The method of determining the function of traffic costs for the node with regulated flows of requests

Natalya Naumova

Applied Mathematics Department, Kuban State Technological University, 2-A Moskovskaya Street, Krasnodar, 350072, Russia
Email: Nataly_Naumova@mail.ru

Abstract. The problem of modelling of transportation networks with the purpose of optimization is very important. The difficulties of numerical solution of optimization problems for networks mainly depend on the analytical definition of the function of traffic costs. We developed a mathematical model of transportation network based on the generalized Erlang time distribution and introduced our classification of nodes. We obtained a function of traffic costs for the node with regulated intersection of multichannel lines and described its analytical realization. We also proposed a method of determining the parameters of the generalized Erlang law from experimental data.
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Introduction

A transportation network is one of the vivid examples of network operation. Mathematical models applied for analysis of transportation networks vary according to the problems solved, mathematical apparatus, data used, and specification of traffic description [1-6]. The first macroscopic model was suggested by M. Lighthill and G. Whitham in the middle of the last century [7]. At that time there also appeared the first microscopic models ('follow-the-leader' theory) which explicitly derived an equation of motion for each individual vehicle (A. Reshel, L. Pipes, D. Gazis and others) [8, 9]. Frank A. Haigt was the first to establish the mathematical investigation of traffic flow as a separate section of applied mathematics [10]. At present there is voluminous literature on the subject. The problem of efficient management of transportation networks is, however, still topical.

At present the problems of rational operation of existing transportation networks in the centers of population as well as those of planning new ones in housing developments are undisputedly very essential. There are a lot of macro-models and micro-models of network flows distribution. The problems of macro-modelling are aimed at searching the equilibrium distribution of flows while micro-modelling solves the problems of traffic capacity of local sections of networks. Hypotheses underlying macro-models are of different character from those of micro-models, and the problem of information exchange between the models have been not solved theoretically, neither in the form of software.

Modelling and research of traffic flows often employ the theory of competitive non-cooperative equilibrium which describes quite an adequate mechanism of operation of urban transportation networks [9]. Such models allow us to obtain forecasts of congestion of transportation network components. They are one of the tools of determining the efficiency of projects of the transportation network reorganizing.

The problem of flow equilibrium resolves itself into routing the traffic in the network in an optimal manner minimizing the traffic costs. The difficulty of numerical solution of such problems substantially depends on the analytical definition of the function of traffic costs.

Development of a model of network operation that will make it possible to adequately forecast the efficiency of network flow distribution from minimal initial data seems to be topical.

1. Graph representation of the network

Let us define the basic notions we use in this article.

We will refer to the network flows as 'non-conflict' if they are not crossed in the given sector of the network, and as 'conflict' otherwise. We will consider the node-points – the points of sources or consumption of information and those of conflict flows crossing – to be the vertices of a graph. The node-points are formed at the intersections of multichannel lines.

In our previous works [11, 12] we gave the following classification of node-points (NP). A number of flows (the main ones) are freely passing the NP. The customers of the rest of flows (the secondary ones) are waiting for sufficient time intervals between arrivals of main flows for their turn to cross the NP. We will call such a node-point a 'type 1 node' or 'unregulated intersection of flows'.

Now let us consider a node-point (NP) at which traffic is alternately blocked for one of the non conflict flow groups for a fixed time to enable the crossing of the NP. We call such a node-point a 'type 2 node' or 'regulated intersection of flows'.

We will consider a network in a traditional form of oriented graph [13]. A network is a graph each arc of which is assigned to a certain number. A flow in the graph is a group of homogeneous objects (requests) sent from one node to another. Therefore, a flow is a certain function prescribed for the graph arcs. In the developed model we show a flow in the graph as a function of density of arrival distribution (arrival times of successive service requests).

Earlier we considered a model of network operation based on the Erlang time distribution for each flow. In this paper we extend the application of the model to the case when time intervals are distributed according to the generalized Erlang law. A proper selection of parameters by the generalized Erlang law will help approximate almost any distribution of a random variable.

