# Proof of an infinite amount of pairs of four primes

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**Abstract.** We created super-sieve of Eratosthenes for pairs of consecutive primes $\mathrm{p}_{n+1}-\mathrm{p}_n=4$ (pairs of fours), a buffer zone containing only those pairs after analyzing previous primes is introduced. It is analytically shown, that the increasing zone number, the number of pairs of fours increases also. Every buffer zone give rise to new buffer zone. This indicates to the infinite number of pairs-fours. Numerical experiments were carried out.

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## Introduction

Question about the number of pairs of twin primes (i.e. this number is finite or infinite) in modern number theory is open [1-4]. Similarly, the question about finite or infinite other pairs of primes - fours (sixes, quartets, etc.) is not resolved. In the recent article by Druzhinin [5], it was showed by method of creating super- sieve of Eratosthenes that the number of pairs of twin is indefinitely. Journal Annals of Mathematics [6] announced the results of the analysis of Ytang Zhang, where he proved that the number of consecutive pairs of primes, which have the distance 70000000 between each other (i.e. $p_{n+1}-p_n=70000000$), is infinite. At the time of this writing, the authors were not familiar with the work of Ytang Zhang, because it will be release in May 2014. But the our analysis is independent and, apparently, proves by a different method for the first time an infinite number of pairs-fours.

The pair of fours (PF) primes mean two consecutive primes $p_n$ and $p_{n+1}=p_n+4$. Do not confuse with pairs of twins of primes $p_{n+1}=p_n+2$ and with a quartet of primes, consisting of four consecutive primes $p_{n+3}=p_{n+2}+2=p_{n+1}+6=p_n+8$

In literature, the quartet is sometimes called "four" [7]. The first PF in a series of primes are the following:

$\{(7;11),(13;17),(19;23),(37;41),(43;47),\dots\}$

In this article we use Druzhinin's method the existence of super-sieve of Eratosthenes [5] for pairs of twins.

It turns out, that there exist good centers $\widehat{m}$ (natural numbers) for PF, that create this pair by the formula

$$(p_n;p_{n+1}=p_n+4)=(6\widehat{m}+3)\mp 2. \quad (1)$$

In addition, there are bad $\overline{m}$, not creating PF. For example, $\widehat{m}=17$ creates PF $(103;107)$, but $\overline{m}$ = 22 gives a pair $(133;137)$, that is not PF, because $s=133=7\cdot 19$ is composite number. To identify bad number $\overline{m}$, we made up a super-sieve of Eratosthenes for bad centers. Every good center $\widehat{m}$ creates four infinite arithmetic sequence (AS) giving only bad $\overline{m}$, which must be removed (cut) from the natural numbers. These AS have the form

$$\overline{m}=\widehat{m}+[(6\widehat{m}+3)\pm 2]k;$$

$$\overline{m}=(5\widehat{m}+2+2)+[(6\widehat{m}+3)+2](k-1). \quad (2)$$

In (2), all $k$ are natural numbers. For example, $\widehat{m}=1$ gives PF $(7;11)$ and removes bad $\overline{m}$ by four equations: $\overline{m}=1+7k,\ \overline{m}=5+7(k-1),\ \overline{m}=1+11k,\ \overline{m}=9+11(k-1)$.

Besides AS (2), there is also two contributions to bad $\overline{m}$. The first contribution is connected with a virtual PF $(1;5)$ with $\widehat{m}=0$. Actually there is not such PF, but removing bad $\overline{m}$ takes place. Here prime p = 5 take places and arise by (2) two the very first AS

$$\overline{m}=5k;\quad \overline{m}=4+5(k-1). \quad (3)$$

The second contribution to the bad number $\overline{m}$ is connected with bad $\overline{m}$ themselves. Thus, to protect the method from the appearance of hidden PF is necessary for each prime p from sequence of primes, starting with p = 5, that included or not included in the PF, to compose two AS

$\overline{m} = m(p) + p(k-1); \overline{m} = (p - m(p) - 1) + p(k-1)$. (4)

The initial number of AS $m(p)$ is found from the equation $((p-3 \pm 2)/6) \in \mathbb{N}$. For example, $p = 29$ not created PF, but $(29 - 3 \pm 2)/6$ with sign "$-$" is «4», i.e. $m(p) = 4$. In this case we have two AS $\overline{m} = 4 + 29(k-1)$ и $\overline{m} = 24 + 29(k-1)$.

