A note on s-topological groups

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Abstract: In this paper, properties of s-topological groups are investigated. s-regularity and s-compactness have been studied for s-topological groups. Relation between topologized groups has been established. Counter examples are given to show that reverse implications are not true in general.


Keywords: Semi homeomorphism, s-topological groups, S-topological groups, Irr-topological groups, irresolute topological groups, and s-compact spaces.

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1. Introduction

To study the properties of topologized groups by weakening the continuity conditions in the sense of N. Levine is the basic interest of the article. Semi continuity [9] and irresolute mapping [6] were the consequence of the study of semi open sets in topological spaces. In [2] Bohn defined and investigated the notion of s-topological groups and in [15] Siddique et al. defined the notion of S-topological groups. In [14] Siab et al. defined and studied the notions of irresolute topological groups and Irr-topological groups by using irresolute mappings [6]. In this paper we will continue the study of the properties of s-topological groups. Investigations on s-compactness and s-regularity are presented on s-topological groups and irresolute topological groups. Relations between S-topological groups, s-topological groups, irresolute topological groups and Irr-topological groups are investigated. Counter examples are given to show that the reverse implications are not true in general.

2. Definitions and Preliminaries

Throughout this paper X and Y are always topological spaces on which no separation axioms are assumed. For a subset A of a space X the symbols Int(A) and Cl(A) are used to denote the interior of A and the closure of A respectively. If f: X → Y is a mapping between topological spaces X and Y and B is a subset of Y, then f⁻¹(B) denotes the pre image of B. Our other topological notations and terminology are standard as in [7]. If (G, ∗) is a group, then e denotes its identity element, and for a given x ∈ G, ℓₓ : G → G, y ↦ x ∗ y, and rₓ : G → G, y ↦ y ∗ x, denote the left and the right translations by x, respectively. The operation ∗ we call the multiplication mapping m: G × G → G, and the inverse operation x ↦ x⁻¹ is denoted by i.

In 1963, N. Levine [9] defined semi open sets in topological spaces. Since then many mathematicians explored different concepts and generalized them by using semi open sets (see [1, 6, 8, 12, 13]). A subset A of a topological space X is said to be semi open if there exists an open set U in X such that U ⊂ A ⊂ Cl(U), or equivalently if A ⊂ Cl(Int(A)). SO(X) denotes the collection of all semi open sets in X. The complement of a semi open set is said to be semi closed; the semi closure of A ⊂ X, denoted by sCl(A), is the intersection of all semi closed subsets of X containing A [4, 5]. x ∈ sCl(A) if and only if for any semi open set U containing x, U ∩ A ≠ ∅.

Clearly, every open (closed) set is semi open (semi closed). It is known that the union of any collection of semi open sets is again a semi open set, while the intersection of two semi open sets need not be semi open. The intersection of an open set and a semi open set is semi open. If A ⊂ X and B ⊂ Y are semi open in spaces X and Y, then A × B is semi open in the product space X × Y. Basic properties of semi open sets are given in [9], and of semi closed sets and the semi closure in [4, 5].

Recall that a set U ⊂ X is a semi open neighbourhood of a point x ∈ X if there exists A ∈ SO(X) such that x ∈ A ⊂ U. A set A ⊂ X is semi open in X if and only if A is a semi open neighbourhood of each of its points. If a semi open neighbourhood U of a point x is a semi open set we say that U is a semi open neighbourhood of x. Let (G, τ) be a topological space and A ⊂ G. Then x ∈ G is called a semi interior point of A if there exists a semi open set U such that x ∈ U ⊂ A. The set of all semi interior points of A is called a semi interior of A, denoted by sht(A). A
topological space \((G, \tau)\) is semi-T₂ [10] if for every distinct points \(x, y \in G\), there are semi open neighbourhoods \(U\) containing \(x\) and \(V\) containing \(y\) such that \(U \cap V = \emptyset\). A topological space \((G, \tau)\) is s-regular [11] if for any \(x \in G\), any closed set \(F \subset G\) such that \(x \notin F\), there is a semi open neighbourhood \(U\) of \(x\) and a semi open neighbourhood \(V\) containing \(F\) such that \(U \cap V = \emptyset\). A space \(G\) is s-compact [3], if every semi open cover of \(G\) has a finite subcover. Every compact space is \(s\)-compact but converse is not always true. For some applications of semi open sets see [12].

A mapping \(f: X \rightarrow Y\) between topological spaces \(X\) and \(Y\) is called:

1. (semi) continuous (resp. irresolute) if for each open (resp. semi open) set \(V \subset Y\), the set \(f^{-1}(V)\) is semi open in \(X\). Equivalently, the mapping \(f\) is semi continuous (irresolute) if for each \(x \in X\) and each open (semi open) neighbourhood \(V\) of \(f(x)\), there exists a semi open neighbourhood \(U\) of \(x\) such that \(f(U) \subset V\). Every irresolute mapping is semi continuous;
2. semi open [6] if for every semi open set \(A \subset X\), \(f(A)\) is semi open in \(Y\);
3. semi homeomorphism [6] if \(f\) is bijective, irresolute and pre semi open;
4. (S-)homeomorphism [15] if \(f\) is bijective, semi continuous and pre semi open. Every semi homeomorphism is an \((S-)\)homeomorphism.

