Modeling of Human Postural Balance Using Neutral Delay Differential Equation to Solvable Lie Algebra Classification

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Abstract: The application of second order neutral delay differential equation (NDDE) to solvable Lie algebra classification in modeling human postural balance of ankle joint is reported. The process of human walking controlled via this model is of immense importance for computational neuroscientists. The model is based on inverted pendulum which introduces time–delayed feedback. These delays being intrinsic components of neural control can greatly influence the interpretation of human tasks such as stick balancing at the fingertip and postural sway during quiet standing and makes the problem unsolvable. The postural balance is examined using group analysis where the control force is activated only for motions exceeding some thresholds. NDDE that originates in the setting of an inverted pendulum is represented by solvable Lie algebra. The classification is completed following the second order extension of the general infinitesimal generator acting on second order NDDE. This is further used for achieving the determining equations for infinitesimal symmetry group. The equations are then solved and the Lie algebras spanned by these corresponding parameters in infinitesimal are obtained. The obtained Lie algebras satisfying the inclusion property render a solvable Lie algebra. All the model properties described by such NDDE with constant coefficients are determined. This model may contribute towards the understanding of many of real problems described by NDDE.

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1. Introduction

Modeling of human postural balancing (HPB) including ankle joint is an outstanding issue for computational neuroscientists (Milton et al., 2008; Milton, 2011). This might be the key to the successful understanding of human walking with numerous medical implications (Piiroinen and Dankowicz, 2005). The existing model depending on inverted pendulum introduces time–delayed feedback. The observation greatly influences the interpretation of human balancing tasks because these time delays are intrinsic components of neural control such as stick balancing at the fingertip and postural sway during quiet standing (Mehta and Schall, 2002; Cabrera and Milton, 2002; Loram et al., 2005; Milton et al., 2009; Asai et al., 2009; Stepan, 2009). Generally, these interpretations are based on a proportional–derivative (PD) controller where the corrective movements depend on the angular position and velocity (Mehta and Schall, 2002; Stepan, 2009; Milton et al., 2009; Bingham et al., 2011; Kowalczyk et al., 2012; Paoletti et al., 2012). It is demonstrated that the control is highly benefited by various inputs including mechano–receptive sensors (tactile or force detectors), proprioceptive sensors (muscle spindle) and vestibular labyrinth (otoliths and semicircular canals). In addition to sensory inputs, information on acceleration can also be obtained from the mathematical model of inverted pendulum (Bottaro et al., 2008). These suggest that an extension of PD to proportional–derivative–acceleration (PDA) controller is essential (Gomi and Kawato, 1992; Peterka, 2003; Sieber and krauskopf, 2005; Welch and Ting, 2008).

In computational neuroscience, the PDA feedback without time delay for motor plants is already been introduced (Gomi and Kawato, 1992; Gomi and Kawato, 1993). The governing equations for PDA controller are NDDE because the delay is present in the argument of the highest derivative (acceleration). A NDDE possesses several roots with positive real part. Thus, apart from the usual engineering objectives of using the 'noisy' acceleration signals in the feedback loops, the control design of such systems requires special care (Insperger et al., 2014). Lately, the complexities of NDDE triggered enormous interests in mathematical modeling (Kolmanovskii and Nosov, 1986; Stepan, 1989; Hale and Lunel, 1993; Malakhovski and Mirkin, 2006). Despite few studies the PDA controller based methods are far from being developed. The difficulty in solving such system allowed the researchers to just examine their stability (Insperger et al., 2014; Lafortune and Lake, 1995; Coveney et al., 2001; Kyrychko et al., 2006a; Kyrychko et al., 2006b). In our view, the best way to study the properties of the solution of these equations is group analysis.

The NDDE that arises in the setting of an inverted pendulum are studied using the classification to solvable Lie algebra. A model for human postural balance of ankle joint is developed. The second order extension of the general infinitesimal generator acting on such NDDE is used to develop the model. The determining equations for infinitesimal symmetry group are obtained and solved. Lie algebras spanned by the parameters satisfying the inclusion property provided a solvable Lie algebra. The model properties are determined, analyzed and understood. This paper is organized as follows. Section 2 described the mathematical model of HPB. The details of Lie algebra and DDE are highlighted in Section 3. A classification of HPB equation to solvable Lie algebra is provided in Section 4. Section 5 concludes the paper.

2. Mathematical Model

Figure 1 displays a postural balance model for human body by using a rod of mass *m* pivoted on joint *A* (Loram and Lakie, 2002). The distance between the center of gravity *C* and the suspension point *A* is denoted by ℓ_{AC} and J_C is the moment of inertia with respect to the normal line via the center of gravity. The passive but insufficient resistance of the ankle joint against falling is modeled by a torsional spring of stiffness w_t and a torsional dashpot of damping b_t. Furthermore, the elements attributing to the foot such as Achilles' tendon and aponeurosis cannot be regulated neurally during quiet standing. The stiffness increases slightly with ankle torque (y) confirming the linear spring model.