2. Method of determining the function of traffic costs in the network for the node with regulated flows of requests

2.1. Generalized Erlang distribution of a random variable

For the generalized Erlang distribution the time interval between two requests in succession has k stages $T_0, T_1, \ldots, T_{k-1}$, the duration of which has exponential distribution with parameters $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ correspondently [14, 15]. The Laplace transform of the function of density distribution $f_k(t)$ holds (even for the case when some parameters coincide λ_i):

$$f^{(*)}(s) = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{(s+\lambda_0)(s+\lambda_1)\cdot \ldots \cdot (s+\lambda_{k-1})}.$$

If all parameters λ_i are different, the function of distribution of the generalized Erlang law holds:

$$f_k(t) = (-1)^{k-1} \cdot \prod_{i=0}^{k-1} \lambda_i \cdot \sum_{i=0}^{k-1} \frac{e^{-\lambda_i t}}{\prod_{\substack{n=0 \\ n \neq j}}^{k-1} (\lambda_j - \lambda_n)}.$$

Or in a simple form:

$$f_k(t) = \sum_{i=0}^{k-1} a_i \lambda_i e^{-\lambda_i t},$$

if we introduce the following notation: $a_i = \prod_{\substack{n=0 \\ n \neq i}}^{k-1} \frac{\lambda_n}{\lambda_n - \lambda_i}$, where $\sum_{i=0}^{k-1} a_i = 1$.

The function of density distribution for generalized Erlang in case of coincidence of some parameters has a different form and can be derived from function $f^{(*)}(s)$ by expansion into simple fractions, however, we are not going to consider such cases when using the method of obtaining the parameters of distribution from experimental data presented below.

So, if all parameters λ_i are different, the integral distribution function must hold:

$$F_k(t) = 1 - \sum_{i=0}^{k-1} a_i e^{-\lambda_i t}.$$

The mathematical expectation for the generalized Erlang law can be obtained subject to the definition of Erlang flow:

$$M(T) = M\left(\sum_{i=0}^{k-1} T_i\right) = \sum_{i=0}^{k-1} \frac{1}{\lambda_i}.$$

The variance for generalized Erlang can be obtain with the definition of Erlang flow:

$$D(T) = D\left(\sum_{i=0}^{k-1} T_i\right) = \sum_{i=0}^{k-1} \frac{1}{(\lambda_i)^2}.$$

The n-th initial moment:

$$\nu_n = M(T^n) = \left(\sum_i a_i \lambda_i \cdot \frac{n!}{\lambda_i^{n+1}}\right).$$

2.2. Function of simple restoration of the generalized Erlang law

The Laplace transform of the function of density distribution $f_k(t)$ holds:

$$f^{(*)} = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{(s+\lambda_0)(s+\lambda_1)\cdot \ldots \cdot (s+\lambda_{k-1})}.$$

Then the Laplace transform of simple restoration must hold [14]:

$$H^{(*)}(s) = \frac{f^{(*)}(s)}{s\left(1 - f^{(*)}(t)\right)}$$ or

$$H^{(*)}(s) = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{s \cdot \left[(s+\lambda_0)(s+\lambda_1)\cdot \ldots \cdot (s+\lambda_{k-1}) - \lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}\right]}.$$

Consider the form of the function and restore its original form by image for the generalized Erlang orders $k \in \{2;3;4\}$ (with $k = 1$ we obtain exponential distribution with parameter λ_0). In order to find the original form by image we must calculate the roots of the fraction denominator $H^{(*)}(s)$:

$$s = 0 \quad \vee \quad f^{(*)}(s) = 1$$

The roots of equation $f^{(*)}(s) = 1$ can be expressed only by parameters λ_i in the solution of equation:

$$(\lambda_0 + s)(\lambda_1 + s)(\lambda_2 + s)\cdot\ldots\cdot(\lambda_{k-1} + s) = \lambda_0\lambda_1\cdot\ldots\cdot\lambda_{k-1}.$$

