Higher-Order Iterative Methods for Solving Nonlinear Equations

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Abstract: In this report, we presented three high-order iterative methods for solving nonlinear equations of the form \( f(x) = 0 \). These proposed iterative methods are obtained by combining a fourth-order iterative method with the classical Newton’s method and approximating the first derivative in the third step by three different approaches of combinations of previously evaluated function values. The convergence analyses of the new methods are discussed, and several examples are given to illustrate the methods’ efficiency.

Keywords: Newton’s method; Iterative methods; Efficiency index; Order of Convergence; Optimal Iterative methods.

AMS subject classification: 65H05, 49M99.

1. Introduction

One of the most important and challenging problems in science and engineering applications is to solve nonlinear equations of the form \( f(x) = 0 \). The Newton method is most likely the best-known iterative method for solving nonlinear equations and is given as follows:

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]  

Recently, several modifications of the Newton method have been proposed and analyzed, which have either an equal or better performance than the Newton method; for examples, refer to (Kou et al., 2010) and (Neta and Petkovic, 2010). Researchers have devoted a significant amount of attention to developing three-step iterative methods with an eighth-order of convergence for solving nonlinear equations of the form \( f(x) = 0 \); for examples, refer to (Kou et al., 2010), (Neta and Petkovic, 2010), (Sharma and Sharma, 2010), (Siyyam et al., 2011), (Siyyam et al., 2011), (Soleymani et al., 2012).

Combining two methods of third order modifications of Newton’s methods (Chun, 2006), (Chun, 2005); for solving nonlinear equations of the form \( f(x) = 0 \) by R. Ezzati and F. Saleki (Ezzati and Saleki, 2011); gives the iterative method as follows:

\[
x_{n+1} = A[x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f'(x_n)}{f(x_n)}],
\]

When \( A = -1 \) and \( B = 2 \), the fourth-order iterative method can be determined as follows:

\[
y_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

The error equation corresponding to the above method is given as follows:

\[
e_{n+1} = (-c_n - 3c_n) e_n + O(e_n^2),
\]

where \( e_n = x_n - \alpha \) and \( c_k = \frac{f^{(k)}(\alpha)}{k!} \) for \( k = 2,3,\ldots \).

Combining the iterative method (2) with Newton’s method, the iterative method can be obtained as follows:

\[
y_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

The error equation corresponding to the above method is given as follows:

\[
e_{n+1} = (-c_n - 3c_n) e_n + O(e_n^2),
\]

where \( e_n = x_n - \alpha \) and \( c_k = \frac{f^{(k)}(\alpha)}{k!} \) for \( k = 2,3,\ldots \).

According to the following theorem, the iterative method (3) has an eighth-order of convergence.

Theorem 2.1: (Traub, 1982) Let \( \Phi_1(x) \) and \( \Phi_2(x) \) be two iterative methods with an order of convergence \( P \) and \( Q \), respectively; then, the order
of convergence of the iterative method \( \phi(x) = \phi_n(\phi(x)) \) is \( pq \).

Furthermore, per one cycle, the iterative method (3) requires three evaluations of the functions and two evaluations of its first derivatives, \( s = 5 \). Therefore, the efficiency index is given as follows:

\[
EI = \rho^{\frac{1}{s}} = 8^{\frac{1}{5}} \approx 1.5157.
\]

An iterative method is an optimal method if its order of convergence is \( 2^{\frac{1}{s}} \) and efficiency index is \( 2^{\frac{1}{s}} \) (Kung and Traub, 1974). Thus, method (3) is not optimal and does not meet the Kung and Traub conjecture (Kung and Traub, 1974). However, (1) and (2) are optimal iterative methods.

The goal of this study is to simultaneously increase the order of convergence and efficiency index as high as possible. Therefore, we have to replace the first derivative in the last step of (3), i.e., \( f'(z_s) \), by a combination of previously evaluated function values, \( f(x_n), f'(x_n) \) and \( f'(y_n) \).