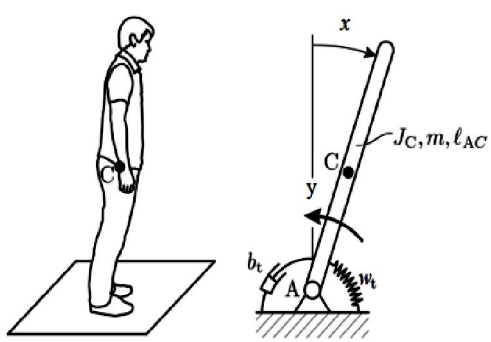

Figure 1: Mathematical model for postural balance.

Denoting the angle between body and vertical by x, the equation of motion takes the form,

$$J_A x''(t) + b_t x'(t) + w_t x(t) - mg\ell_{AC}\sin(x(t)) = -y(t), \quad (1)$$

where $J_A = J_C + m\ell_{AC}^2$ is the moment of inertia of the body with respect to the normal line via the pivot point *A*, and g = 9.81 m/s^2 is the gravitational acceleration. The control torque $y(t)$ is assumed to be a linear combination of the angular position x, the angular velocity x' and the angular acceleration x''. They are obtained from mechano-receptive and proprioceptive sensors, vestibular organs and visual inputs.

The presence of a sensory dead zone is considered by assuming that the actuating forces occur only if the input signals exceed some threshold values (Milton et al., 2009; Kowalczyk et al., 2012; Eurich and Milton, 1996; Delignières et al., 2011). Furthermore, the overall reaction time is modeled as a feedback delay. Considering different thresholds for various sensation inputs, the control torque is expressed as,

$$y(t) = y_p(t) + y_d(t) + y_a(t), \quad (2)$$

with

$$y_p(t) = \begin{cases} 0 & if\,|x(t-\tau)| < x_s \\ -W_p x(t-\tau) & if\,|x(t-\tau)| \ge x_s \end{cases} \quad (3)$$

$$y_d(t) = \begin{cases} 0 & if\,|x'(t-\tau)| < x'_s \\ -W_d x'(t-\tau) & if\,|x'(t-\tau)| \ge x'_s \end{cases} \quad (4)$$

$$y_a(t) = \begin{cases} 0 & if\,|x''(t-\tau)| < x''_s \\ -W_a x''(t-\tau) & if\,|x''(t-\tau)| \ge x''_s \end{cases} \quad (5)$$

where W_p , W_d and W_a are the proportional, derivative and acceleration gain, respectively. τ is the time delay also appears for threshold conditions and x_s, x'_s and x''_s are the sensory threshold values for the angular position, angular velocity and the angular acceleration, respectively. It is worth noting that the control force is activated only for motions exceeding some thresholds (Melton et al., 2009; Eurich and Milton, 1996). For small motions, when the state variables are within the sensory dead zone (without control) equation (1) can be combined with (2) – (5) in the absence of domain of attraction around $(x, x', x'') = (0,0,0)$ to achieve a linear system (Haller and Stepan, 1998; Enikov and Stepan, 1998) for digital balancing given by,

$$J_A x''(t) + b_t x'(t) + (w_t - mg\ell_{AC})x(t) = \\ -W_p x(t-\tau) - W_d x'(t-\tau) - W_a x''(t-\tau), \quad (6)$$

The nonlinear system has an attractor (a limit cycle or a chaotic attractor) around $(x, x', x'') = (0,0,0)$ with $w_t - mg\ell_{AC} < 0$ (Loram and Lakie, 2002). Present controller is intermittent because the control force is switched on and off depending on the size of the sensory inputs. The corresponding switching condition is defined in the phase plane via $x(t-\tau)(x'(t-\tau) - wx(t-\tau)) >$ 0,(with $w \le 0$) for the controller to be on and off otherwise. Following Asai et al.(2009), the sensory dead zone is modeled through the appearance of intermittency. By rescaling the time and dropping the tilde immediately (Insperger et al., 2014), equation (6) is transformed as,

$$x''(t) + bx'(t) - ax(t) = -w_p x(t-1) - w_d x'(t-1) \\ -w_a x''(t-1), \quad (7)$$

Where

$$b = \frac{b_t \tau}{J_A}, \qquad a = \frac{(mg\ell_{AC} - w_t)\tau^2}{J_A} > 0, \quad (8)$$

$$w_p = \frac{W_p \tau^2}{J_A}, \quad w_d = \frac{W_d \tau}{J_A}, \quad w_a = \frac{W_a}{J_A}, \quad (9)$$