Case I) $k = 2$

$$\lambda_0\lambda_1 = (\lambda_0 + s)(\lambda_1 + s)$$
$$\Leftrightarrow s^2 + s(\lambda_0 + \lambda_1) = 0$$
$$\Leftrightarrow s = 0 \quad \vee \quad s = -(\lambda_0 + \lambda_1).$$

Case II) $k = 3$

$$\lambda_0\lambda_1\lambda_2 = (\lambda_0 + s)(\lambda_1 + s)(\lambda_2 + s)$$
$$\Leftrightarrow s^2 + s(\lambda_0 + \lambda_1 + \lambda_2) + (\lambda_1\lambda_2 + \lambda_0\lambda_1 + \lambda_0\lambda_2) = 0$$
$$\vee \quad s = 0$$
$$\Leftrightarrow s = \frac{-(\lambda_0 + \lambda_1 + \lambda_2) \pm \sqrt{(\lambda_0 + \lambda_1 + \lambda_2)^2 - 4(\lambda_1\lambda_2 + \lambda_0\lambda_1 + \lambda_0\lambda_2)}}{2}$$
$$\vee \quad s = 0.$$

Case III) $k = 4$.

$$\lambda_0\lambda_1\lambda_2\lambda_3 = (\lambda_0 + s)(\lambda_1 + s)(\lambda_2 + s)(s + \lambda_3)$$
$$\Leftrightarrow s^3 + s^2(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) + s(\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_0\lambda_3 + \lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2) +$$
$$+ (\lambda_1\lambda_2\lambda_3 + \lambda_0\lambda_2\lambda_3 + \lambda_0\lambda_1\lambda_3 + \lambda_0\lambda_1\lambda_2) = 0$$
$$\vee \quad s = 0$$

Otherwise,

$$s^3 + s^2 \cdot \left(\sum_i \lambda_i\right) + s \cdot \left(\sum_{i \neq j} \lambda_i\lambda_j\right) + \left(\sum_{\substack{i \neq j \\ i \neq l \\ j \neq l}} \lambda_i\lambda_j\lambda_l\right) = 0$$
$$\vee \quad s = 0.$$

Let $b_1 = \sum_i \lambda_i$, $b_2 = \sum_{i \neq j} \lambda_i\lambda_j$, $b_3 = \sum_{\substack{i \neq j \\ i \neq l \\ j \neq l}} \lambda_i\lambda_j\lambda_l$.

The roots of the third power polynomial $x^3 + b_1x^2 + b_2x + b_3 = 0$ can be obtained by Cardano's formulas:

1) substitute the unknown $x = y - \frac{b_1}{3}$; the equation takes the form: $y^3 + py + q = 0$;
2) the equation roots are:

$$y_1 = \alpha_1 + \beta_1,$$
$$y_2 = -\frac{1}{2}(\alpha_1 + \beta_1) + i\frac{\sqrt{3}}{2}(\alpha_1 - \beta_1),$$
$$y_3 = -\frac{1}{2}(\alpha_1 + \beta_1) - i\frac{\sqrt{3}}{2}(\alpha_1 - \beta_1)$$

where $\alpha = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$, $\beta = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$, and $\alpha_1\beta_1 = -\frac{p}{3}$.

With all $k \in \{2;3;4\}$, one may, thus, expand $H^*(s)$ into simple fractions with the following terms:
1) from pole $s = 0$;
2) from non-zero poles in the points which are the roots of the equation $f^*(s) = 1$.

That is

$$H^*(s) = \frac{1}{\mu}\frac{1}{s^2} + \frac{\sigma^2 - \mu^2}{2\mu^2}\frac{1}{s} + R^*(s),$$

where $R^*(s)$ is a rational function.

Define the form of $R^*(s)$ according to the rots of the equation $f^*(s) = 1$.

