Zero divisors in rings with involution

Usama A. Aburawash

Department of Mathematics and Computer Sciences, Faculty of Science, Alexandria University, Alexandria, Egypt
aburawash@alex-sci.edu.eg

Abstract: In rings with involution, the concept of *-zero divisors is introduced and the relation with zero divisors in rings without involution is discussed. This definition, however, is compatible with the category of involution rings; since it preserves the involution. Moreover, closely related definitions; such as *-completely prime ideals and *-rings and *-cancellation law are introduced. Finally, *-prime and *-completely prime *-ideals are characterized using *-zero divisors.

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Throughout this paper, a ring will always mean an associative ring. A ring A is said to be an *involution ring* or briefly **-ring* if on A there is defined an *involution* * subject to the identities

$a^{**} = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$,

for all $a, b \in A$ (see [2] and [4]). Considering the category of involution rings, all morphisms (and also embeddings) must preserve involution. For this reason, we are looking here for a particular concept of zero divisors which appropriate for the category of involution rings.

By a **- ideal* (or *self-adjoint ideal*), we mean an ideal I of A closed under involution; that is $I^* = I$, and will be denoted by $I \lhd^* A$.

We start by introducing the concept of *-zero divisor.

Definition 1 A nonzero element a of a *-ring A is said to be a **-zero divisor* if there exists a nonzero element $b \in A$ such that $ab = 0$ *and* $a^*b = 0$.

Remark 1 If Definition 1 defines left *-zero divisors then, by taking involution, we get $b^*a^* = 0$ and $b^*a = 0$ which mean that a is a right *-zero divisor, too. By symmetry, a right *-zero divisor is also a left *-zero divisor. Thus, as expected in the category of involution rings, we have only the concept of *-zero divisor. So that, this new concept of *-zero divisor preserves the involution and therefore is appropriate for the category of rings with involution.

It is evident that a *-zero divisor is a zero divisor, but the converse is not always true as shown from the following example.

Example 1 Consider the direct sum $A = D \oplus D^{op}$, where D is an integral domain and D^{op} is its opposite domain. A is a *-ring with the exchange involution given by $(a,b)^* = (b,a)$ for all $(a,b) \in A$. For any $0 \neq a \in D$, the element $(a,0)$ of A is a zero divisor since $(a,0)(0,b) = 0 = (0,b)(a,0)$ for every $0 \neq b \in D$. Because neither a nor b are zero divisors, from $(0,a)(0,b) \neq 0$, we conclude that $(a,0)$ is not a *-zero divisor.

In particular, if a is a symmetric ($a^* = a$) or a skew symmetric ($a^* = -a$) element of a *- ring A, then a is a zero divisor if and only if it is a *-zero divisor. Moreover, we can construct symmetric or skew symmetric *-zero devisors from given *-zero devisors as in the following proposition.

Proposition 1 *Let A be a *-ring and $a \in A$. If a is a *-zero divisor, then there exists a (nonzero) symmetric or skew symmetric *-zero divisor in A.*

Proof If a is a symmetric or skew symmetric element, then it is done. If a is not symmetric, then $a - a^* \neq 0$ is a skew symmetric element in A such that, for an appropriately chosen $b \in A$, we have $(a - a^*)b = ab - a^*b = 0$ and

$(a - a^*)^*b = (a^* - a)b = a^*b - ab = 0$. ∎

Nevertheless, the existence of zero divisors which are also *-zero divisors is illustrated by the next example.

Example 2 In the involution ring of all 2×2 matrices over the integers Z with the transpose as involution, the element $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is both zero and *-zero divisor. In fact, the matrix $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ satisfies

$ab = ba = 0$ and $ab = a^*b = 0$.

As zero divisors is used to define integral domains in rings without involution, we may use *-zero divisors to define *-integral domains in rings with involution as in the following definition.

Definition 2 A commutative *-ring without *-zero divisors is said to be a **-integral domain*.

Example 3 The following are *-integral domains:

1. Each *-division ring.
2. The ring $A = D \oplus D^{op}$ in Example 1.

Since a commutative *-ring is an integral domain if it has no zero divisors, then it has no *-zero divisors and consequently it is a *-integral domain. Moreover, Example 1 shows that not every *-integral domain is an integral domain; since the *-ring $A = D \oplus D^{op}$ is *-integral domain but not integral domain.