This report is organized as follows: In Section 2, we describe the concept of the composition of the iterative methods and, consequently, prove that the iterative method defined in (3) has an eighth-order of convergence. Three different estimations for the first derivative in the last step of (3) are presented in Section 2. Furthermore, Section two, proves that the order of convergence of the last two new resulting iterative methods is eight with an efficiency index of \( 8^{\frac{1}{8}} \approx 1.6818 \). Therefore, these last two new iterative methods defined in (18) and (27) are optimal and satisfy the conjecture of Kung and Traub (1974). In Section 3 various numerical examples are presented, to illustrate and, confirm the performance and accuracy of our proposed iterative methods as well as compare methods of the same order of convergence.

2. Construction of Higher-Order Iterative Methods

To improve the efficiency index of method (3), several estimations for the first derivative in the last step \( f'(z_s) \) of (3) are proposed by using a combination of previously evaluated function values. In Section (2.1), we presented the first method, which is the estimation based on Cordero et al. (2010); and proved in Theorem (2.1.1) that the order of convergence of the resulting iterative method is seven. More, two estimations of \( f'(z_s) \) are described based on Neta and Petkovic (2010), and Siyyam et al. (2011). The last two methods will be proved to have an eighth-order of convergence.

2.1 Method One

A three-step family of iterative methods based on the modified Kou’s method (Kou, Li and Wang, 2007); is considered by Cordero et al. (2010).

A second degree Taylor polynomial of \( f(z_s) \) is used to approximate \( f'(y_n) \), and substituting this with an appropriate approximation of \( f''(y_n) \) in \( f'(z_s) \) is given as follows:

\[
f'(z_s) \approx f[z_n, y_n] + f[z_n, x_n, x_n](z_s - y_n).
\]

By substituting the estimation obtained in (4) into (3), the following new iterative method can be obtained as follows:

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[
z_n = y_n + \frac{f(y_n)}{f'(x_n)} - 2 \frac{f(x_n)f(y_n)}{f'(x_n)[f(x_n) - f(y_n)]} \]
\[
x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}.
\]

Per one full iteration, the method requires three evaluations of the function and one evaluation of its first derivative; therefore, its efficiency index is \( \rho = 7^{\frac{1}{8}} \approx 1.6266 \). Furthermore, as \( \rho \neq 2^{\frac{1}{s}} \), the method is not optimal.

**Theorem 2.1.1:** Let \( \alpha \in I \) be a simple zero of a sufficiently differentiable function \( f : I \to \mathbb{R} \) for an open interval \( I \). Then, the method that is defined by equation (5) has a seventh-order of convergence and satisfies the error equation as follows:

\[
e_{n+1} = (2c_1^4 c_2^2 - 6c_1^2 c_2^2) e_{n+1} + O(e_{n+1}^5),
\]

where \( e_n = x_n - \alpha \) and \( c_k = \frac{f^{(k)}(\alpha)}{f^{(k)}(\alpha)} \), for \( k = 2,3,... \).

**Proof:** Let \( \alpha \) be a simple zero of the nonlinear equation \( f(x) = 0 \), and \( x_n = \alpha + e_n \).

By the Taylor expansion, we have the equation as follows:

\[
f(x_n) = f'(\alpha)[e_n + c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6 + c_7 e_7 + O(e_7^8)],
\]

and

\[
f'(x_n) = f''(\alpha)[1 + 2c_1 e_1 + 3c_2 e_2 + 4c_3 e_3 + 5c_4 e_4 + 6c_5 e_5 + 7c_6 e_6 + 8c_7 e_7 + O(e_7^8)].
\]

Dividing (7) by (8), the equation becomes as follows:
\[ f(x_n) = e_n - c_1 e_1^i + (-2c_2 + 2c_1^2)e_1^{i'} + \ldots \]
\[ \frac{f(x_n)}{f'(x_n)} = \frac{c_2}{c_1} \frac{e_1^i}{e_1^{i'}} + \frac{c_1^2}{c_1^2} \frac{e_1^{i'}}{e_1^i} + \ldots \]