As an example, we present an initial set of good and bad $m$.

$\{\hat{1}; \hat{2}; \hat{3}; \overline{4}; \overline{5}; \hat{6}; \hat{7}; \overline{8}; \overline{9}; \overline{10}; \hat{1}1; \overline{12}; \hat{1}3; \overline{14}; \overline{15}; \hat{1}6; \hat{1}7; \hat{1}8; \overline{19}; \overline{20}; \ldots\}$.

Underlined numbers are buffer zone $D(\hat{3}) = [\hat{3},\ 5 \cdot \hat{3} - 1]$ (see further).

We verified this super-sieve of Eratosthenes to numerically verified prior to $m = 10^8$ and obtained results completely coincide with the formulas (1-4). We obtained following values for the last five PF (see table # 1)

**Table # 1 Example PF for large values of $\hat{m}$**

| $p_n$ | $p_{n+1}$ | $\hat{m}$ |
|---|---|---|
| 599997193 | 599997197 | 99999532 |
| 599997313 | 599997317 | 99999552 |
| 599997913 | 599997917 | 99999652 |
| 599998519 | 599998523 | 99999753 |
| 599999467 | 599999471 | 99999911 |

Note some features of obtained super-sieve of Eratosthenes for PF.

**1**. Calculation of the PF starts with $(7; 11)$.

**2**. In the process of division m into good and bad values, we verify all primes without exception, both included in the PF, and not included in a pair. This means that the proposed super-sieve of Eratosthenes, as well as the sieve of Eratosthenes generates all primes.

**3**. The first $m$ that is not removed when sequential selection, starting with the first prime numbers $\{p_1 = 5;\ p_2 = 7; p_3 = 11; \ldots; p_n\}$, take place, is a good $\hat{m}$, forming a PF. This is due to the following circumstance. Since we cut out all composite numbers at this set of primes, then if the number $\hat{m}$ is bad, it gives a pairs which contains composite number. It is the product of prime numbers from the specified set. By selection (2), there cannot be it. So the first not removed $m$, as well as in the classical sieve of Eratosthenes, is a good number and creates PF.

For example, $\hat{m} = 0$ to (3) removes $\overline{m} = \{4; 5; 9; 10; 14; 15; \ldots\}$. The first not removed $m = 1$ is the center of PF (7;11). Further, this number $\hat{m} = 1$ to (2) removes bad $\overline{m} = \{5; 8; 9; 12; 15; 19; \ldots\}$. The first not removed $m = 2$ is the second center PF (13; 17). This number $\hat{m} = 2$ to (2) removes bad $\overline{m} = \{10; 14; 15; 19; \ldots\}$. The first not removed $m = 3$ is the third center PF (19; 23).

**4**. Every new $m$ not only is $\hat{m}$ (i.e. that creates PF), but also creates a buffer zone - interval $D(\hat{m}) = [\hat{m},\ 5\hat{m} - 1]$, in which all numbers $m$, not deleted by previous analysis, are good $\hat{m}$, i.e. not deleted m are centers of new the PF. Why does it happen? According to (2) the deletion of bad $\overline{m}$ begins as: $\{5\hat{m};\ 5\hat{m} + 4; 7\hat{m} + 1; 7\hat{m} + 5; \ldots\}$. Therefore, at the interval $[\hat{m},\ 5\hat{m} - 1]$ there are no new deleted $m$ and all $m$, not cutted earlier, are good $\hat{m}$.

