Computer analysis and visualisation of non-linear dynamic system (NDS) in plane

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Abstract. The research of non-linear dynamic system (non-linear DS,NDS) behavior is a widespread task in many fields of human activity. For effective management of NDS it is necessary to know its free behavior (without influence of external effects), conditions which allows NDS to reach stable state, necessary parameter values for the system to provide the required functional characteristics. A phase-plane portrait describes a behavior of DS in the most obvious way. The modern development of computer engineering allows usage of computer heuristic devices as addition to classic analytical methods to create the phase-plane portrait.

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Introduction

Complex processes in various fields of science, technology and natural phenomena develop dynamically. Majority of processes in systems are often non-linear. Analysis of the behavior of these systems necessitates development special techniques and algorithms for non-linear dynamic systems (non-linear DS, NDS) research.

Modern control theory and Synergetics provide the ability to describe systems behavior by abstraction of the their nature behind generalized language. This method establishes a kind of semblance of the phenomena that explored with tools of different sciences, making phenomena reducible to a common model [1,2]. Successful identification of models' unity enables Synergetics to share achievements of one scientific domain among completely different domains [3,4].

Given that, actual place of processes in the world becomes irrelevant, they only need to be represented by the same model [5]. V. I. Vernadsky wrote about unusual similarity in laws of development of various phenomena of life sciences [6] and physics (population dynamics, epidemic, etc. [7]).

In many fields of science and technology there are objects can be considered NDS. Mathematical model of DS are differential equations, presented in the form of Cauchy

$$\frac{dx}{dt}=F(x) \qquad (1)$$

where x — vector of system state variables, t — time.

History of NDS research

Problems of NDS research are closely linked with several areas of science and technology. NDS research operates with theory of differential equations and vector fields.

Since the XVII century many scientists has been involved in the study of NDS, among them: I. Newton, G. W. Leibniz, J. Bernoulli, L. Euler, J.-L.Lagrange, B. Taylor, J. D'Alembert, J. Bull, A.-M. Ampere, C. Jacobi, G. Darboux, H. Poincare, Cauchy and at a later time — .E. Lorenz, H. Haken, V. I. Arnold, S. P. Kapitza and other researchers.

U. S. Il'yashenko divides history of the study of DS on three periods:

Newton Period: There is a differential equation. Solve it.

Poincare Period: There is a differential equation. Describe the properties of its solutions without solving the equation, only using the properties of the right side.

Andronov period: There no any differential equation. Describe the properties of its solutions [8] (ie, describe the general solutions' properties of any differential equations in plane).

During Poincare period problem of solving a system of differential equations and finding explicit solution functions has been replaced by a proposition to conduct a qualitative analysis of the behavior of the system of equations in the state space, without solving the equations, but using properties of their right side. One of the basic tools in the theory of qualitative analysis was the study of special solutions of differential equations, in particular — stationary or singular points [9,10].

The next step (Andronov period) was the study of the most important properties of differential equations on the plane and the limiting behavior of all solutions [11].

During Andronov period the approach was changed from the solution of a particular problem to the exploration a general form of abstract dynamic system. These systems require using particular cases.

Breakthrough in computer technologies permitted a new approach to the research wider range of analytically unexplorable NDS. Such systems require the use of numerical methods [12, 13].

In recent scientific publications on the subject following methods can be found: qualitative analysis techniques, possible signs and types of separatrixes, possible types of singular points neighborhood, numerical methods for solving differential equations. Those methods were used to create interactive methodology and software for the explore dynamical systems.

S. M. Ulam, who got much experience during work with first electronic computers, praised the continuous interactive cooperation between the machine and its operator, noting the synergy arising from analysis carried out by man on displayed information [14].

Today the most significant software tools for NDS research are software tools for analysis and forecasting based on symbolic computation [15-17] and/or simulations [18-19], but modern tools have many shortcomings, which gives the opportunity to improvement in this field.

The mathematical approaches of description of the NDS dynamics.

It is well known the relations between the parameters of dynamic system can be described by means of differential equations in Cauchy form (1), and the personal behavior of NDS is visually represented in the state space in the form of orbits (we will identify it as L). It means a set of sequential locations of x point in the phase space, where x point represents the system.

$$L = \{x(t)\}, t \in \square \qquad (2)$$

where

$$x(t) = x_0 + \left(\int_{t_0}^{t} F(x(\tau))d\tau \right)_{x(t_0)=x_0} \qquad (3)$$

t_0 — some initial time moment, x_0 — the state of system in t_0 moment.

The orbit L of system must be divided by two parts [9]. A positive half-orbit L^+ describes the development of system state at $t \to +\infty$, a negative half-orbit L^- describes the prehistory of the system at $t \to -\infty$.

The topologically similar orbits make the singular areas (or basins) on the phase plane that describes the state space (fig. 1).