(10)

Substituting (7), (8) and (9) into \( y_n \) in (5) gives the expression as follows:

\[ y_n = \frac{\alpha}{c_1^i} + \frac{2c_2}{c_1^2} e_1^{i'} + (3c_1 - 7c_2 c_1 + 4c_1^2) e_1^i + \ldots \]

(11)

By expanding \( f(y_n) \) with respect to \( \alpha \), the expression is given as follows:

\[ f(y_n) = f'(\alpha)[c_1^i e_1^{i'} + (2c_2 - 2c_1^2) e_1^i + \ldots + (-31c_1 c_1 + 73c_1^2 e_1^{i'} + 176c_1^3 e_1^i + 297c_1^4 e_1^{i''} - 134c_1^5 e_1^i)] + 54c_1^2 e_1^{i'} - 147c_1^3 e_1^i + 582c_1^4 e_1^{i''} - 552c_1^5 e_1^i + 144c_1^6 e_1^{i'''} - 19c_1^7 e_1^i + 7c_1^8 - 455c_2 c_1^i + 134c_1 c_1^i e_1^{i'} e_1^i + O(e_1^i). \]

(12)

Substituting (7), (8), (10), and (11) into \( x_n \) in (5), the equation is given as follows:

\[ x_n = \frac{\alpha}{c_1^i} + (c_1 + 3c_1^2) e_1^{i'} + \ldots + (164c_1 c_1 - 965c_1^2 c_1^i + 5c_1 c_1^2 - 13c_1 + 50c_1^2 c_1^3 - 17c_1 c_1^i + 29c_1^2 c_1^i + 122c_1^3 c_1^i + 91c_1^4 c_1^i + 799c_1^5 c_1^i - 395c_1^6 c_1^i + 1863c_1^7 c_1^i - 2076c_1^8 c_1^i + 624c_1^9 c_1^i + O(e_1^i). \]

(13)

By using (10), (11), (12) and (13), the equation is obtained as follows:

\[ f[y_n, x_n] = f'(\alpha)[1 + c_1^i e_1^{i'} + (2c_2 c_1 - 2c_1^2) e_1^i + \ldots + (-18c_1 c_1 + 86c_1^2 e_1^{i'} - 16c_1^3 c_1^i + 5c_1 c_1^2 e_1^i + 4c_1^3 c_1^4 e_1^{i''}] + 52c_1^2 c_1^i - 167c_1^3 c_1^i) e_1^i + O(e_1^i). \]

(14)

By using \( f[x_n, x_n] = f'(\alpha)[1 + c_1 e_1^{i'} + c_1 c_1^2 e_1^i + c_1 e_1^{i''} + \ldots + (-12c_1 c_1^2 - 257c_1^2 e_1^{i'} - 53c_1 c_1^3 - 1279c_1^4 c_1^i - 22c_1 c_1^2 e_1^i + 288c_1^2 c_1^3 e_1^{i''} - 18c_1 c_1^2 e_1^i - 13c_1^2 c_1^i - 738c_1^3 c_1^2 e_1^{i''} + 624c_1^2 e_1^i + 28c_1^2 c_1^i e_1^i + O(e_1^i)]. \)

(15)

By using (6), (8), (12), and (15), the expression \( f[z_n, x_n] \) can be expressed in terms of \( e_n \) as follows:

\[ f[z_n, x_n] = f'(\alpha)[c_1 + 2c_1 e_1^{i'} + 3c_1 c_1^2 e_1^i + 4c_1 e_1^{i''} + \ldots - 2c_1 e_1^{i'} + O(e_1^i)]. \]

(16)

Substituting (10), (12), (13), (14) and (15) into \( x_{n+1} \) in equation (5), the equation can be expressed as follows:

\[ x_{n+1} = \alpha + (2c_1^2 c_1^2 - 6c_1 c_1^2) e_1^{i'} + O(e_1^i). \]