A) Each *simple non-zero root* s_p of the equation $f^{(*)}(s) = 1$ in expansion $H^*(s)$ corresponds to the fraction [1]:

$$\frac{-1}{s_p \cdot \left(f^*(s_p)\right)' \cdot (s - s_p)}.$$

The derivatives of $\left(f^{(*)}(s)\right)'$ for each case with $k \in \{2, 3, 4\}$:

$k = 2$:

$$\left(f^{(*)}(s)\right)' = \frac{\lambda_0\lambda_1}{\lambda_1 - \lambda_0} \cdot \frac{-1}{(s + \lambda_0)^2} + \frac{\lambda_0\lambda_1}{\lambda_0 - \lambda_1}\frac{-1}{(s + \lambda_1)^2}$$

$k = 3$:

$$\left(f^{(*)}(s)\right)' = \frac{\lambda_0\lambda_1\lambda_2}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)}\frac{-1}{(s + \lambda_0)^2} + \frac{\lambda_0\lambda_1\lambda_2}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)}\frac{-1}{(s + \lambda_1)^2} + \frac{\lambda_0\lambda_1\lambda_2}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}\frac{-1}{(s + \lambda_2)^2}$$

$k = 4$:

$$\left(f^{(*)}(s)\right)' = \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)(\lambda_3-\lambda_0)}\frac{-1}{(s+\lambda_0)^2} + \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{(\lambda_0-\lambda_1)(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\frac{-1}{(s+\lambda_1)^2} + \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\frac{-1}{(s+\lambda_2)^2} + \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{(\lambda_0-\lambda_3)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\frac{-1}{(s+\lambda_3)^2}$$

Each *simple fraction* $\dfrac{-1}{s_p \cdot \left(f^*(s_p)\right)' \cdot (s-s_p)}$ in expansion $H^*(s)$ corresponds to the original $\dfrac{-1}{s_p \cdot \left(f^*(s_p)\right)'} e^{s_p t}$.

B) Each pair of *multiple real roots* s_p of the equation $f^{(*)}(s) = 1$ in expansion $H^*(s)$ corresponds to the sum $\dfrac{A_1}{s-s_p} + \dfrac{A_2}{(s-s_p)^2}$. Therefore, the original holds: $A_1 e^{s_p t} + A_2 t e^{s_p t}$.

C) Each pair of *complex conjugate roots* $s_p = \alpha \pm i\beta$ of the equation $f^{(*)}(s) = 1$ corresponds to a fraction in expansion $\dfrac{A\cdot(s-\alpha) + B\cdot\beta}{(s-\alpha)^2+\beta^2}$. In this case the original holds: $A\cdot e^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$.

With $k = 2$ the form of the restoration function is completely determined.

Note that with $k = 3$ either only case B, or only case C, or only case A (two simple real roots) is possible. Hence, there may be no more than two unknown parameters.

With $k = 4$ one of the roots is simple real and the two others are either simple real (case A) or multiple real (case B), or complex conjugate (case C). Hence, there also may be no more than two unknown parameters.

We can apply the method of arbitrary values to find the two unknown coefficients:

- reduce the right part of the equality $H^*(s) = \dfrac{1}{\mu}\dfrac{1}{s^2} + \dfrac{\sigma^2-\mu^2}{2\mu^2}\dfrac{1}{s} + R^*(s)$ to a common denominator;
- equate the numerator of the obtained fraction to the numerator of the fraction reduce to a common denominator reduce to a common denominator

$$H^{(*)}(s) = \frac{\lambda_0\lambda_1\cdot\ldots\cdot\lambda_{k-1}}{s\cdot\left[(s+\lambda_0)(s+\lambda_1)\cdot\ldots\cdot(s+\lambda_{k-1}) - \lambda_0\lambda_1\cdot\ldots\cdot\lambda_{k-1}\right]}$$

- substitute two arbitrary values in the equation and solve the system of two linear equations in two unknowns, for example, by Kramer's formulas.

Then, $H(t) = \dfrac{t}{\mu} + \dfrac{\sigma^2-\mu^2}{2\mu^2} + R(t)$.

2.3. Determining the mean delay of requests in the type 2 node for the true generalized Erlang distribution

We assume that a delay in the type 2 node is the time of waiting when the motion in the given direction is not allowed.