The occurrence of second derivative of the state variable (x'') with both the actual and delayed arguments of NDDE in equation (7) makes it difficult to study. Researchers are mostly interested in the stability analyses of this equation (Insperger et al., 2014). However, this equation is very significant for neuroscientists because it might be the key to control of human postural balance. Determination of the properties of this equation may reveal HPB. Therefore, it is vital to classify it to solvable Lie algebra to achieve the solution easily (for solvable Lie algebra is easy to solve see (Bluman and Kumei, 1989)). The classification is performed in few steps. Firstly, the second order extension of the general infinitesimal generator acting on NDDE is used to obtain the determining equations for infinitesimal symmetry group. Secondly, the equations are solved and the Lie algebras spanned by these corresponding parameters in infinitesimal are acquired. Finally, the obtained Lie algebras satisfying the inclusion property provide the solvable one. Now, it is customary to underscore a glimpse on Lie algebra and DDEs.

3. Methodology

The classification of equation (7) to solvable Lie algebra is proposed. Some striking features of Lie algebra and DDEs are hereunder.

Definition 3.1 (Andreas, 2009)**:** A Lie algebra L is an n-dimensional solvable Lie algebra if a sequence exists that yields,

$$L_1 \subset L_2 \subset \ldots \subset L_n = L,$$

where L_k is a k-dimensional Lie algebra and L_{k-1} is an ideal of $L_k, k = 1,\ldots,n$ in which two-dimensional Lie algebra are solvable.

Definition 3.2 (Bluman and Kumei, 1989): Let $Q_i = \zeta_s \frac{\partial}{\partial x_s}$ and $Q_j = \eta_s \frac{\partial}{\partial x_s}$, $i, j = 1,\ldots,r$ and $s = 1,\ldots,n$ be two infinitesimal generator. The commutator $[Q_i, Q_j]$ of Q_i and Q_j is the first order operator such that,

$$[Q_i, Q_j] = Q_i Q_j - Q_j Q_i = \\ \sum_s^n \sum_m^n \left(\zeta_m \frac{\partial \eta_s}{\partial x_m} - \eta_m \frac{\partial \zeta_s}{\partial x_m}\right) \frac{\partial}{\partial x_s}.$$

Definition 3.3 (Humi and Miller, 1988): A finite set of infinitesimal generator $\{Q_1, Q_2, \ldots, Q_r\}$ is said to be a basis for the Lie algebra L if $Q_i \in L$ and

1. $Q_1, Q_2, \ldots, Q_r$ from a basis of the vector space L,
2. $[Q_i, Q_j] = c_{ijk} Q_k$.

Where the coefficients c_{ijk} for $i, j, k = 1,2,\ldots,r$, are called the structure constant of the Lie algebra.

Theorem 2.1 (Second Fundamental Theorem of Lie) (Bluman and Kumei, 1989): Any two infinitesimal generators of an r-parameter Lie group satisfying commutation relation of the form $[Q_i, Q_j] = c_{ijk} Q_k$ form a basis for Lie algebra.

Theorem 3.5 (Third Fundamental Theorem of Lie) (Bluman and Kumei, 1989): The structure constants satisfy the following relations,

$$c_{ijk} = -c_{jik},$$

$$c_{ijk} c_{klm} + c_{jlk} c_{kim} + c_{lik} c_{kim} = 0.$$

In other word, the structure constants determine the Lie algebra and hence the Lie group.

Lemma 3.6 (Muhsen and Maan, 2014): The second order NDDEs containing the infinitesimal generator ξ that obeys periodic property is given by $\xi(t, x) = \xi(t-\tau, x_\tau)$.

Lemma 3.6 implies that ξ is independent of x.

Assume,

$$x'' = f(t, x, x_\tau, x', x'_\tau, x''_\tau), \quad (10)$$

where $x = x(t), x' = x'(t), x'' = x''(t), x_\tau = x_\tau(t-\tau)$, $x'_\tau = x'_\tau(t-\tau)$, and $x''_\tau\, x''_\tau(t-\tau)$. Equation (10) is a second order NDDE. Using Lemma 3.6, the determining equation for (10) is derived in the form of,

$$X^{(2)}\left(x'' - f(t,x,x_\tau,x',x'_\tau,x''_\tau)\right)\Big|_{(10)} = 0,$$

where

$$X^{(2)} = \xi\frac{\partial}{\partial t} + \eta^x\frac{\partial}{\partial x} + \eta^{x_\tau}\frac{\partial}{\partial x_\tau} + \eta^{x'}\frac{\partial}{\partial x'} + \eta^{x'_\tau}\frac{\partial}{\partial x'_\tau} + \eta^{x''}\frac{\partial}{\partial x''} + \eta^{x''_\tau}\frac{\partial}{\partial x''_\tau}, \qquad (11)$$