Therefore, the equation is given as follows:

\[ e_{n+1} = (2c_1^2 c_1^2 - 6c_1 c_1^2) e_1^{i'} + O(e_1^i). \]

(17)

Equation (17) establishes the seventh-order convergence of the method that is defined by equation (5). □

2.2 Method Two

A general technique is given by Neta and Petkovic (2010); to construct such methods using inverse interpolation and any optimal two point methods and present an approximation of the third step, which can be placed into the last step in the method (3). Thus, the new iterative method to solve the nonlinear equation of the form \( f(x) = 0 \) is given as follows:

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)} \]

\[ z_n = y_n + \frac{f(y_n)}{f'(x_n)} \left( \frac{f(x_n) - f(y_n)}{f'(x_n)(f(x_n) - f(y_n))} \right) \]

\[ x_{n+1} = y_n + c[f(x_n)]^2 - d[f(x_n)]^3. \]

(18)

where

\[ c = \frac{1}{(f(x_n) - f(y_n))}, \quad d = \frac{1}{(f(x_n) - f(y_n))}. \]

(19)

\[ \frac{1}{(f(x_n) - f(y_n))} = \frac{1}{(f(x_n) - f(y_n))(f(x_n) - f(z_n))} \]

and

\[ \frac{1}{(f(x_n) - f(y_n))(f(x_n) - f(z_n))} = \frac{1}{(f(x_n) - f(y_n))(f(x_n) - f(z_n))}. \]

(20)

Per one full iteration, the method requires three evaluations of the function and one evaluation of its first derivative; therefore, its efficiency index is \( 8^{1/4} \approx 1.6818 \), which implies that the method is an
optimal eighth-order method according to Kung and Traub’s conjecture (1974). In the next theorem, we will show that the order of convergence of the iterative method (18) is eight.

Theorem 2.2.1: Let \( \alpha \in I \) be a simple zero of a sufficiently differentiable function \( f : I \rightarrow R \) for an open interval \( I \). Then, the method that is defined by equation (18) has an eighth-order of convergence and satisfies the error equation as follows:

\[
e_{n+1} = (-c,c,c,c_1^1 +15 c_2^1 +3 c,c_3^1 +5 c_4^1 c_1^1 -20 c,c_3^2) e_n^8 + O(e_n^9).
\]

\[\text{Proof:}\] Let \( \alpha \) be a simple zero of \( f \) and \( x_n = \alpha + e_n \). Using (6), (7), (10) and (11), we obtain the expression as follows:

\[
f[y_n,x_n] = f'(\alpha)(1+c,c_1 +c,c_2 +...+(-8c_1,c_1 c_2 -93c_1^2 c_1^2 +8c_1 c_2 +45c_1 c_1^3 -313c_1 c_1 c_2 -40c_1 c_3 c_2 +56c_2 c_1 c_2 +116c_1 c_2 c_3 -36c_2 c_3 c_4 +8c_3 c_4 -8c_4) e_n^8 + O(e_n^9).
\]

The parameters \( c \) and \( d \) can be expressed in terms of \( e_n \) as follows:

\[
c = \frac{1}{f'(\alpha)} (-c_1 + (6c_1^2 -3c_1) e_n + (350c_1^3 c_1 -166c_1^2 c_1 e_n -189c_1^3 c_1^2 -530c_1^2 c_1^2 -1034c_1 c_1^3 c_1 +382c_1^2 c_1^3 c_1 -13c_1 +58c_1 c_2 +54c_1 c_1^2 -209c_1 c_2 c_3 +72c_1 c_3 -376c_1 c_1 c_3 +76c_1 c_1 +1084c_1^2 c_1 c_1 e_n + O(e_n^9)).
\]

and

\[
d = \frac{1}{f'(\alpha)} (-c_1 +2c_1 +(-10c_1 -2c_1 +10c_1 c_1) e_n + (27c_1^2 -3c_1 +9c_1^2 +15c_1 c_1 -48c_1 c_1 e_n + \cdots +(-6c_1 +264c_1^2 +186c_1 c_1 -92c_1 c_2 -90c_1 c_1 e_n -176c_1 c_1 c_2 -216c_1 c_1 c_2 -316c_1 c_1 c_1 c_1 -30c_1 c_1 -100c_1 c_1^2 +36c_1 c_1 +36c_1 c_1 -192c_1 c_1 c_1 +480c_1 c_1 c_1 e_n + O(e_n^9)).
\]