Next, we define the *-cancellation law to work with *-zero divisors as follows.

Definition 3 The **-cancellation law* is said to be hold in a *-ring A if $ab = ac$ and $a^*b = a^*b$ imply $b = c$, for any $0 \neq a \in A$.

Similar to remark 1, if one defines left *-cancellation law to be hold in A as in Definition 3, we can easily show that the right *-cancellation law holds also in A. Therefore, we have only the *-cancellation law as expected for *-rings.

It is well-known that the cancellation laws hold in a ring A if and only if A has no zero divisors. The following similar result for *-ring, can now be given.

Proposition 2 *The *-cancellation law holds in a *-ring A if and only if A has no *-zero divisors.*

Proof Suppose that the *-cancellation law holds in A. If $0 \neq a \in A$ is such that $ab = 0$ *and* $a^*b = 0$, then $b = 0$ follows and consequently A has no *-zero divisors. Conversely, let A have no *-zero divisors. For $0 \neq a \in A$, if $ab = ac$ and $a^*b = a^*c$, then $a(b - c) = 0$ and $a^*(b - c) = 0$ which forces $b - c = 0$. Thus $b = c$ and the *-cancellation law holds in A. ■

It is obvious that if the left (right) cancellation law holds in a *-ring A, then the *-cancellation law holds in A, too. The converse is not always true as shown

in example 1, where the *-ring $A = D \oplus D^{op}$ has zero divisors but does not contain *- zero divisors.

Remind that an ideal P of a ring A is called a *completely prime* ideal if $ab \in P$ implies $a \in P$ or $b \in P$ for all $a, b \in A$ (see for instance [5] or [6]). The involutive version of this definition can now be formulated by the following definition.

Definition 4 An ideal P of a *-ring A is said to be a **-completely prime* ideal if $ab \in P$ and $a^*b \in A$ imply $a \in P$ or $b \in P$ for all $a, b \in A$. The *-ring A is called a **-completely prime *-ring* if the zero ideal is a *-completely prime ideal.

From the definition it follows that the *-ring A is *- completely prime if and only if it has no *-zero divisors. We remind also that a ring A is completely prime if and only if it has no zero divisors. By this remark, a completely prime *-ring A is also *-completely prime, since A has no zero divisors implies that A has no *-zero divisors. By the way, the converse is not true; since the *-ring $A = D \oplus D^{op}$ in example 1 is *-completely prime, but not completely prime.

Following [3], an ideal P of a *-ring A is called a **-prime ideal* if $JK \subseteq P$ implies $J \subseteq P$ or $K \subseteq P$, for any $J, K \lhd^* A$. A *-ring A is a **-prime ring* if the zero ideal is a *-prime ideal. By the way, Birkenmeier and Groenewald gave in [3] the following equivalences for *-primeness of ideals.

Proposition 3 ([3], Proposition 5.4) *Let A be a *-ring and $P \lhd^* A$. The following conditions are equivalent:*

*(i) P is a *-prime *-ideal of A.*

(ii) If $0 \neq a, b \in A$ are such that $aAb \subseteq P$ and $a^*Ab \subseteq P$, then $a \in P$ or $b \in P$.

(iii) If $I \lhd A$ and $K \lhd^* A$ such that $JK \subseteq P$, then $I \subseteq P$ or $K \subseteq P$.

For *-prime rings without nonzero nilpotent elements, we claim that they have no *-zero divisors.

Proposition 4 If *A is a *-prime *-ring having no nonzero nilpotent elements, then A has no *-zero divisors.*

Proof Let $0 \neq a, b \in A$ be such that $ab = 0$ and $a^*b = 0$. Then $(ba)^2 = b(ab)a = 0$. Since A has no nonzero nilpotent elements, it follows that $ba = 0$. Thus for all $x \in A$, we get $(axb)^2 = ax(ba)xb = 0$, whence $axb = 0$ and consequently $aAb = 0$. Similarly, we have $a^*Ab = 0$. Because A is *-prime, we deduce from Proposition 3 that $b = 0$, from which A has no *-zero divisors. ■

From the definitions, it is easy to check that a *-completely prime *-ideal of A is also a *-prime *-ideal. The converse is true only in particular cases; for instance if A possesses identity. For commutative *-rings, we have the following equivalences.

