_1^*, J_2^*)$, E is a connecting homomorphism.

Proof. The exact sequences

$$0 \to {}_{\alpha}CD^{n}(\theta_X) \xrightarrow{J} {}_{\alpha}CD^{n}(\theta_{X_1}) \oplus {}_{\alpha}CD^{n}(\theta_{X_2})$$

$$\xrightarrow{I} {}_{\alpha}CD^{n}(\theta_{X_1} \cap \theta_{X_2}) \to 0, \qquad \alpha = \pm 1$$

induce the following long exact sequence of dihedral groups:

$$\cdots \longrightarrow {}_{\alpha}HD_n(X_1 \cup X_2) \xrightarrow{J^*} {}_{\alpha}HD_n(X_1) \oplus {}_{\alpha}HD_n(X_2)$$

$$\xrightarrow{I^*} {}_{\alpha}HD_n(X_1 \cap X_2) \xrightarrow{E} {}_{\alpha}HD_{n-1}(X_1 \cup X_2) \to \cdots.$$

Since $E \circ J^* = 0$.

4.The relation between cyclic and dihedral homology of schemes

In this part we study the relations between the cyclic and dihedral homology to all schemes over ring with identity and involution. We shall prove the following assertion.

Theorem 4.1. Let $\Im$ be a sheaf of θ_X-modules, and let X be a scheme over unital ring k with an involution. Then the relation between the cyclic and dihedral homology groups is given by:

$$... \to {}_{-\alpha}HD_n(X,\Im) \xrightarrow{i^*} HC_n(X,\Im) \xrightarrow{j^*}$$

$${}_{\alpha}HD_n(X,\Im) \longrightarrow {}_{\alpha}HD_{n-1}(X,\Im) \longrightarrow ...$$

where j^* *is* a connecting homomorphism.

Proof. For a scheme X, let $\zeta(X,\Im)$ be the total complex of Connes double complex [7]. We embed the complex $\zeta(X,\Im)$ in the tricomplex $D(X,\Im)$ (see [6]). Passing to the total complexes associated with $\zeta(X,\Im)$ and $D(X,\Im)$, we get the following short exact sequence

$$0 \to Tot\,\zeta(X,\Im) \to Tot\,{}^{\alpha}D(X,\Im) \to Tot\,{}^{-\alpha}D(X,\Im)[-4] \to 0$$

This sequence induces the following long exact sequence which relates the cyclic and dihedral homology groups:

$$... \to {}_{-\alpha}HD_n(X,\Im) \xrightarrow{i^*} HC_n(X,\Im) \xrightarrow{j^*}$$

$${}_{\alpha}HD_n(X,\Im) \longrightarrow {}_{\alpha}HD_{n-1}(X,\Im) \longrightarrow ...$$

when 2 is invertible in the ground ring k we get the following exact sequence

$$0 \to {}_{-\alpha}HD_n(X,\Im) \to HC_n(X,\Im) \longrightarrow {}_{\alpha}HD_n(X,\Im) \to 0$$

Remark:

From the last sequence we get the following fact:

$$HC_n(X,\Im) \cong {}_{-\alpha}HD_n(X,\Im) \oplus {}_{\alpha}HD_n(X,\Im)$$

5. Cohomology groups of schemes

In this part we are concerned with the dihedral cohomology groups. It's necessary to translate the definitions and results of a pervious discussion in the cohomological framework because there is an interesting pairing between homology and cohomology groups. It's well known, in cyclic cohomology case, that if A is a unital associated k-algebra and $A^* = Hom(A,k)$, then its cochain complex is $C^n(A) = Hom(A^{\otimes(n+1)},k)$. The dualizing of the Connes bicomplex $CC_*(A)$ gives a bicomplex of cochains $CC^{**}(A)$ and its homology gives the cyclic cohomology group. The dihedral cohomology group can be defined in the same manner. A chive this, we replace the category ΔD^{op} by ΔD in the definitions 1.6, 1.7 and 2.2, then we get the dihedral cohomology group $HD^n(M) = Ext^n_{k[\Delta D]}(M, K^D)$, $n \geq 0$, where K^D is trivial dihedral k-module. Also the dihedral cohomology of schemes X over a commutative ring k is the hypercohomology of the total complex of the tricomplex of sheaves $({}^{\alpha}CD^n(\theta_X, \delta_i))$: ${}^{\alpha}HD^*(X) = H^*(Tot({}^{\alpha}CD^n(\theta_x), X))$, where