and

$$\eta^x(t,x) = \eta(t,x),$$

$$\eta^{x_\tau}(t,x_\tau) = \eta(t-\tau,x_\tau),$$

$$\eta^{x'}(t,x,x') = \eta_1(t,x,x') = \eta_t(t,x) + [\eta_x(t,x) - \xi_t(t,x)]x' - \xi_x(t,x)(x')^2,$$

$$\eta^{x'_\tau}(t,x_\tau,x'_\tau) = \eta_1(t-\tau,x_\tau,x'_\tau) = \eta_t(t-\tau,x_\tau) + [\eta_x(t-\tau,x_\tau) - \xi_t(t-\tau,x_\tau)]x'_\tau - \xi_x(t-\tau,x_\tau)(x')^2,$$

$$\eta^{x''}(t,x,x',x'') = \eta_2(t,x,x',x'') = \eta_{tt}(t,x) + [2\eta_{tx}(t,x) - \xi_{tt}(t,x)]x' + [\eta_{xx}(t,x) - 2\xi_{tx}(t,x)](x')^2 - \xi_{xx}(t,x)(x')^3 + [\eta_x(t,x) - 2\xi_t(t,x)]x'' - 3\xi_x(t,x)x'x'',$$

$$\eta^{x''_\tau}(t,x_\tau,x'_\tau,x''_\tau) = \eta_2(t-\tau,x_\tau,x'_\tau,x''_\tau) = \eta_{tt}(t-\tau,x_\tau) + [2\eta_{tx}(t-\tau,x_\tau) - \xi_{tt}(t-\tau,x_\tau)]x'_\tau + [\eta_{xx}(t-\tau,x_\tau) - 2\xi_{tx}(t-\tau,x_\tau)](x'_\tau)^2 - \xi_{xx}(t-\tau,x_\tau)(x'_\tau)^3 + [\eta_x(t-\tau,x_\tau) - 2\xi_t(t-\tau,x_\tau)x''_\tau - 3\xi_x(t-\tau,x_\tau)x'_\tau x''_\tau, \qquad (12)$$

4. Classification of HPB Equation to solvable Lie algebra

Consider that w_a is not equal to zero, equation (7) becomes second order NDDE. Now, rewrite equation (7) in the solvable form as,

$$x''(t) + bx'(t) - ax(t) + w_p x(t-1) + w_d x'(t-1) + w_a x''(t-1) = 0. \qquad (13)$$

The general infinitesimal generator of HPB can be written as,

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \eta^\tau\frac{\partial}{\partial x_\tau}, \qquad (14)$$

where $\eta^\tau = \eta(t-\tau,x_\tau)$.

The second extension of (14) acting on neutral delay is required because HPB is a second order NDDE. The second order extension of equation (11) is given by,

$$X^{(2)} = X + \eta^{x'}\frac{\partial}{\partial x'} + \eta^{x'_\tau}\frac{\partial}{\partial x'_\tau} + \eta^{x''}\frac{\partial}{\partial x''} + \eta^{x''_\tau}\frac{\partial}{\partial x''_\tau},$$

(15)

where $x = x(t)$, $x' = x'(t), x'' = x''(t)$, $x'_\tau = x'(t-1), x''_\tau = x''(t-1)$.

Combination of Equation (15) with (13) yields the invariance condition as,

$$\eta_2 + b\eta_1 - a\eta + w_p\eta^\tau + w_d\eta_1^\tau + w_a\eta_2^\tau = 0.$$

Now, substituting Equation (12) in this equation one gets,

$$\eta_{tt} + [2\eta_{tx} - \xi_{tt}]x' + [\eta_{xx} - 2\xi_{tx}](x')^2 - \xi_{xx}(x')^3 + [\eta_x - 2\xi_t]x'' - 3\xi_x x'x'' + b[\eta_t + [\eta_x - \xi_t]x' - \xi_x(x')^2] - a\eta + w_p\eta^\tau + w_d[\eta_t^\tau + [\eta_x^\tau - \xi_t^\tau]x'_\tau - \xi_x^\tau(x'_\tau)^2] + w_a[\eta_{tt}^\tau + [2\eta_{tx}^\tau - \xi_{tt}^\tau]x'_\tau + [\eta_{xx}^\tau - 2\xi_{tx}^\tau](x'_\tau)^2 - \xi_{xx}^\tau(x'_\tau)^3 + [\eta_x^\tau - 2\xi_t^\tau]x''_\tau - 3\xi_x^\tau x'_\tau x''_\tau = 0,$$

where $\xi^\tau = \xi(t-\tau,x_\tau)$.

