Geometric visualization of singularities on Weingarten surfaces

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Abstract In this work, we investigate ridges and ravines of Weingarten surfaces in the Euclidean space R^3. The necessary and sufficient conditions for existence of ridges and ravines are obtained. Finally, an application is constructed, plotted and its ridges are shown through figures.
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1. Introduction:

On a smooth surface, we define ridges to be the local positive maxima of the maximal principal curvature along its associated curvature line, therefore it encode important information used in surface analysis[4].

Beside pure mathematical studies discovering the beauty of ridges and associated structures, the ridge have been studied in connection with research on accommodation of eye lens, image and data analysis, and quality control of free-form surface.

The set of solutions of the equation defining Weingarten surface is called curvature diagram or W-diagram of the surface.The study of W-surface is a classical topic in differential geometry.

2. Preliminaries:

In this section we fix some notations on local classical differential geometry of surfaces. Let M be a surface in R^3, and consider $x = x(u^1, u^2)$ a local parameterization of M. Let N denote the unit normal vector field on M given by Do Carmo, Gray[5],[6]:

$$N = \frac{x_1 \times x_2}{|x_1 \times x_2|} \quad , \quad x_i = \frac{\partial x}{\partial u^i}, \quad i = 1,2.$$

In each tangent plane, the induced metric $\langle \, , \rangle$ is determined by the first fundamental form :

$$I = \langle dx, dx \rangle = g_{ij} du^i du^j \quad , where \quad g_{ij} = \langle x_i, x_j \rangle$$

The second fundamental form :

$$II = -\langle dN, dx \rangle = L_{ij} du^i du^j \quad , where \ L_{ij} = \langle N, x_{ij} \rangle$$

Under the parameterization x, the mean curvature H and the Gauss curvature K have the classical expressions :

$$H = \frac{L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)} \quad , \quad K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

The principal curvatures k_1 and k_2 are given by :

$$k_1 = H + \sqrt{H^2 - K} \quad , \quad k_2 = H - \sqrt{H^2 - K}$$

Defenition1:

A point is umbilical point if and only if $H^2 = K$ at this point.

We have an important theorem about umbilical points :

Lemma1:

If all points of a continuous surface S are umbilical, then S is either part of a plane or a part of a sphere

Thus an umbilical point is either a planner point or a spherical point. Usually, a point p on a smooth surface has either only two principal directions, or an infinite number of principal directions with the same normal curvature. Therefore,

Lemma2:

p is a non-umbilical point if and only if there exist only two orthogonal principal directions at p.

A line of curvature is a curve on a surface whose tangent are the principal directions at all of its points.

For a non-umbilical point $p \in M$, let us choose coordinates in the space so that p is at the origin, the xy-plane is the tangent plane to the surface at p, and the principal directions coincide with x and y axis. The surface is then expressible, in the

Monge form, as the graph of a smooth function from xy-plane to the z axis with the equation :

$$z = \frac{1}{2!}\left(k_1 x^2 + k_2 y^2\right) + \frac{1}{3!}\left(b_0 x^3 + 3b_1 x^2 y + 3b_2 xy^2 + b_3 y^3\right) + \frac{1}{4!}\left(c_0 x^4 + 4c_1 x^3 y + 6c_2 x^2 y^2 + 4c_3 xy^3 + c_4 y^4\right) + O(x,y)^5 \quad (1)$$

This patch is called principal patch[4],[3].

3. Weingarten Surfaces:

A surface M in R^3 is called a Weingarten surface if there is some relation between its two principal curvatures k_1 and k_2, that is to say, there is a smooth function φ of two variables such that $\varphi(k_1, k_2) = 0$,

or equivalently $\psi(H, K) = 0$ [7],[8],[9].

These surfaces were introduced by the very Weingarten in the context of the problem of finding all surfaces isometric to a given surface of revolution. Weingarten surfaces are important in computer aided design and shape investigation.

If the function φ is linear, that is :

$$ak_1 + bk_2 = c \quad , \quad where \quad a, b, and\ c\ const.$$

The surface is called linear Weingarten surface(LW-surface), typical examples of LW-surfaces are:

- Surface with constant Gauss curvature.
- Surface with constant mean curvature.