$$Tot({}^{\alpha}CD^n(\theta_X)) = \frac{\theta_X^n}{Im(1-R)+Im(1-T)},$$

$$n = 0,1,2,..., \quad \alpha = \pm 1.$$

The theorem 3.1 of the Mayer-Vietoris sequence for dihedral homology and theorem 4.1 of the relation between cyclic and dihedral homology of schemes can be translated to cohomology case.

Similar arguments as those used in the proof of theorem 3.1 give the following theorem.

Theorem 5.1. If $X = X_1 \cup X_2$ where X_1 and X_2 are open subsets of scheme X and the diagram

$$\begin{array}{ccc} X & \leftarrow & X_1 \\ \uparrow & & \uparrow \\ X_2 & \leftarrow & X_1 \cap X_2 \end{array}$$

is commutative, then there exist the following long exact sequence:

$$\ldots \to {}^{\alpha}HD^n(X_1 \cup X_2) \xrightarrow{I^*} HD^n(X_1) \oplus HD^n(X_2)$$
$$\xrightarrow{J^*} HD^n(X_1 \cap X_2) \xrightarrow{E} HD^{n+1}(X_1 \cup X_2) \to \ldots.$$

where $I^* = (I_1^*, I_2^*)$, $J^* = (J_1^*, J_2^*)$, E is a connecting homomorphism.

Similar arguments as those used in the proof of theorem 4.1 give the following theorem 5.2.

Theorem 5.2. Let $\Im$ be a sheaf of θ_X-modules, and let X be a scheme over unital ring k with involution. Then the relation between the cyclic and dihedral cohomology groups is given by:

$$\ldots \to {}^{-\alpha}HD^n(X,\Im) \xrightarrow{i^*} HC^n(X,\Im) \xrightarrow{j^*}$$
$${}^{\alpha}HD^n(X,\Im) \to {}^{\alpha}HD^{n+1}(X,\Im) \to \ldots$$

where j^* is a connecting homomorphism.

Remark.

1-when 2 is invertible in the ground ring k we get the following exact sequence

$$0 \to {}^{-\alpha}HD^n(X,\Im) \to HC^n(X,\Im) \longrightarrow {}^{\alpha}HD^n(X,\Im) \to 0.$$

2- $HC^n(X,\Im) \cong {}_{-\alpha}HD^n(X,\Im) \oplus {}_{\alpha}HD^n(X,\Im)$.

In what follows we are concerned with the cohomology theory of involutive schemes.

6.The Connes-Tsygan long exact sequence for dihedral cohomology for schemes

In this part we get the Connes-Tsygan long exact sequence for dihedral cohomology for schemes. The main result is theorem 6.4.

Consider the following Helmmski exact sequence [9]

$$0 \to CC(X) \xrightarrow{I} C(X) \xrightarrow{N} \tilde{C}(X) \xrightarrow{M} CC(X) \to 0, \quad (3)$$

Where $C(X), CC(X)$ is defined above and $\tilde{C}(X) = (C^n(X), b'^n)$,

$b^n f(a_o, a_1, ..., a_{n-1}) =$

$b'^n f(a_o, a_1, ..., a_{n-1}) + (-1)^n f(a_n a_o, a_1, ..., a_{n-1})$,

$N = 1 - t_n$, $M = 1 + t_n + t_n^2 + ... + t_n^n$, where $t_n : CC(X) \to CC(X)$ is a cyclic operator and I is an inclusion.

To get the Connes-Tsygan long exact sequence for dihedral cohomology of involutive schemes we split the sequence (3) into two exact sequences. Firstly we need the following lemma.

Lemma 6.1. The sequence (3) is exact.