Equating the first and second order coefficients of the various monomials of x and x_τ, the determining equations for the symmetry group of Equation (13) as listed in Table 1 are obtained.

Table 1. The determining equations for the symmetry group obtained using Equation (13)

Monomial	**Coefficient**	**Number of equation**
1	$\eta_{tt} + b\eta_t - a\eta + w_p\eta^\tau + w_d\eta_t^\tau + w_a\eta_{tt}^\tau = 0$	(a[1])
x'	$2\eta_{tx} - \xi_{tt} + b[\eta_x - \xi_t] = 0$	(a[2])
$(x')^2$	$2\eta_{tx} - \xi_{tt} - b\xi_x = 0$	(a[3])
$(x')^3$	$\xi_{xx} = 0$	(a[4])
x''	$\eta_x - 2\xi_t = 0$	(a[5])
$x'x''$	$\xi_x = 0$	(a[6])
x'_τ	$w_a[2\eta_{tx}^\tau - \xi_{tt}^\tau] - w_d[\eta_x^\tau - \xi_t^\tau] = 0$	(a[7])
$(x'_\tau)^2$	$w_a[\eta_{xx}^\tau - 2\xi_x^\tau] - w_d\xi_x^\tau = 0$	(a[8])
$(x'_\tau)^3$	$w_a\xi_{xx}^\tau = 0$	(a[9])
x''_τ	$w_a[\eta_x^\tau - 2\xi_t^\tau] = 0$	(a[10])
$x'_\tau x''_\tau$	$w_a\xi_x^\tau = 0$	(a[11])

From (a[6]) it is clear that ξ is independent of x and (a[3]) exhibits η is linear in x. Here,

$\eta = g(t)x + h(t)$, with $g(t)$ and $h(t)$ as arbitrary functions of t. Using, (a_5) one obtains,

$$\xi_t = \frac{1}{2} g \tag{16}$$

Since w_a is not equal to zero, then (a_{11}) shows that ξ^τ is independent of x.

Using Lemma 3.6 we get $\xi = \xi^\tau$, then $\xi_t = \xi_t^\tau$ implying that $\xi_t^\tau = \frac{1}{2} g$. However, ($a_{10}$) renders $\eta_x^\tau = 2\xi_t^\tau$ and $\eta_x^\tau = g$. Thus, $\eta^\tau = g(t)x + k(t-\tau)$ with $k(t-\tau)$ is as arbitrary function.

Using the coefficients (a_1) one gets,

$$g_{tt}x + h_{tt} + bg_t x + bh_t - agx - ah + w_p gx + w_p k$$
$$+ w_d g_t x + w_d k_t + w_a g_{tt} x + w_a k_{tt} = 0.$$

Different cases are analyzed below.

Case 1: if $a \neq w_p$

By equating the coefficients of the various monomials one obtains,

$$(1+w_a)g_{tt} + (b+w_d)g_t + (w_p - a)g = 0\,, \tag{17}$$

$$h_{tt} + bh_t - ah + w_p k + w_d k_t + w_a k_{tt} = 0\,, \tag{18}$$

which means that $g(t), h(t)$ and $k(t-\tau)$ are the solutions of (13). Equation (17) results,

$$g = \frac{1+w_a}{a-w_p} g_{tt} + \frac{b+w_d}{a-w_p} g_t\,,$$

Equation (16) produces,

$$2\xi_t = \frac{2(1+w_a)}{a-w_p} \xi_{ttt} + \frac{2(b+w_d)}{a-w_p} \xi_{tt}\,,$$

By integration one gets,

$$\xi = \frac{1+w_a}{2(a-w_p)} g_t + \frac{b+w_d}{2(a-w_p)} g + c_1,$$

and

$$g = c_2 - \frac{(1+w_a)}{b+w_d} g_t + \frac{2(a-w_p)}{b+w_d} \xi.$$

From above we get,

$$\eta = \left(c_2 - \frac{(1+w_a)}{b+w_d} g_t + \frac{2(a-w_p)}{b+w_d} \xi \right) x + h,$$

and

$$\eta^\tau = \left(c_2 - \frac{(1+w_a)}{b+w_d} g_t + \frac{2(a-w_p)}{b+w_d} \xi \right) x + k, \text{ where}$$

c_1, c_2 are arbitrary constants.