By substituting (7), (10), (22) and (23) into \( x_{n+1} \) in (18), the equation is obtained as follows:

\[
x_{n+1} = \alpha + (-c,c,c_1^1 +15 c_2^1 +3 c,c_3^1 +5 c_4^1 c_1^1 -20 c,c_3^2) e_n^8 + O(e_n^9).
\]

Therefore, the expression becomes as follows:

\[
e_{n+1} = (-c,c,c_1^1 +15 c_2^1 +3 c,c_3^1 +5 c_4^1 c_1^1 -20 c,c_3^2) e_n^8 + O(e_n^9).
\]

Equation (24) establishes the eighth-order convergence of the method that is defined by equation (18). □

2.3 Method Three

Siyyam et al. (2011); considers composing the two-step family of a fourth-order equation based on Chun (2007), with the classical Newton’s method to obtain a three-step iterative method. A new estimation for \( f'(z_n) \), also proposed in (Siyyam et al., 2011), is provided by the third-degree polynomial as follows:

\[
R(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + a_4(x-x_0)^4 (x-x_0)^5
\]

where \( \tilde{f} \) is interpolated at \( x_n, y_n \) and \( z_n \); and \( f'(x_n) = \tilde{P}(x_n) \) is satisfied. Therefore, the expression becomes as follows:

\[
f'(x_n) = \tilde{P}(x_n) = f'(x_n) + f(x_n,y_n,z_n) - f(x_n,y_n,z_n) \ni x_n - x_n \ni + 2f(x_n,y_n,z_n) - f(x_n,y_n,z_n)(z_n - y_n).
\]

Substituting (26) into the last step of equation (3), we obtain the new iterative method as follows:

\[
y_n = x_n - \frac{f'(x_n)}{f'(x_n)} z_n = y_n + \frac{f(y_n)}{f'(x_n)} - \frac{2f(x_n,y_n)}{f'(x_n)(f(x_n) - f(y_n))}
\]

\[
x_{n+1} = z_n - \frac{f'(z_n)}{P'(z_n)}.
\]

The next theorem shows that the order of convergence of the iterative method defined in (27) is eight. Moreover, per one full iteration, the method requires three evaluations of the function and one evaluation of its first derivative; therefore, its efficiency index is \( \frac{8}{4} \approx 1.6818 \), which implies that the method is an optimal eighth-order method according to Kung and Traub’s conjecture (1974).

Theorem 2.3.1: Let \( \alpha \in I \) be a simple zero of a sufficiently differentiable function \( f : I \rightarrow R \) for an open interval \( I \). Then, the method that is defined by equation (27) has an eighth-order of convergence.

\[\text{Proof:}\] Use equations (6), (12), (14) and (21), (23) to write the expression \( f[x_n,y_n,z_n] \) in terms of \( e_n \) as follows:

\[
f[x_n,y_n,z_n] = f'(\alpha) \ni c_1, e_n^1 + e_n^2 c_1 + e_n^3 c_1 c_2 + ... + (-16c_1 c_1 c_2 -8c_1^2 c_1 c_2 +3c_2 c_1 c_1 c_3 +3c_2 c_1 c_2^2 +6c_1 c_1 c_2 c_3 +40c_1^2 c_2^2 -26c_1 c_2 c_1 c_3 +c_2 c_1 c_3 e_n^1 + 0(e_n^9).
\]

Using (6), (8), (10) and (21), the expression \( f[x_n,x_n,y_n] \) can be expressed in terms of \( e_n \) as follows:

\[
f[x_n,x_n,y_n] = f'(\alpha)(c_1 + 2c_1 e_n + (3c_1 + c_2) e_n^2 + ...
\]

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By substituting equations (12), (13) and (30) into iterative methods.