Calculate the conjectured total delay of the flow requests for T_i seconds – the time interval when the motion in the given direction is not allowed. We assume that at the moment of time $t = 0$ there is no queue at the node.

Split the interval $(0;T_i)$ into n pieces by points $t_1, t_2, \ldots, t_n$; $\Delta t_i = t_i - t_{i-1}$. Then the total delay of all requests of the given flow for one cycle of regulation $T = T_1 + T_2$ approximately is:

$W(T_i,\lambda) \approx \sum_{i=1}^{n} H(t_i)\cdot\Delta t_i$. Proceeding to the limit with $\max\{\Delta t_i\} \to 0$, we obtain: $W(T_i,\lambda) = \int_0^{T_1} H(t)dt$ (req / sec).

$$W(T_i,\lambda) = \int_0^{T_i}\left(\frac{t}{\mu} + \frac{\sigma^2-\mu^2}{2\mu^2} + R(t)\right)dt = \left(\frac{1}{\mu}\frac{t^2}{2} + \frac{\sigma^2-\mu^2}{2\mu^2}t\right)\Bigg|_0^{T_i} + \int_0^{T_i} R(t)dt$$

.

It was proved above that the function $R(t)$ can only consist of the following terms:

1) $Ae^{s_p t}$;
2) $A_1 e^{s_p t} + A_2 t e^{s_p t}$;
3) $A\cdot e^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$.

The corresponding terms are in the function $W(T_i,\lambda)$:

1) $\int_0^{T_i} Ae^{s_p t}dt = \left(\dfrac{A}{s_p}e^{s_p t}\right)\Bigg|_0^{T_i} = \dfrac{A}{s_p}\left(e^{s_p T_i} - 1\right)$;

2)

$$\int_0^{T_i}\left(A_1e^{s_pt}+A_2te^{s_pt}\right)dt=\left(\frac{A_1}{s_p}e^{s_pt}\right)\Bigg|_0^{T_i}+A_2\left(\frac{t}{s_p}e^{s_pt}-\frac{1}{s_p^{\,2}}e^{s_pt}\right)\Bigg|_0^{T_i}=$$

$$=\frac{A_1}{s_p}\left(e^{s_pT_i}-1\right)+\frac{A_2}{s_p}\left(e^{s_pT_i}-1\right)-\frac{A_2}{s_p^{\,2}}\left(e^{s_pT_i}-1\right);$$

3)

$$\int_0^{T_i}\left(A\cdot e^{\alpha t}\cos\beta t+Be^{\alpha t}\sin\beta t\right)dt=A\frac{\alpha^2}{\alpha^2+\beta^2}\left(e^{\alpha t}\cdot\left(\frac{1}{\alpha}\cos\beta t+\frac{\beta}{\alpha^2}\sin\beta t\right)\right)\Bigg|_0^{T_i}+$$

$$+B\frac{\alpha^2}{\alpha^2+\beta^2}\left(e^{\alpha t}\cdot\left(\frac{1}{\alpha}\sin\beta t-\frac{\beta}{\alpha^2}\cos\beta t\right)\right)\Bigg|_0^{T_i}=$$

$$=A\frac{\alpha^2}{\alpha^2+\beta^2}\left(e^{\alpha T_i}\cdot\left(\frac{1}{\alpha}\cos(\beta T)_i+\frac{\beta}{\alpha^2}\sin(\beta T_i)\right)-\frac{1}{\alpha}\right)\Bigg|+$$

$$+B\frac{\alpha^2}{\alpha^2+\beta^2}\left(e^{\alpha T_i}\cdot\left(\frac{1}{\alpha}\sin(\beta T_i)-\frac{\beta}{\alpha^2}\cos(\beta T_i)\right)+\frac{\beta}{\alpha^2}\right)\Bigg|.$$

The total delay of all requests of the given flow for the time interval of one hour is expressed as:

$$W(T_i,\lambda)\cdot\frac{3600}{T}\cdot\frac{1}{3600}=\frac{W(T_i,\lambda)}{T}\ \text{(req / sec)}.$$

If the number of arrivals at the node for one regulated cycle is less than the number of requests passing through the node for the interval during which the motion in the given direction is allowed, the queue will not grow and will become empty for one cycle.