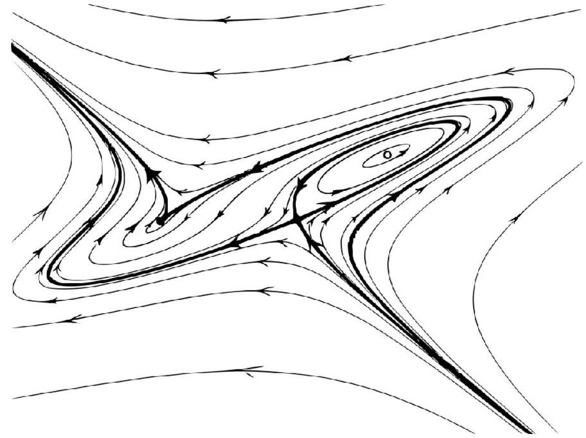

Fig.1. The four basins with different dynamics.

In accordance with [9], the border of any basin (separatrix) consists of total orbit unstable orbits. Every can be:

2. An equilibrium point (a singular point)
3. a limit cycle
4. an orbit which has at least one half-orbit, that is a separtrix of some equilibrium point

To get a high-quality visualization of behavior of NDS it is necessary to calculate and render separtrixes (the boundaries of basins) and to designate the dynamics of every basin.

Key provisions used in visualisation of the NDS phase portrait

Visualization of the phase portrait requires building of a certain set of system's orbits and separatrixes between different basins. To find which orbits must be built, is determined a specific parameter with 0 to 1 range – weight of orbit [19]. Expert rating have shown a high (close to 1) weight of separatrixes.

For visualization of the systems' phase portrait in accordance with this rating and known basin boundaries structure it is necessary:

5. first determine the type of singular points. It gives particulary quantity of basins and positioning of separatrixes around each one equilibrium point
6. then find the number and relative positioning of limit continuums, particularly limit cycles
7. finally, find positioning of non-limiting separatrixes, that means to find a limiting set which a separatrix of that equilibrium point approaches with $t \to +\infty$ and $t \to -\infty$.

According this idea, the new method of topological structure determination must use separatrix search algorithms near equilibrium points and in random points of orbit. One or several orbits is necessary for visualization of dynamics inside a basin, therefore here is also building of any orbits from user defined points.

Analysis of system's behavior near its singular points

As long as the singular points and their separatrixes may form part of the basin boundary, it is necessary to establish the position of singular points within the subject area and to investigate the dynamics of the system in the vicinity of each of them.

Moreover, the type the singular points lying inside basin, but not in the borders (topological nodes and centers) determines the dynamics of their basins. Accordingly, for the visualization of the phase portrait of the system it is important to analyze the behavior of the system near each of its singular points.

Position of the system's (1) singular points can be determined by numerical solution of the system:

$$F(x) = 0 \tag{4}$$

The system dynamics near regular singular point can be found from the eigenvalues of the Jacobi matrix of the system. It can be classified as a saddle, node, focus or center in the Poincare classification [9].

Irregular isolated singular point, as in the theory of Andronov period's qualitative analysis, is either a topological node (similar to node or focus), or a center or it's neighborhood consists of a finite number of qualitatively different sectors [9]. The boundaries between these sectors usually are, but not always, separatrixes.

Thereby for any particular, even a regular singular point, if we consider a sufficiently small (characteristic) neighborhood, there are three possibilities:

8. neighborhood consists of a finite number of elliptic, nodal and saddle sectors (in the case of the saddle — only saddle sectors);
9. point is a topological node (this category includes not only the nodes and node-like irregular points, but focus);
10. point is the center.

Even if the Jacobian of the system is close to zero and its numerical estimation wrongly classifies a point as irregular, the analysis of the neighborhood will correctly detect the system dynamics near this point.

It can be used as the basis of the first of our proposed methods for searching separatrixes — edge detection of qualitatively different sectors in the neighborhood of an isolated singular point. For each of singular points:

1) Determine is it irregular or not (by numerical estimation of the Jacobian of the system at this point);

for regular singular point determine its type and, according to the above Poincare classification determine the presence of separatrixes and their directions;

for irregular singular point explore it's neighborhood.

Finding of limit continuums

If the singular point determines the position of a separatrix and lies outside the defined area or separatrix and is a limit cycle, to find this separatrix it is necessary to use another features. It is known the separatrix as limit orbit for positive and negative half of at least one of the shared basins is always orbitally unstable [9]. Accordingly, comparing the behavior of neighboring orbits in some part of the study area, it is possible to determine the position of the separatrix passing through this part.

Fig. 2 shows an example of such a comparison.