Iterative methods using several numerical examples confirm the accuracy of our developed methods. We tested these theoretical results and illustrated the efficiency and the eighth-order iterative method defined in (5), which we will refer as (SQM1). To confirm the eighth-order iterative method of Kou et al., (KM1) in (Kou et al., 2010), which are defined as follows:

\[ x_{n+1} = z_n - [(1 + H_2(x_n, y_n)) z_n] + (1 + 4H_4(x_n, y_n)) \]
\[ \times H_4(y_n, z_n) \frac{f(z_n)}{f'(x_n)} \]

where
\[ H_2(x_n, y_n) = \frac{f(y_n)}{f(x_n) - 2f(y_n)} \]
\[ H_4(y_n, z_n) = \frac{f(z_n)}{f(z_n) - 2f(y_n)} \]

with \( \beta = 0 \).

Thus, using equations (8), (10) and (12), (28) and (29), we can write the estimation of Siyyam et al. in terms of \( e_n \) as follows:

\[ f'(z_n) \approx P_7(z_n) \]

in the last step of (27) in terms of \( e_n \) as follows:

\[ f'(z_n) \approx P_7(z_n) \]

\[ = f'(x)[1 + (-2c_2 e_1 + c_4 e_1 + 6c_4 e_1)]e_1 + \ldots \]
\[ + (-20c_2 c_1 + 3c_2 c_1 + 140c_4 c_1 + 6c_4 c_1 + 4c_4 c_1) e_1 \]
\[ + 84c_2 c_2 c_1 + 260c_2 c_2 c_1 - 9c_2 c_2 c_1 + 3c_2 c_2 c_1) e_1] + O(e_1^2). \]

By substituting equations (12), (13) and (30) into (27), the equation becomes as follows:

\[ x_{n+1} = \alpha + (-c_2 c_2 e_1 + c_4 c_2 e_1 + 6c_4 c_2 e_1 + 9c_2 c_2 e_1) \]
\[ + O(e_1^2). \]

Therefore, the expression can be written as follows:

\[ e_n = \alpha + (-c_2 c_2 e_1 + c_4 c_2 e_1 + 6c_4 c_2 e_1 + 9c_2 c_2 e_1) \]
\[ + O(e_1^2). \]

Equation (32) establishes the eighth-order convergence of the method that is defined by equation (27). □

3. Numerical Examples

Three high-order iterative methods have been derived for solving nonlinear equations of the form \( f(x) = 0 \), namely the seventh-order iterative method defined in equation (5), which we will refer to as (SQM1), the eighth-order iterative method defined in (18), which we will refer to as (SQM2) and the eighth-order iterative methods defined in (27), which we will refer to as (SQM3). To confirm our theoretical results and illustrate the efficiency and accuracy of our developed methods, we tested these iterative methods using several numerical examples and compared them with other existing eighth-order iterative methods.

We can compare our iterative methods, (SQM1), (SQM2) and (SQM3), with the eighth-order iterative methods of Kou et al., (KM1), (KM2) in (Kou et al., 2010) which are defined as follows:

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \]
\[ z_n = y_n - H_2(x_n, y_n)(x_n - y_n), \]

\[ x_{n+1} = z_n - [(1 + H_2(x_n, y_n)) z_n] + (1 + 4H_4(x_n, y_n)) \]
\[ \times H_4(y_n, z_n) \frac{f(z_n)}{f'(x_n)} \]

where
\[ H_2(x_n, y_n) = \frac{f(y_n)}{f(x_n) - 2f(y_n)} \]
\[ H_4(y_n, z_n) = \frac{f(z_n)}{f(z_n) - 2f(y_n)} \]

with \( \beta = 0 \).