Proof. We shall only show that the sequence (3) is exact on $C(X)$, that is $Ker\ M = Im N$. Clearly $Ker M \supset Im N$, Since

$$MN = M(1-t_n) = (1 + t_n + t_n^2 + ... + t_n^n)(1-t_n) = 0.$$

It remains to show that if $x \in Ker\ M$, then there exist $y \in C(X)$, such that $My = x$. Suppose

$y_n = (1+t_n+t_n^2+...+t_n^n)\, x$,

$t_n y_n = (t_n + t_n^2 + ... + t_n^n)x$, then

$y_n - t_n y_n = (1-t_n^n)x$. Let $y = y_1 + y_2 + ... + y_n$ then

$y + t_n y = (n+1)x = (1-t_n)y$.

Hence

$y = 1/(n+1)\{n + +(n-1)t_n + (n-2)t_n^2 + ... + t_n^{n-1}\}\} \in C(A)$. So $Ker\ M \subset Im\ N$. The lemma is proved.

If the group $Z/2Z$ acts on the sequence (3) as follows, on $CC(X)$, by the reflexive operator $r_n : {}^{\alpha}CD(X) \to {}^{\alpha}CD(X)$ and by $r_n t_n$ on $C(X)$, we get the following assertion:

Proposition 6.2. The sequence (3) induces the following commutative diagram:

$$\begin{array}{ccc} 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ CC(X) & \xrightarrow{\phi} {}^{+}CD(X) \oplus & {}^{-}CD(X) \\ \downarrow & \downarrow & \downarrow \\ C(X) & \xrightarrow{\phi} {}^{+}CR(X) \oplus & {}^{-}CR(X) \\ \downarrow & \downarrow & \downarrow \\ \tilde{C}(X) & \xrightarrow{\phi'} {}^{-}C\tilde{R}(X) \oplus & {}^{+}C\tilde{R}(X) \\ \downarrow & \downarrow & \downarrow \\ CC(X) & \xrightarrow{\phi''} {}^{-}CD(X) \oplus & {}^{+}CD(X) \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \qquad (4)$$

where $\phi(f) = \frac{1}{2}((1+r_n)(f), (1-r_n)(f)), \quad f \in CC(X),$

$$\phi`(f) = \frac{1}{2}((1-r_n t_n)(f), (1+r_n t_n)(f)), \quad f \in \tilde{C}(X),$$

$$\phi``(f) = \frac{1}{2}((1-r_n)(f), (1+r_n)(f)), \quad f \in CC(X),$$

Proposition 6.3. The following sequences are exact:

$$0 \to {}^{+}CD\ (X) \xrightarrow{I} {}^{+}CR(X) \xrightarrow{N} {}^{-}C\tilde{R}(X) \xrightarrow{M} {}^{-}CD(X) \to 0$$
$$0 \to {}^{-}CD\ (X) \xrightarrow{I} {}^{-}CR(X) \xrightarrow{N} {}^{+}C\tilde{R}(X) \xrightarrow{M} {}^{+}CD(X) \to 0 \quad (5)$$

Note that, in (5), if we take an element $x \in Ker\ M$, invariant with respect to $r_n t_n$, that is $r_n t_n(x) = \alpha x$, then there exist an element $y \in CR(X)$, such that $r_n y = y$.

Using the short exact sequence (5) we can consider the following sequences:

$$0 \to {}^{\alpha}CD\ (X) \xrightarrow{I} {}^{\alpha}CR(X) \xrightarrow{N} {}^{-\alpha}CSR(X) \to 0$$

$$0 \to {}^{-\alpha}CSR(X) \xrightarrow{N} {}^{-\alpha}C\tilde{R}(X) \xrightarrow{M} {}^{\alpha}CD(X) \to 0$$

Where $\alpha = \pm 1$, ${}^{\alpha}CSR(A) = Ker\ M$. They induce the following two long exact sequences group cohomologies:

$$\begin{aligned}
&{}^{\alpha}HD^{o}(X) \to {}^{\alpha}HR^{o}(X) \to {}^{-\alpha}HSR^{o}(X) \xrightarrow{\xi_{\alpha}^{o}} {}^{\alpha}HD^{1}(X) \to \\
&{}^{-\alpha}HR^{1}(X) \to \dots \to {}^{\alpha}HD^{n}(X) \to {}^{\alpha}HR^{n}(X) \to {}^{-\alpha}HSR^{n}(X) \\
&\xrightarrow{\xi_{\alpha}^{n}} {}^{\alpha}HD^{n+1}(X) \to {}^{\alpha}HR^{n+1}(X) \to \dots \qquad (6)
\end{aligned}$$