Let $nA1$ be the number of non-conflict flows in direction A of Line 1,

h be the mean time interval (in seconds) between two requests of one flow arriving at the node;

$H_i(t)$ be the restoration function for the i-th flow of direction A of Line 1;

$W_i(T_i,\lambda)$ be the total delay of all requests of i-th flow for one cycle of regulation;

T_1 be the time interval during which the motion in direction A of Line 1 is not allowed (sec);

T_2 be the time interval during which the motion of flows of Line 2 is not allowed (sec);

$T = T_1+T_2$.

If $\sum_{i=1}^{nA1}H_i(T)-\frac{T_2}{h}nA1<0$, then the queue in this direction will become empty for one cycle and the total delay of all requests for one hour equals:

$$(T_\Sigma)=\left(\sum_{i=1}^{nA1}W_i(T_1,\lambda)\right)\frac{1}{(T_1+T_2)}$$

In much the same way is the chaining for the flows of other directions.

If $\sum_{i=1}^{nA1}H_i(T)-\frac{T_2}{h}nA1>0$, then то for the interval during which the motion in direction A is allowed, the queue will not become empty. In this case we have congestion in the given direction. ***Congestion*** is assumed as unlimited growth of the queue of the arrivals at the node.

The chaining is similar for the flows of other directions.

2.3. Determining the function of traffic costs

According to the theory of the flow equilibrium [9], in order to obtain the numerical values of equilibrium flow distribution it is necessary first to solve the construction of function of traffic costs problem. The most common assumption on the properties of the function of traffic costs is that of its additive dependence on particular links and nodes travel costs.

The function of traffic costs for the node depending on the aim of optimization can be chosen as:

1) $\vec{\mu}(z_n)$ – the weight of vertex z_n (a node-point) for a flow of the given direction;

2) $\mu(z_n)$ – the integrated weight of vertex z_n (a node-point);

3) $\omega_M(z_n)$ – the mean delay of requests of the chosen directions.

For a type 2 node:

1) $$\mu(z_n)=\frac{\sum_i W_i(T_1,\lambda)+\sum_j W_j(T_2,\lambda)}{T};$$

2) $$\vec{\mu}(z_n)=\frac{\sum_{i\in M}W_i(T_a,\lambda)}{T},$$ where M is the set of the chosen directions, $a\in\{1;2\}$;

3) $$\omega_M(z_n)=\frac{\sum_{i\in M}W_i(T_a,\lambda)}{\sum_{i\in M}H_i(T_a)},$$ where M is the set of the chosen directions, $a\in\{1;2\}$.

We developed Delpi-based computer programs, defining the form of the function of traffic costs in the node depending on the parameters of the generalized Erlang law.

Let Ψ be the set of modes of flow distribution for the given node of the graph. The optimal flow distribution at the node is the solution of the problem (according to the aim):

1) $\vec{\mu}(z_n)\_opt = \min\limits_{\Psi}\{\vec{\mu}(z_n)\}$;

2) $\mu(z_n)\_opt = \min\limits_{\Psi}\{\mu(z_n)\}$;

3) $\omega_M(z_n)\_opt = \min\limits_{\Psi}\{\omega_M(z_n)\}$.

3. Determining the parameters of the generalized Erlang law from experimental data

The above model applied, the parameters of the generalized Erlang law can be determined by the method of moments, i. e. by equating theoretical and empirical values of the mathematical expectation and variance.

Approximate to the whole number the variable $k^* = \frac{\bar{x}_B^2}{\hat{s}^2}$, choose parameter k.