Therefore, using equations (8), (10) and (12), (28) and (29), we can write the estimation of Siyyam et al. in terms of \( e_n \) as follows:

\[ f'(z_n) \approx P_7(z_n) \]

in the last step of (27) in terms of \( e_n \) as follows:

\[ f'(z_n) \approx P_7(z_n) \]

\[ = f'(x)[1 + (-2c_2 e_1 + c_4 e_1 + 6c_4 e_1)]e_1 + \ldots \]
\[ + (-20c_2 c_1 + 3c_2 c_1 + 140c_4 c_1 + 6c_4 c_1 + 4c_4 c_1) e_1 \]
\[ + 84c_2 c_2 c_1 + 260c_2 c_2 c_1 - 9c_2 c_2 c_1 + 3c_2 c_2 c_1) e_1] + O(e_1^2). \]

By substituting equations (12), (13) and (30) into (27), the equation becomes as follows:

\[ x_{n+1} = \alpha + (-c_2 c_2 e_1 + c_4 c_2 e_1 + 6c_4 c_2 e_1 + 9c_2 c_2 e_1) \]
\[ + O(e_1^2). \]

Therefore, the expression can be written as follows:

\[ e_n = \alpha + (-c_2 c_2 e_1 + c_4 c_2 e_1 + 6c_4 c_2 e_1 + 9c_2 c_2 e_1) \]
\[ + O(e_1^2). \]

Equation (32) establishes the eighth-order convergence of the method that is defined by equation (27). □

The particular eighth-order iterative method of Sharma et al. (SM1) in (Sharma and Sharma, 2010), is defined as follows:

\[ w_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}, \]
\[ z_n = w_{n+1} - \frac{f(w_{n+1})}{f(x_n) - 2f(w_{n+1})} \frac{f(x_n)}{f'(x_n)}, \]
\[ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \]

where
\[ \mu_n = \frac{f(z_n)}{f'(x_n)}, \]
\[ \beta = 0 \]
f_1(x) = xe^{-x} + \log(1 + x + x^4) \\
\alpha \approx 0.22045246 21592135 \\
f_2(x) = \sqrt{x} - \frac{1}{x} - 3 \\
\alpha \approx 9.63359556 28326952 \\
f_3(x) = \log(x) + \sqrt{x} - 5 \\
\alpha \approx 8.30943269 42315718 \\
f_4(x) = \arcsin(x) - x \\
\alpha \approx 1.74613953 04080124 \\
\alpha \approx 0.73908513 32151606 \\
\alpha = 0.73908513 32151606 \\
f[x, z] = \frac{f(z) - f(x)}{z - x} \\
f[x, w] = \frac{f(w) - f(x)}{w - x} \\
f[w, z] = \frac{f(z) - f(w)}{z - w} \\
W \text{ can be expressed as follows:} \\
W(t) = 1 + t + t^2. \\
The same method was used by Sharma et al.(SM2) in (Sharma et al., 2011), to produce another eighth-order method as follows: \\
w_i = x_i - \frac{f(x_i)}{f'(x_i)} \\
z_i = w_i - \frac{f(w_i) - f(x_i)}{f(x_i)} f'(x_i) - 2 f(w_i) \\
x_{i+1} = z_i - (\phi)^i f'(x_i). \\
(38)
\phi = [f(w_i)f(w_i) - f(x_i)f(z_i)]^{-1} f(x_i) f'(x_i) - 2 f(w_i) \\
- 2 f(w_i)f(z_i) - f(x_i) f(z_i) - f(x_i) f(w_i) f'(w_i) \\
- f(x_i) f(w_i) f'(w_i)]. \\
(39)
All computations were performed using MATLAB 7.11 with 1000 significant digits. Table(1) shows the absolute value of the function at \( x_0 \), \( |x_n - x_0| \) and \( |x_n - \alpha| \) for the test functions for all the iterative methods mentioned in this section, where \( n \) represents the number of iterations and is taken in the table to be 3, \( \alpha \) is the zero of the function and \( x_0 \) is the initial estimation of \( \alpha \). Furthermore, the computational order of convergence for all the iterative methods mentioned is displayed, where the computational order of convergence (COC) can be approximated by using the formula as follows (Weerakoon and Fernando, 2000):
\[
COC \approx \frac{\ln |(x_n - \alpha)(x_{n+1} - \alpha)|}{\ln |(x_n - \alpha)(x_n - \alpha)|}.
\]
The following test problems are used to demonstrate the performance of the new developed iterative methods and compare with other existing eighth-order iterative methods.