$$\begin{aligned}
&{}^{\alpha}HSR^{o}(X) \to {}^{\alpha}H\tilde{R}^{o}(X) \to {}^{\alpha}HD^{o}(X) \xrightarrow{\eta_{\alpha}^{o}} {}^{\alpha}HSR^{1}(X) \\
&\to \dots \to {}^{\alpha}H\tilde{R}^{n-1}(X) \to {}^{\alpha}HD^{n-1}(X) \xrightarrow{\eta_{\alpha}^{n-1}} \\
&{}^{\alpha}HSR^{n}(X) \to {}^{\alpha}H\tilde{R}^{n}(X) \to \dots\dots \qquad (7)
\end{aligned}$$

Suppose that, the connection map η_{α}^{n} $(n \geq 0)$ in (7) is a topological vector space isomorphism, then by using this isomorphism in (6) we get the following exact sequence:

$$\begin{aligned}
&\dots \to {}^{\alpha}HD^{n}(X) \to {}^{\alpha}HR^{n}(A) \xrightarrow{\phi_{\alpha}^{n}} \\
&{}^{\alpha}HD^{n-1}(X) \xrightarrow{\beta_{\alpha}^{n}} {}^{\alpha}HD^{n+1}(X) \to \qquad (8) \\
&{}^{-\alpha}HR^{n+1}(X) \xrightarrow{\phi_{\alpha}^{n+1}} {}^{\alpha}HD^{n}(X) \to \dots,
\end{aligned}$$

Where

$\phi_{\alpha}^{n} = (\xi_{\alpha}^{n-1})^{-1} \circ H^{n}(N),\ \beta_{\alpha}^{n} = \xi_{\alpha}^{n} \circ \eta_{\alpha}^{n-1}.$

The sequence (8) is the Connes–Tsygan long exact sequence for dihedral cohomology of schemes.

Now considering the Connes–Tsygan exact sequence from and the relation, that, relates the cyclic and dihedral cohomologies of schemes, we get the following infinite commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\dots \to & & {}^{\varepsilon}HD^{n}(X) & \to & {}^{\varepsilon}HR^{n}(X) & \to & {}^{-\varepsilon}HD^{n-1}(X) & \to & {}^{\varepsilon}HD^{n+1}(X) \to \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\dots \to & & HC^{n}(X) & \to & {}^{\varepsilon}H^{n}(X) & \to & HC^{n-1}(X) & \to & HC^{n+1}(X) \to \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\dots \to & & {}^{-\varepsilon}HD^{n}(X) & \to & {}^{-\varepsilon}HR^{n}(X) & \to & {}^{\varepsilon}HD^{n-1}(X) & \to & {}^{-\varepsilon}HD^{n+1}(X) \to \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array} \qquad (9)$$

Theorem 6.4. Let H be a Hilbert schemes space and $A = L(H)$ is algebra of all bounded operators on H. Then ${}^{\alpha}HD^{n}(A) = 0,\quad n\ ?\ ,\ \varepsilon = \pm 1.$

(for more information about Hilbert schemes, see [5])

Proof. Clearly the algebra A is C*-algebra and has no bounded traces. So $H^{0}(A) = HC^{n}(A) = 0$. Following [5] $HC^{n}(A) = 0, n \geq 0$. Considering these facts in (9) we have:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\dots \to & & {}^{\alpha}HD^{0}(A) & \to & {}^{\alpha}HR^{0}(A) & \to \dots \to & {}^{-\alpha}HD^{n-1}(A) & \to & {}^{\alpha}HD^{n+1}(A) \to \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\dots \to & & 0 & \to & 0 & \to \dots \to & 0 & \to & 0 \to \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\dots \to & & {}^{-\alpha}HD^{0}(A) & \to & {}^{-\alpha}HR^{0}(A) & \to \dots \to & {}^{\alpha}HD^{n-1}(A) & \to & {}^{-\alpha}HD^{n+1}(A) \to \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array} \qquad .\ (10)$$

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