The presence of buffer zones plays a key role in the proof of an infinite number of PF. The fact is that if the buffer zones have at least two centers of PF, then the second center forms a new buffer zone in which there are other centers. The chain of overlapping buffer zones covers all the natural numbers and therefore the number of PF is infinite. Real buffer zones for the considered examples is the following: $D(1) = [1,\ 4]$; $D(2) = [2,\ 9]$; $D(3) = [3,\ 14]$; $D(6) = [6,\ 29]$. In a set of good and bad m (see above) the buffer zone $D(3)$ were underlined.

Let's discuss the amount of PF in a particular buffer zone. Suppose, we have analyzed $n$ primes

$a_n - \{p_1 - 5; p_2 - 7; p_3 - 11; p_4 - 13; \dots; p_n\}$

Expunging bad $\overline{m}$ are carried out by $2n$ AS: $\{m(p_k) + p_k \cdot t;\ (p_k - m(p_k) - 1) + p_k \cdot t\}$

All primes $p_k$ in a row get over, $m(p_k)$ is found by the formula

$$m(p_k) = \frac{p_k - 3 \pm 2}{6} \in \mathbb{N}. \qquad (5)$$

There is a Chinese Remainder Theorem about imposing of several AS with two different initial terms and mutually simple differences. At the interval of $T_n = [3, B_n]$, where $B_n = \prod_{k=1}^{n} p_k$, at the same time there are no removed (not crossed out) $S_n = \prod_{k=1}^{n}(p_k - 2)$ numbers. Therefore, the probability of finding not crossed out $m$ (we do not know yet good or bad it) equals $W_n = S_n / B_n = \prod_{k=1}^{n}(1 - 2/p_k)$. This formula is well known and used in the study of pairs of twins [8-10]. Since in the buffer zone, all, not have been crossed out earlier numbers are new centers of PF, then according to the theory of probability the number of new the PF $N_{n\ TB} = (4\hat{m})W_n$. Here $4\hat{m}$ is the number of favorable events — the length of the buffer zone.

To assess $\ln W_n = \sum_{k=1}^{n} ln(1 - 2/p_k)$ use the inequality

$(-2{,}6/p_k) < ln(1 - 2/p_k)$, that leads to the relation

$$\ln W_n > -2{,}6 \sum_{k=1}^{n} \left(\frac{1}{p_k}\right) \approx -2{,}6\,[ln(lnp_n) + lnC] - ln\frac{1}{C^{2,6} ln^{2,6}(p_n)} . \qquad (6)$$

Here we used the formula of the Lagrange-Chebyshev ( $C \approx 0{,}82)$ for the partial sum of inverse primes. From (6) it is followed that the number of new PF in the buffer zone

$$N_{n\ TB} > \frac{(4\hat{m})}{C^{2,6} ln^{2,6}(p_n)} > \frac{2(p_n - 5)}{3C^{2,6} \cdot ln^3(p_n)}. \qquad (7)$$

Right side of inequality with increasing $p_n$ increases and in the limit tends to infinity. This indicates the growth of PF in the buffer zones and their infinite amount at a number line. Numerical calculations up to $m = 10^8$ show that the real number $N_n$ PF in the buffer zones also in middle grows and slightly exceeds the probability value. Table # 2 shows a fragment of a direct calculation of the number of PF $N_n$ in consequence the buffer zones

**Table #2. Increase of the number of PF $N_n$ in $D(\hat{m})$**

| $\hat{m}$ | $5\hat{m} - 1$ | $N_n$ |
|---|---|---|
| 553 | 2764 | 213 |
| 557 | 2784 | 215 |
| 576 | 2879 | 219 |
| 577 | 2884 | 218 |
| 588 | 2939 | 223 |
| 602 | 3009 | 224 |
| 612 | 3059 | 227 |
| 616 | 3079 | 230 |

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