After earlier works in the fifties due Chern, Hopf,Hartman,amongst others, there has been progress in this theory, especially when the Weingarten relation is of the type $H = f\left(H^2 - K\right)$ and f is elliptic, the surface satisfy a maximum principal that allows a best knowledge of the shape of such surface[7],[8],[9].

For the study of these surfaces, **kühnel** investigated ruled W-surfaces in Euclidean 3-space[8]. kühnel [8], and Yoon[7] gave a classification of ruled W-surface and ruled LW-surfaces in Minkowski 3-space.

M.H.Kim and D.W.Yoon gave a classification of Weingarten quadratic surfaces in Euaclidean 3-space[7].

4. Ridges and Ravines:

Surfaces creases provide us with important information about the shape of objects and can be intuitively defined as curves on a surface alng which the surface bends sharply. Mathematical description of surface creases is based on study of extrema of the principal curvatures along their curvature lines [10],[11].

Assume that the maximal and minimal curvature $k_{\max} = k_1$ and $k_{\min} = k_2$ of M at p. The associated tangent directions $t_{\max} = t_1$ and $t_{\min} = t_2$ of M at p. The integral curves of the principal direction fields are called the curvature lines. A point at which one of the principal curvature vanishes is called parabolic.A point at which the principal curvature are equal to each other is called umbilic[1],[2],[3].

Definition2:

A non-umbilic point $p \in M$ is called ridge point if $k_{\max}$ attains a local positive maximum at p along the associated curvature line.

A non-umbilic point $q \in M$ is called ravine point if $k_{\min}$ attains a local negative minimum at q along the associated curvature line.

Lemma3[12]:

The necessary condition for a surface to have a ridge is to satisfy :

$$\frac{\partial k_i}{\partial u^j} = o \quad , \frac{\partial^2 k_i}{\left(\partial u^j\right)^2} > 0 \quad , i, j = 1,2$$

Lemma 4[12]:

The necessary condition for a surface to have a ravine is to satisfy :

$$\frac{\partial k_i}{\partial u^j} = o \quad , \frac{\partial^2 k_i}{\left(\partial u^j\right)^2} < 0 \quad , i, j = 1,2$$

Lemma5[13],[14]:

The necessary condition for the surface $z = f(x, y)$ to have a ridge (ravine) is to satisfy the differential form $d^3 f = o$.

Lemma6[13],[14]:

The sufficient condition for $z = f(x, y)$ to have ridge (ravine) is to satisfy the inequality $\frac{\partial^4 f}{\partial x_i^4} < 0$ ($\frac{\partial^4 f}{\partial x_i^4} > 0$).

5. Ridges and Ravines on Weingarten Surface:

Let us consider the Weingarten surface satisfying the equation :

$$k_2 = F(k_1) \quad , where\, k_1 = k_1(u^1, u^2), k_2 = k_2(u^1, u^2) \quad (2)$$

Differentiating k_2 with respect to u^1, u^2 we get :

$$\frac{\partial k_2}{\partial u^1} = F' \frac{\partial k_1}{\partial u^1} \quad (3)$$

$$\frac{\partial k_2}{\partial u^2} = F' \frac{\partial k_1}{\partial u^2} \quad (4)$$

At ridge or ravine we have :

$$\frac{\partial k_1}{\partial u^1} = 0 \,, \frac{\partial k_1}{\partial u^2} = 0 \quad , F' \neq 0 \quad (5)$$

Thus : $\frac{\partial k_2}{\partial u^1} = 0 \,, \frac{\partial k_2}{\partial u^2} = 0$,hence:

Theorem1:

On Weingarten surface M , if there is a point $p \in M$ at which k_1 max (min) then k_2 is max(min).

Remark:

1- In the case of the sphere with radius r , $k_1 = k_2 = \frac{1}{r} = const.$

i.e. F is the constant linear function, that is $F' = 0$ .Then the sphere is a LW-surface.

2- In the case of cylinder, $k_1 = \frac{1}{r}, k_2 = 0$, so $F(k_1) = 0$, and hence

$F' = 0$. It is a LW- parabolic surface.