Recalling Equation (14) as the general infinitesimal generator for (13) one achieves,

$$X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \eta^\tau \frac{\partial}{\partial x_\tau}.$$

Let $x_\tau = u$, then

$$X = \left(\frac{1+w_a}{2(a-w_p)} g_t + \frac{b+w_d}{2(a-w_p)} g + c_1 \right) \frac{\partial}{\partial t} +$$
$$\left((c_2 - \frac{(1+w_a)}{b+w_d} g_t + \frac{2(a-w_p)}{b+w_d} \xi) x + h \right) \frac{\partial}{\partial x}$$
$$+ \left((c_2 - \frac{(1+w_a)}{b+w_d} g_t + \frac{2(a-w_p)}{b+w_d} \xi) x + k \right) \frac{\partial}{\partial u}.$$

The Lie algebra of the HPB equation is spanned by the following three infinitesimal generators corresponding to each parameter c_1, c_2 to attain,

$$Q_1 = \frac{\partial}{\partial t}, \quad Q_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right).$$

$$Q_3 = \left(\frac{1+w_a}{2(a-w_p)} g_t + \frac{b+w_d}{2(a-w_p)} g \right) \frac{\partial}{\partial t} +$$
$$\left((\frac{2(a-w_p)}{b+w_d} \xi - \frac{1+w_a}{b+w_d} g_t) x + h \right) \frac{\partial}{\partial x} +$$
$$\left((\frac{2(a-w_p)}{b+w_d} \xi - \frac{1+w_a}{b+w_d} g_t) x + k \right) \frac{\partial}{\partial u},$$

where Q_3 is an infinite dimensional Lie sub-algebra.

Definition 3.1 renders $L_2 = \{Q_1, Q_2\}$ as solvable Lie algebra of the HPB equation. In fact, the expression for L_2 with the Lie sub-algebra Q_3 encloses all the properties of HPB equation. To study this equation one needs Q_1, Q_2, Q_3. Now, the space is solvable signifying their easy implementation.

Case 2: If $a = w_p$.

By equating the coefficients of the various monomials we get,

$$(1+w_a)g_{tt} + (b+w_d)g_t = 0, \tag{19}$$

$$h_{tt} + bh_t - ah + w_p k + w_d k_t + w_a k_{tt} = 0, \tag{20}$$

which implies $g(t), h(t)$ and $k(t-\tau)$ are the solutions of (13). From (19), we have

$$g_t = -\frac{(1+w_a)}{(b+w_d)} g_{tt}. \tag{21}$$

Substituting Equation (16) in (21) we get,

$$\xi_{tt} = -\frac{(1+w_a)}{(b+w_d)}\xi_{ttt},$$

Integrating both side to obtain,

$$\xi = -\frac{(1+w_a)}{(b+w_d)}\xi_t + c_1 t + c_2,$$

$$\xi = -\frac{(1+w_a)}{2(b+w_d)}g + c_1 t + c_2,$$

$$g = c_4 t + c_3 - \frac{2(b+w_d)}{(1+w_a)}\xi.$$

Combining the above equations we gets,

$$\eta = (c_4 t + c_3 - \frac{2(b+w_d)}{(1+w_a)}\xi)x + h,$$

and

$$\eta^\tau = (c_4 t + c_3 - \frac{2(b+w_d)}{(1+w_a)}\xi)x + k,$$

where $c_i, i = 1,\ldots,4$ are arbitrary constants.

Recalling Equation (14) as the general infinitesimal generator for (13) one achieves,

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \eta^\tau\frac{\partial}{\partial x_\tau}.$$

Let $x_\tau = u$, then

$$X = \left(c_1 t + c_2 - \frac{(1+w_a)}{2(b+w_d)}g\right)\frac{\partial}{\partial t} + \left((c_4 t + c_3 - \frac{2(b+w_d)}{(1+w_a)}\xi)x + h\right)\frac{\partial}{\partial x} + \left((c_4 t + c_3 - \frac{2(b+w_d)}{(1+w_a)}\xi)x + k\right)\frac{\partial}{\partial u}.$$

The Lie algebra of the HPB equation is spanned by the following three infinitesimal generators corresponding to each parameter c_i.

$$Q_1 = t\frac{\partial}{\partial t},\ Q_2 = \frac{\partial}{\partial t},\ Q_3 = x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}),$$

$$Q_4 = tx(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}),$$

with infinite dimensional Lie sub-algebra

$$Q_5 = \left(-\frac{(1+w_a)}{2(b+w_d)}g\right)\frac{\partial}{\partial t} + \left(h - \frac{2(b+w_d)}{(1+w_a)}\xi x\right)\frac{\partial}{\partial x} + \left(k - \frac{2(b+w_d)}{(1+w_a)}\xi x\right)\frac{\partial}{\partial u}.$$

To prove the solvability of the Lie algebra $L_4 = \{Q_1, Q_2, Q_3, Q_4\}$ the following commutator table is generated by using definition 3.2.