With $k = 2$.

$$\begin{cases} \frac{1}{\lambda_0} + \frac{1}{\lambda_0} = \bar{x}_B \\ \frac{1}{(\lambda_0)^2} + \frac{1}{(\lambda_1)^2} = (\sigma_B)^2 \end{cases}$$

Then, if $(\sigma_B)^2 < \left(\bar{x}_B\right)^2 < 2(\sigma_B)^2$ the values of the parameters are:

$$\lambda_0 = \frac{2}{\bar{x}_B + \sqrt{2(\sigma_B)^2 - (x_B)^2}},$$

$$\lambda_1 = \frac{2}{\bar{x}_B - \sqrt{2(\sigma_B)^2 - (x_B)^2}}.$$

With $k = 3$.

$$\begin{cases} \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \bar{x}_B \\ \frac{1}{(\lambda_0)^2} + \frac{1}{(\lambda_1)^2} + \frac{1}{(\lambda_2)^2} = (\sigma_B)^2 \end{cases}$$

Let $\lambda_1 = x \cdot \lambda_0, \lambda_2 = x \cdot \lambda_1 = x^2 \cdot \lambda_0$, then

$$x = \frac{\left((\bar{x}_B)^2 + (\sigma_B)^2\right) + \sqrt{\left(-(\bar{x}_B)^2 + 3(\sigma_B)^2\right)\cdot\left(3(\bar{x}_B)^2 - (\sigma_B)^2\right)}}{2\left((\bar{x}_B)^2 - (\sigma_B)^2\right)}$$

, with $(\sigma_B)^2 < \left(\bar{x}_B\right)^2 < 3(\sigma_B)^2$.

After that, we calculate parameter λ_0:

$$\lambda_0 = \frac{x^2 + x + 1}{x^2} \cdot \frac{1}{\bar{x}_B}.$$

With $k = 4$.

$$\begin{cases} \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \bar{x}_B \\ \frac{1}{(\lambda_0)^2} + \frac{1}{(\lambda_1)^2} + \frac{1}{(\lambda_2)^2} + \frac{1}{(\lambda_3)^2} = (\sigma_B)^2 \end{cases}$$

Let

$\lambda_1 = x \cdot \lambda_0, \lambda_2 = x \cdot \lambda_1 = x^2 \cdot \lambda_0, \lambda_3 = x \cdot \lambda_2 = x^3 \cdot \lambda_0$.

Then, if $(\sigma_B)^2 < \left(\bar{x}_B\right)^2$ the values of the parameters will be as follows:

$$x = \frac{y \pm \sqrt{y^2 - 4}}{2},$$ where

$$y = \frac{(\sigma_B)^2 + \sqrt{\left((\bar{x}_B)^2 - (\sigma_B)^2\right)^2 + (\bar{x}_B)^4}}{(\bar{x}_B)^2 - (\sigma_B)^2};$$

$$\lambda_0 = \frac{(x^2+1)(x+1)}{x^3} \cdot \frac{1}{\bar{x}_B}.$$

Note that if $k^* = \frac{\bar{x}_B^2}{\hat{s}^2}$ is a whole number, then for all $k \in \{2, 3, 4\}$ the variable $x = 1$, and, therefore, all λ_i coincide. We obtain the Erlang special distribution which we have considered in detail in our previous works [11, 12].

Conclusions

The above results is our generalized research on optimization of traffic flow distribution in the network. The hypothesis on Erlang distribution of time intervals between the requests allowed us to develop the mathematical model providing satisfactory accuracy of estimation of the results of the network efficiency [16]. Besides, the minimal number of initial parameters made less expensive the development of the database for evaluation of quality of reorganization within the network.

A proper selection of parameters by the generalized Erlang law will allow one to approximate almost any distribution with the sufficient accuracy, and, therefore, will enable to extend the application of the model to the flows of higher density passing through a number of nodes. In its turn, this will allow for a greater accuracy when solving optimization problems in the theory of traffic flow.

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Corresponding Author:
Dr.Naumova Natalya
Applied Mathematics Department, Kuban State Technological University
2-A Moskovskaya Street, Krasnodar, 350072, Russia
e-mail: Nataly_Naumova@mail.ru

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