Now, differentiating (3),(4) with respect to u^1, u^2 we get :

$$\frac{\partial^2 k_2}{(\partial u^1)^2} = F'' \left(\frac{\partial k_1}{\partial u^1} \right)^2 + F' \frac{\partial^2 k_1}{(\partial u^1)^2} \quad (6)$$

$$\frac{\partial^2 k_2}{\partial u^2 \partial u^1} = F'' \frac{\partial k_1}{\partial u^1} \frac{\partial k_1}{\partial u^2} + F' \frac{\partial^2 k_1}{\partial u^2 \partial u^1} \quad (7)$$

$$\frac{\partial^2 k_2}{(\partial u^2)^2} = F'' \left(\frac{\partial k_1}{\partial u^2} \right)^2 + F' \frac{\partial^2 k_1}{(\partial u^2)^2} \quad (8)$$

Hence, equations (6),(7),(8) can be written in the form :

$$Hess(k_2) = F'' \begin{bmatrix} \left(\frac{\partial k_1}{\partial u^1} \right)^2 & \frac{\partial k_1}{\partial u^1} \frac{\partial k_1}{\partial u^2} \\ \frac{\partial k_1}{\partial u^1} \frac{\partial k_1}{\partial u^2} & \left(\frac{\partial k_1}{\partial u^2} \right)^2 \end{bmatrix} + F' Hess(k_1) \quad (9)$$

At ridge, ravine,[from(5)], equation (9) will have the form :

$$Hess(k_2) = F' \; Hess(k_1) \quad , \quad F' \neq 0 \quad (10)$$

The point p is ridge (ravine) if $k_1(p)$ is max (min), that is :

$$Det[Hess\; k_1(p)] > 0 \quad and \quad \frac{\partial^2 k_1}{(\partial u^1)^2} < 0 \,, \left[\frac{\partial^2 k_1}{(\partial u^1)^2} > 0 \right] \quad (11)$$

From Th.1, we can see that if p is a ridge (ravine) then $k_2(p)$ is max (min), using (9) we have :

$$Det[Hess k_2(p)] > 0 \quad and \quad \frac{\partial^2 k_2}{(\partial u^1)^2} < 0 \,, \left[\frac{\partial^2 k_2}{(\partial u^1)^2} > 0 \right]$$

That is :

$$Det[Hess k_2(p)] = (F')^2 Det[Hess k_1(p)] > 0$$

$$\frac{\partial^2 k_2}{(\partial u^1)^2}(p) = F'(k_1(p)) \frac{\partial^2 k_1}{(\partial u^1)^2}(p) < 0 \quad (at\; ridge) \quad (12)$$

From (11), (12) we have:

a- k_2 has max at p if :

$$\frac{\partial^2 k_2}{(\partial u^1)^2} = F' \frac{\partial^2 k_1}{(\partial u^1)^2} < 0$$

Which gives the following :

i- $F' < 0 \,, \frac{\partial^2 k_1}{(\partial u^1)^2} > 0$ at the point p , k_1 has min.

ii- $F' > 0 \,, \frac{\partial^2 k_1}{(\partial u^1)^2} < 0$ at the point p , k_1 has max

b- k_2 has max at p if :

$$\frac{\partial^2 k_2}{(\partial u^1)^2} = F' \frac{\partial^2 k_1}{(\partial u^1)^2} > 0$$

Which gives the following:

i- $F' < 0$, $\frac{\partial^2 k_1}{(\partial u^1)^2} < 0$ at the point p, k_1 has max.

ii- $F' > 0$, $\frac{\partial^2 k_1}{(\partial u^1)^2} > 0$ at the point p, k_1 has min.

Thus we have proved the following :

Theorem 2:

On Weingarten surface (2), where F is decreasing, then :

1- k_2 has max at the point p if k_1 has min at p.

2- k_2 has min at the point p if k_1 has max at p.

Theorem 3:

On Weingarten surface (2), where F is increasing, then :

1- k_2 has max at the point p if k_1 has max at p.

2- k_2 has min at the point p if k_1 has min at p.