$[Q_i, Q_j]$	Q_1	Q_2	Q_3	Q_4
Q_1	0	$-Q_2$	0	Q_4
Q_2	Q_2	0	0	Q_3
Q_3	0	0	0	0
Q_4	$-Q_4$	$-Q_3$	0	0

Here, $L_4 = \{Q_1, Q_2, Q_3, Q_4\}$ is spanned by Q_1, Q_2, Q_3, Q_4 is the Lie algebra of Equation (13).

The sub-space $L_1 = \{Q_1\}$, $L_2 = \{Q_1, Q_2\}$, $L_3 = \{Q_1, Q_2, Q_3\}$ is one, two and three dimensional Lie sub-algebra of L_4, respectively. Furthermore, they obey the inclusion property such that,

$$L_1 \subset L_2 \subset L_3 \subset L_4,$$

and hence L_4 is a solvable Lie algebra of HPB equation.

The solvable Lie algebra L_4 with the Lie sub-algebra Q_5 includes all the properties of HPB equation. To study this equation one needs to examine Q_1, Q_2, Q_3, Q_4, Q_5 where the space is easily solvable.

Case 3: If $w_a = 1$, $b + w_d = 1$, and $w_p = a$.

Then, the Lie algebra of the HPB equation is spanned by

$$Q_1 = t\frac{\partial}{\partial t} + tx(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}),\ Q_2 = \frac{\partial}{\partial t} + x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}),$$

with infinite dimensional Lie sub-algebra $Q_3 = -g\frac{\partial}{\partial t} + (h - \xi x)\frac{\partial}{\partial x} + (k - \xi x)\frac{\partial}{\partial u}$. Hence, by definition 3.1 $L_2 = \{Q_1, Q_2\}$ is solvable Lie algebra of HPB equation.

5. Conclusion

A classification scheme of HPB model to solvable Lie algebra is presented. HPB for the ankle joint is taken as a special case. This model is found to play a significant role in computational neuroscience that controls human walking. The transformation of HPB into second order NDDE is demonstrated where the solution can easily be found. Classification of HPB equation to solvable Lie algebra allows us to determine the models properties. Time–delayed feedback in the model being intrinsic components of neural control is introduced via inverted pendulum motion. The interpretation of human tasks including stick balancing at the fingertip and postural sway during quiet standing is found to be strongly influenced by these delays. Group analysis is performed for the postural balance. The control force is activated only for motions exceeding some thresholds. Second order extension of the general infinitesimal generator acting on NDDE is employed for classification which is later used for achieving the determining equations for infinitesimal symmetry group. The characteristics equations for HBP are solved where the Lie algebras satisfying the inclusion property are spanned by these corresponding parameters in infinitesimal. Three cases with different constant coefficients of NDDE are considered and the model properties are computed. Classification of other models described by higher order NDDE with functional coefficients is worth-looking. The admirable features of the results suggest that our methodology for modeling HPB to solvable Lie algebra may constitute a basis for solving many real problems in neuroscience portrayed by second order NDDE.

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References

1. Milton, J. G., Cabrera, J. L., Ohira, T. Unstable dynamical systems: Delays, noise and control. Europhysics Letters. 2008; 83: 48001.
2. Milton, J. G. The delayed and noisy nervous system: Implications for neural control. J. Neural Eng. 2011; 8: 065005.
3. Piiroinen, P. T., Dankowicz, H. Low-cost control of repetitive gait in passive bipedal walkers. Int. J. Bifurcat. Chaos. 2005; 15: 1959–1973.
4. Mehta, B., Schall, S. Forward models in visuomotor control. J. Neurophysiol. 2002; 88: 942–953.
5. Cabrera, J. L., Milton, J. G. On–off intermittency in a human balancing task. Phys. Rev. Lett. 2002; 89: 158702.
6. Loram, I. D., Maganaris, C. N., Lakie, M. Human postural sway results from frequent, ballistic bias impulses by soleus and gastrocnemius. J. Physiol. 2005; 564: 295–311.
7. Milton, J., Townsend, J. L., King, M. A., Ohira, T. Balancing with positive feedback: the case for discontinuous control. Phil. Trans. R. Soc. 2009; A 367: 1181–1193.
8. Asai, Y., Tasaka, Y., Nomura, K., Nomura, T., Casidio, M., Morasso, P. A model of postural control in quiet standing: robust compensation of delayinduced instability using intermittent activation of feedback control. PLoS ONE. 2009; 4: e6169.
9. Stepan, G. Delay effects in the human sensory system during balancing. Phil. Trans. R. Soc. 2009; A 367: 1195–1212.
10. Milton, J., Cabrera, J. L., Ohira, T., Tajima, S., Tonosaki, Y., Eurich, C. W., Campbell, S. A. The time-delayed inverted pendulum: implications for human balance control. Chaos. 2009; 19: 026110.
11. Bingham, J. T., Choi, J. T., Ting, L. H. Stability in a frontal plane model of balance requires coupled changes to postural configuration and neural feedback control. J. Neurophysiol. 2011; 106: 437–448.
12. Kowalczyk, P., Glendinning, P., Brown, M., Medrano-Cerda, G., Dallali, H., Shapiro, J. Modelling human balance using switched systems with linear feedback control. J. R. Soc. Interface. 2012; 9: 234–245.
13. Paoletti, P., Mahadevan, L. Balancing on tightropes and slacklines. J. R. Soc. Interface. 2012; 9: 2097–2108.
14. Bottaro, A., Yasutake, Y., Nomura, T., Casadio, M., Morasso, P. Bounded stability of the quiet standing posture: An intermittent control model. Hum. Mov. Sci. 2008; 27: 473–495.
15. Gomi, H., Kawato, M. Adaptive feedback control models of the vestibulocerebellum and spinocerebellum. Biol. Cybern. 1992; 68: 105–114.
16. Peterka, R. J. Simplifying the complexities of maintaining balance. IEEE Eng. Med. Biol. Mag. 2003; 22: 63–68.
17. Sieber, J., Krauskopf, B. Extending the permissible control loop latency for the controlled inverted pendulum. Dyn. Syst. 2005; 20: 189–199.
18. Welch, T. D. J., Ting, L. H. A feedback model reproduces muscle activity during human postural responses to support-surface