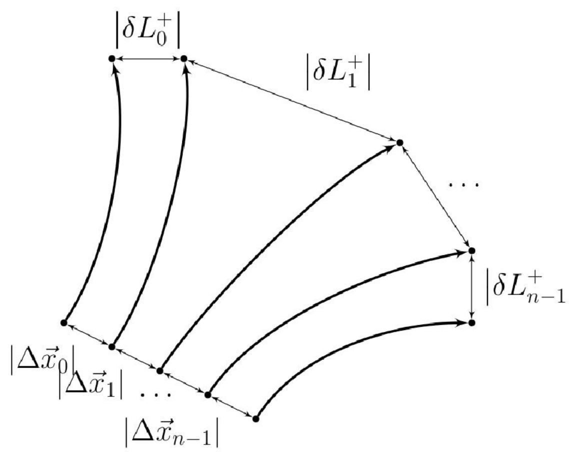

Fig. 2. On [omega]-orbital stability

The test segment is divided into n parts, and the distance $|\delta L_1^+|$ between positive half-orbits L_0^+ and L_1^+ substantially are greater than the distance between other pairs of adjacent half-orbits. According this, it can be concluded that between L_0^+ and L_1^+ passes orbitally unstable at $t \to +\infty$ ([omega]-orbitally unstable) orbit, which is a limit for neighboring orbits as $t \to -\infty$ and it is sought separatrix. Further refinement of its provisions can be done iteratively, with similar reasoning.

Orbital stability analysis has several advantages. Comparing the distances between neighboring orbits it is possible to find attracting ([alpha]-orbitally unstable) and repulsive ([omega]-orbitally unstable) closed orbits (cycles). With the analysis of orbital stability there are no false positives on the orbits with an inflection point. This makes weight of orbits constructed using analysis of orbital stability high and close to 1. Accordingly, the algorithmic reliability of separatrixes search by analyzing the orbital stability of orbits is very high.

At the same time, calculation of the separatrix fragment also requires calculation of set of neighboring orbits fragments of the same length. Therefore, the complexity of building and searching the separatrix with this algorithm significantly higher than the complexity of building an orbit with a corresponding length [20,21].

In many cases, rather than reliable, but time-consuming analysis of orbital stability, one can use indirect signs of separatrixes. One such indirect sign is the curvature of the orbits.

Near the saddle point the separatrix is a straight line directed along one of the eigenvectors of the saddle point, and keeps a straight form far enough away from the singular point, depending on the influence of the nonlinear terms in the right-hand side of equation (1). Remaining orbits near a singular point are hyperbolas, and at opposite sides of the separatrix they have different "curvature", which can be described quantitatively. Thus, the orbit with the minimum curvature in the scope area with a high probability can be separatrix of the nearby saddle singular point [21].

The complexity of such search and the building of separatrix does not exceed the complexity of building orbit of the corresponding length.

This algorithm has its limitations. This algorithm does not allow identifying the cycles that aren't limiting for saddles separatrixes. False positives are also possible if each line of the family of orbits in the area has an inflection point, or close to their characteristic directions of nodes. Because of these errors, weight of lines constructed with this algorithm is less than the line constructed by comparing the distances between the orbits and is about 0.5-0.7.

In practice, these limitations require interactive interference in the analysis by researcher's own intelligence.

Interactive phase portrait visualization technique for a nonlinear dynamical system

Developed algorithms, methods and conclusions based on the theory of qualitative analysis, made possible to develop research technique of nonlinear dynamic system as a whole. It includes five basic stages.

Stage 1, the user specifies the workspace taking into account technological restrictions, the equations defining the behavior of the studied system, its parameters and modeling parameters.

Stage 2. Search for singular points of the system within the given technological constraints is performed. For localization of singular points velocity moduli in various areas of workspace are compared. Then the position of the singular points of the system is iteratively refined. Position found in the work area of the singular points is drawn on the screen and stored in memory for use in the next stages.

Stage 3. The nature of the equilibrium (singular) points is studied. This is done by moving around each singular point in the positive direction (counterclockwise for the finite singular points and clockwise for ∞) and finding if neighboring points belongs to the separatrix. All the separatrix in the neighborhood of each singular point identified, iteratively refined and rendered. Drawing separatrix orbits allows to see limit set, which is approaches. Orbits calculated with numerical methods.

With the results of the stage 3, user can adjust model parameters and set any arbitrary area for separatrixes search (not bound to any singular points). This is especially important if due to technological limitations needed singular points are not available.

Stage 4. All separatrixes in user-defined area are being searched. Then found separatrixes are drawn (if existed). After this search area can be adjusted, and any found separatrix can be tuned to improve accuracy.

Stage 5. User analizes found separatrixes. Separatrixes or auxiliary lines kinks require more detailed study. According to the results of the fifth stage user can adjust parameters or get back to the stage 4.

Note that developed and described above algorithms are efficient for constructing separatrixes regardless of degeneracy and position of singular points.

The user also has the ability to calculate and draw the orbit passing through an user-defined point.

Software implementation of developed technique and NDS research algorithms gives sufficiently complete results both in limited scope area, and with irregular singular points. This feature is unique among tools known today.

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