6. Special Representation For Weingarten Surface:

Considering the surface given in (1),and assuming that the surface is Weingarten surface satisfying (2).Then the surface can be represented by :

$$z = \frac{1}{2!}\left(k_1x^2 + F(k_1)y^2\right) + \frac{1}{3!}\left(b_0x^3 + 3b_1x^2y + 3b_2xy^2 + b_3y^3\right) + O(x,y)^4$$

Where:

$$b_0 = \frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial k_1}{\partial x}$$

$$b_1 = \frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial k_1}{\partial y}$$

$$b_2 = \frac{\partial^3 z}{\partial y^2 \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial k_2}{\partial x} = \frac{\partial F(k_1)}{\partial x} = F'\frac{\partial k_1}{\partial x}$$

$$b_3 = \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial k_2}{\partial y} = \frac{\partial F(k_1)}{\partial y} = F'\frac{\partial k_1}{\partial y}$$

At ridge (ravine) using (5),we get the following:

Teorem 4:

The necessary condition for the Weingarten surface $z = f(x,y)$ to have a ridge (ravine) is to satisfy $d^3 f = o$, $F' \neq 0$.

Now, Taylor expansion of $z = f(x,y)$ at ridge (ravine) has the form:

$$z = \frac{1}{2!}\left(k_1x^2 + k_2y^2\right) + \frac{1}{4!}\left(c_0x^4 + 4c_1x^3y + 6c_2x^2y^2 + 4c_3xy^3 + c_4y^4\right) + O(x,y)^5$$

Where :

$$c_0 = \frac{\partial^2 k_1}{\partial x^2}, \quad c_1 = \frac{\partial^2 k_1}{\partial x \partial y}$$

$$c_2 = F'\frac{\partial^2 k_1}{\partial x^2}, \quad c_3 = F'\frac{\partial^2 k_1}{\partial x \partial y},$$

$$c_4 = F'\frac{\partial^2 k_1}{\partial y^2}.$$

Hence, according to the sign of F' and the hessian matrix of k_1 we can determine whether the point is ridge or ravine.That is :

Teorem5:

On Weingarten surface covered by principal patch, If F is increasing then k_2 has max (min) if $\frac{\partial^4 z}{\partial x^4} < 0$ ($\frac{\partial^4 z}{\partial x^4} > 0$).

Teorem6:

On Weingarten surface covered by principal patch, If F is decreasing then k_2 has max(min) if $\frac{\partial^4 z}{\partial x^4} > 0$ ($\frac{\partial^4 z}{\partial x^4} < 0$).

7. Applications:

We will give illustration to the ridge and ravine using the well known programs in geometry to plot a surface in R^3 .For this perpose we consider the following surfaces:

1-Enneper's surface: it is a self intersecting minimal surface,parametrized by:

$$x(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

On this surface, the parametric lines are lines of curvature, the principal curvatures are :

$k_1 = -k_2 = \dfrac{2}{\left(1+u^2+v^2\right)^2}$, and hence $F(k_2) = -k_2$.

According to the relation between principal curvatures, this surface is minimal linear Weingarten surface.

At $u = 0, v = 0$, k_1 has max, and since F decreasing, then k_2 has min.

Figure 2 shows the surface together with ridge lines.

2- Helicoid surface: it is a minimal surface having a helix as its boundry, it is the only ruled minimal surface other than the plane, parametrized by:

$$x(u,v) = (u\cos v, u\sin v, cv) \quad , \quad c = cont \neq 0$$

On this surface, the parametric lines are lines of curvature, the principal curvatures are :

$k_1 = -k_2 = \dfrac{c}{u^2+c^2}$, and hence $F(k_2) = -k_2$.

According to the relation between principal curvatures, this surface is minimal linear Weingarten surface.

At $u = 0$, k_1 has max if c is positive,and has min if c is negative.

And since F decreasing, k_2 has min if c is positive, and has max if c is negative, Figures 1-a,1-b shows the surface together with ridge lines.