translations. J. Neurophysiol. 2008; 99: 1032–1038.
19. Gomi, H., Kawato, M. Neural network control for a closed-loop system using feedback-error-learning. Neural Netw. 1993; 6: 933–946.
20. Insperger, T., Milton, J., Ste'pa'n, G. Acceleration feedback improves balancing against reflex delay. Journal of the society Interface. June 2014; 10: 20120763. (accepted).
21. Kolmanovskii, V. B., Nosov, V. R. Stability of functional differential equations. London, UK: Academic Press. 1986.
22. Stepan, G. Retarded dynamical systems. Harlow, UK: Longman. 1989.
23. Hale, J. K., Lunel, S. M. V. Introduction to functional differential equations. New York, NY: Springer. 1993.
24. Malakhovski, E., Mirkin, L. On stability of second-order quasi-polynomials with a single delay. Automatica. 2006; 42: 1041–1047.
25. Lafortune, M. A., Lake, M. J. Human pendulum approach to simulate and quantify locomotor impact loading. J. Biomech. 1995; 28: 1111-1114.
26. Coveney, V. A., Hunter, G. D., Spriggs, J. Is the behaviour of the leg during oscillation linear? J. Biomech. 2001; 34: 827-830.
27. Kyrychko, Y. N., Blyuss, K. B., Gonzalez-Buelga, A., Hogan S. J., Wagg, D. J. Stability switches in neutral delay differential equation with application to real-time dynamic substructuring. Journal of Applied Mechanies and Materials. 2006 a; 5-6: 79-84.
28. Kyrychko, Y. N., Blyuss, K. B., Gonzalez-Buelga, A., Hogan, S. J., Wagg, D. J. Real-time dynamic substructuring in a coupled oscillator-pendulum system. Proc. R. Soc. 2006 b; A 462: 1271–1294.
29. Loram, I. D., Lakie, M. Direct measurement of human ankle stiffness during quiet standing: the intrinsic mechanical stiffness is insufficient for stability. J. Physiol. 2002; 545: 1041–1053.
30. Eurich, C. W., Milton, J. G. Noise-induced transitions in human postural sway. Phys. Rev. 1996; E 54: 6681–6684.
31. Deligniêres, D., Torre, K., Bernard, P. L. Transition from persistent to anti-persistent correlations in postural sway indicates velocity-based control. PLoS Comput. Biol. 2011; 7: e1001089.
32. Haller, G., Stepan, G. Micro-chaos in digital control. J. Nonlinear Sci. 1996; 6: 415–448.
33. Enikov, E., Stepan, G. Micro-chaotic motion of digitally controlled machines. J. Vib. Control. 1998; 2: 427–443.
34. Bluman, G. W., Kumei, S. Symmetries and differential equations. Sprinder. New York. 1989.
35. Andreas, C. Lie algebras and representation theory. Universit¨at Wien. Nordbergstr. 2009.
36. Humi, M., Miller, W. Second course in ordinary differential equations for scientists and engineers. Springer. New York. 1988.
37. Muhsen, L., Maan, N. New Approach to Classify Second Order Linear Delay Differential Equations With Constant Coefficients to Solvable Lie Algebra. International journal of mathematical analysis. 2014; 20: 973-993.

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