Table 1: Comparison of various iterative methods with \( n = 3 \)

<p>| Method | (| f(x_0) | | x_n - \alpha | ) | COC |
|--------|-----------------|-----|
| ( f_1(x) ), ( x_0 = 0.25 ) | SQM1 0.214e-338 0.107e-338 7.000 |
| | SQM2 0.335e-522 0.167e-522 8.000 |
| | SQM3 0.670e-515 0.335e-515 8.000 |
| | KM1 0.107e-461 0.535e-462 7.972 |
| | KM2 0.426e-451 0.213e-451 8.029 |
| | NAM 0.412e-497 0.206e-497 8.000 |
| | SM1 0.933e-497 0.467e-497 8.000 |
| | SM2 0.479e-441 0.240e-441 8.000 |
| ( f_2(x) ), ( x_0 = 15.5 ) | SQM1 0.132e-246 0.769e-246 7.000 |
| | SQM2 0.321e-548 0.187e-547 8.000 |
| | SQM3 0.456e-365 0.318e-364 8.000 |
| | KM1 0.120e-491 0.696e-491 7.930 |
| | KM2 0.100e-414 0.584e-414 7.834 |
| | NAM 0.437e-407 0.254e-406 8.000 |
| | SM1 0.359e-518 0.279e-517 8.000 |
| | SM2 0.644e-386 0.375e-385 8.000 |
| ( f_3(x) ), ( x_0 = 11.9 ) | SQM1 0.244e-258 0.832e-258 7.000 |
| | SQM2 0.150e-511 0.510e-511 8.000 |
| | SQM3 0.456e-385 0.213e-384 8.000 |
| | KM1 0.699e-482 0.238e-481 7.915 |
| | KM2 0.437e-501 0.165e-500 7.877 |
| | NAM 0.332e-447 0.113e-446 8.000 |
| | SM1 0.392e-491 0.134e-490 8.000 |
| | SM2 0.472e-433 0.161e-432 8.000 |
| ( f_4(x) ), ( x_0 = 0.5 ) | SQM1 0.286e-515 0.270e-515 7.000 |
| | SQM2 0.234e-796 0.221e-796 8.000 |</p>
<table>
<thead>
<tr>
<th></th>
<th>SQM 3</th>
<th>SQM 2</th>
<th>KM 1</th>
<th>KM 2</th>
<th>NAM</th>
<th>SM 1</th>
<th>SM 2</th>
</tr>
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<tr>
<td>f_1</td>
<td>0.186e-774</td>
<td>0.175e-774</td>
<td>0.298e-777</td>
<td>0.282e-777</td>
<td>0.468e-803</td>
<td>0.839e-775</td>
<td>0.488e-745</td>
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<tr>
<td>f_2</td>
<td>0.103e-724</td>
<td>0.974e-725</td>
<td>0.298e-777</td>
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<tr>
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<td>0.369e-474</td>
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<td>0.242e-674</td>
<td>0.660e-700</td>
<td>0.570e-700</td>
<td>0.294e-738</td>
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<tr>
<td>f_4</td>
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<td>0.570e-700</td>
<td>0.242e-674</td>
<td>0.384e-660</td>
<td>0.445e-660</td>
<td>0.442e-803</td>
<td>0.461e-745</td>
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<tr>
<td>f_5</td>
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<td>0.908e-750</td>
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<td>f_6</td>
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<td>0.304e-674</td>
<td>0.240e-675</td>
<td>0.413e-675</td>
<td>0.247e-675</td>
<td>0.240e-730</td>
</tr>
</tbody>
</table>

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