3- Ellipsoid of revolution:

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Which can parametrized by :

$$x(u,v) = (a\cos u\sin v, a\sin u\sin v, b\cos v)$$

By simple calculation we get $g_{12} = L_{12} = 0$, i.e. parametric lines are lines of curvature.

Principal curvature are givin by:

$$k_1 = \frac{2\sqrt{2}ab}{\left(\sqrt{a^2+b^2+(a^2-b^2)\cos 2v}\right)^3}$$

$$k_2 = \frac{\sqrt{2}b}{a\sqrt{a^2+b^2+(a^2-b^2)\cos 2v}}$$

Which can be written as : $k_1 = \dfrac{a^4}{b^2}k_2^3$, that is,ellipsoid of revolution is Weingarten surface with $k_1 = F(k_2)$, it is not LW-surface.

Ridges or ravine accrues at $v = 0, \pm\dfrac{\pi}{2}, \pm\pi$, and since F is increasing, and k_1 has min at these values, k_2 has min also.which is illustrated in figure 3 taking $a = 5, b = 2$.

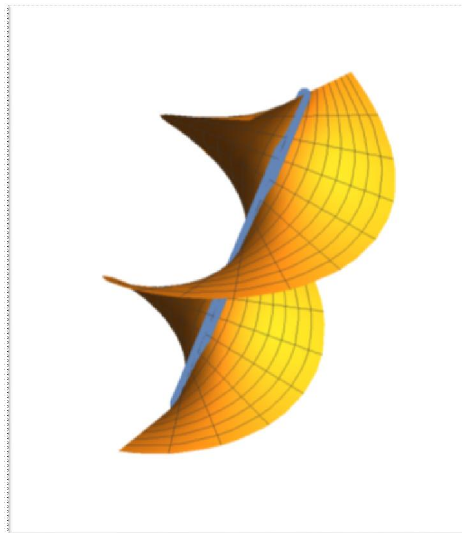

Fig.1.a: Helicod with $c = 2$, and its ridge line

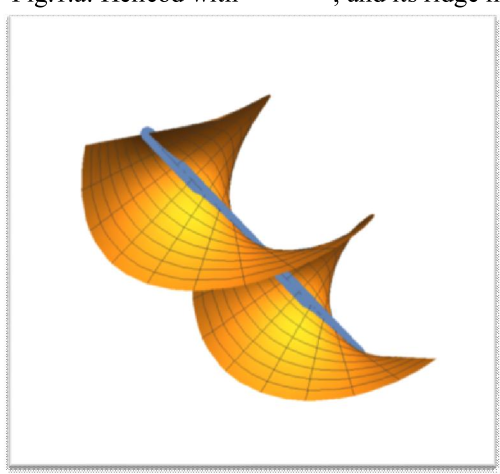

Fig.1.b: Helicod with $c = -2$,and its ridge line

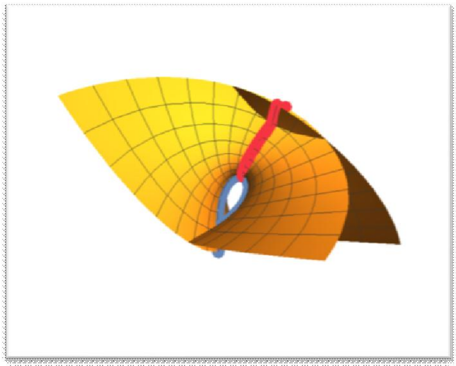

Fig.2.Enneper surface with its ridge lines

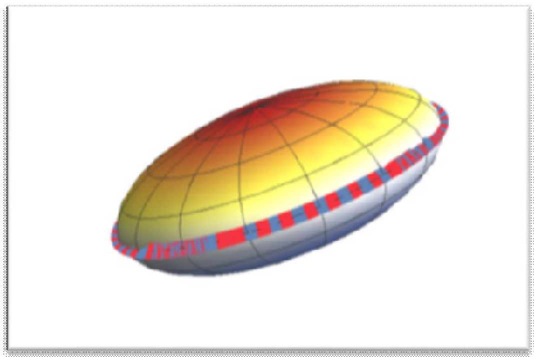

Fig.3. ellipsoid of revolution with its ravine lines

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