Theorem 1 *Let A be a commutative *-ring and $P \lhd^* A$. The following conditions are equivalent:*

*(i) P is a *-prime *-ideal.*

*(ii) P is a *-completely prime *-ideal.*

*(iii) The factor ring A/P is a *-integral domain.*

Proof $(i) \Rightarrow (ii)$. Let $a, b \in A$ such that $ab \in P$ and $a^*b \in A$. Then $aAb \subseteq P$ and $a^*Ab \subseteq P$. Hence, by Proposition 3, $a \in P$ or $b \in P$ and consequently P is a *-completely prime *-ideal.

$(ii) \Rightarrow (iii)$. A/P is commutative because A is commutative. Since P is a *-completely prime *-ideal, then $ab \in P$ and $a^*b \in A$ imply $a \in P$ or $b \in P$ for all $a, b \in A$. In other words, $(a + P)(b + P) = P$ and $(a + P)^*(b + P) = P$ imply $a + P = P$ or $b + P = P$, whence A/P is a *-integral domain.

$(iii) \Rightarrow (i)$. Suppose that $aAb \subseteq P$ and $a^*Ab \subseteq P$. By the commutativity of A, we get $(ab)b \in P$, $(ab)^*b \in P$ and $(a^*b)b \in P$, $(a^*b)^*b \in P$. Since A/P has no

*-zero divisors, it follows that $ab \in P$ or $b \in P$ and $a^*b \in P$ or $b \in P$. If $b \notin P$, then $ab \in P$ and $a^*b \in A$, from which $a \in P$ follows. Thus P is a *-prime *-ideal, by Proposition 3. ■

Proposition 5 *For a commutative *-ring A, the following are satisfied:*

*(i) The set K={all *-zero divisors of A}∪{0} is a *-ideal of A.*

*(ii) The factor ring A/K is a *-integral domain.*

Proof *(i)* Let $a, b \in K$ and $r \in A$, then there exist nonzero elements $c, d \in A$ such that $ac = a^*c = 0$ and $bd = b^*d = 0$. Hence, we get $(a-b)cd = 0$ and $(a-b)^*cd = (a^* - b^*)cd = 0$, $rac = 0$ and $(ra)^*c = 0$. Thus $a - b,\ ra \in K$. Moreover $a^* \in K$, since $a^*c = a^{**}c = ac = 0$.

(ii) Since A/K is commutative and has no *-zero divisors, it is a *-integral domain. ■

The following proposition gives a necessary condition for an element in the center of a *-ideal to be in the center of the ring.

Proposition 6 *Let N be a *-ideal of a *-ring A and $c \in C(N)$; the center of N. If c is not a *-zero divisor, then $c \in C(A)$; the center of A.*

Proof $C(N) = \{n \in N | nx = xn, for\ all\ x \in N\}$ is a *-subring of A, since for $n \in C(N)$, $x \in N$, we have $nx^* = x^*n$. Hence $n^*x = xn^*$ and $n^* \in C(N)$. Now, for every $y \in A$, we have $cy, yc, c^*, c^*y, yc^* \in N$. Hence $c(cy - yc) = c(cy) - cyc = cyc - cyc = 0$ and

$$c^*(cy - yc) = c^*(cy) - c^*yc = (c^*c)y - c^*yc = c(c^*y) - c^*yc = c^*yc - c^*yc = 0.$$

But c is not a *-zero divisor, whence $cy - yc = 0$ and $c \in C(A)$ follows. ■

Finally, since a *-ring without zero divisors has no *-zero divisors, we conclude the following immediate result from Proposition 3 in [1].

Proposition 7 *Every *-ring A without zero divisors is embeddable as a *-ideal (up to isomorphism) into one and only one involution ring $\overline{A^1}$ with identity and without *-zero divisors such that $\overline{A^1}$ is a minimal *-extension of A possessing identity.*

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