3. Properties of topologized groups.

Definition 3.1. [15] A triple \((G, \ast, \tau)\) is said to be an \((S-)\)topological group if \((G, \ast)\) is a group, \((G, \tau)\) is a topological space, and (a) the multiplication mapping \(m: G \times G \rightarrow G\) defined by \(m(x, y) = x \ast y\) for every \(x, y \in G\), is semi continuous, (b) the inverse mapping \(i: G \rightarrow G\) defined by \(i(x) = x^{-1}\), for every \(x \in G\), is semi continuous.

Definition 3.2. [2] An \((S-)\)topological group is a group \((G, \ast)\) with a topology \(\tau\) such that for each \(x, y \in G\) and each neighbourhood \(W\) of \(x \ast y^{-1}\) there are semi open neighbourhoods \(U\) of \(x\) and \(V\) of \(y\) such that \(U \ast V^{-1} \subset W\).

Definition 3.3. [14] A triple \((G, \ast, \tau)\) is said to be an \((Irr-)\)topological group if \((G, \ast)\) is a group, \((G, \tau)\) is a topological space, and (a) the multiplication mapping \(m: G \times G \rightarrow G\) defined by \(m(x, y) = x \ast y\) for every \(x, y \in G\), is irresolute, (b) the inverse mapping \(i: G \rightarrow G\) defined by \(i(x) = x^{-1}\), for every \(x \in G\), is irresolute.

Definition 3.4. [14] A triple \((G, \ast, \tau)\) is an irresolute-topological group with a group \((G, \ast)\) and a topology \(\tau\) such that for each \(x, y \in G\) and each semi open neighbourhood \(W\) of \(x \ast y^{-1}\), there are semi-open neighbourhoods \(U\) of \(x\) and \(V\) of \(y\) such that \(U \ast V^{-1} \subset W\).

Lemma 3.5. [2] If \((G, \ast, \tau)\) is an \((S-)\)topological group, then

1. (i) \(A \in SO(G)\) if and only if \(A^{-1} \in SO(G)\);
2. (ii) \(A \in SO(G)\) and \(B \subset G\), then \(A \ast B\) and \(B \ast A\) are both semi open in \(G\).

Lemma 3.6. [14] If \((G, \ast, \tau)\) is an irresolute topological group, then

1. (i) \(A \in SO(G)\) if and only if \(A^{-1} \in SO(G)\);
2. (ii) \(A \in SO(G)\) and \(B \subset G\), then \(A \ast B\) and \(B \ast A\) are both semi open in \(G\).

Lemma 3.7. [14] Every irresolute topological group is an \((Irr-)\)topological group.


Lemma 3.9. [15] Each left (right) translation of \((S-)\)topological groups is \((S-)\)homeomorphism.

Lemma 3.10. [15] Every left (right) translation of irresolute topological groups is semi homeomorphism.

Proof. Let \((G, \ast, \tau)\) be an irresolute topological group. We prove the statement only for left translation. Of course, left translation is bijective mapping. We prove directly that for any \(x \in G\) the left translation \(\ell_x : G \rightarrow G\) is irresolute. Let \(y\) be an arbitrary element in \(G\) and let \(W\) be a semi open neighbourhood of \(\ell_x(y) = x \ast y = x \ast (y^{-1})^{-1}\). By definition of irresolute topological group, there are semi open neighbourhoods \(U\) and \(V\) containing \(x\) and \(y^{-1}\), respectively, such that \(U \ast V^{-1} \subset W\). In particular, we have \(x \ast V^{-1} \subset W\). By Lemma 3.6, the set \(V^{-1}\) is a semi open neighbourhood of \(y\), so that the last inclusion actually says that \(\ell_x\) is irresolute at \(y\). Since \(y \in G\) was an arbitrary element in \(G\), therefore, \(\ell_x\) is irresolute on \(G\). We prove now that \(\ell_x\) is pre semi open mapping. Let \(A\) be a semi open set in \(G\). Then by Lemma 3.6, the set \(\ell_x(A) = x \ast A\) is semi open in \(G\), which means that \(\ell_x\) is a pre semi open mapping. Thus left translation is semi homeomorphism. Similarly we can prove that the right translation is semi homeomorphism.

Theorem 3.11. Let \((G, \ast, \tau)\) be an \((S-)\)topological group and let \(\beta_e\) be the base at identity element \(e\) of \(G\). Then:

1. (i) for every \(U \in \beta_e\), there is an element \(V \in SO(G, e)\) such that \(V^2 \subset U\);
2. (ii) for every \(U \in \beta_e\), there is an element \(V \in SO(G, e)\) such that \(V^{-1} \subset U\);
3. (iii) for every \(U \in \beta_e\), there is an element \(V \in SO(G, e)\) such that \(V \ast x \subset U\), for each \(x \in U\).

Proof (i). Let \(U \in \beta_e\). This implies that \(e \in U \subset G\) and \(U \ast e^{-1} = U\). Since \((G, \ast, \tau)\) is an \((S-)\)topological group therefore, there exists \(V \in SO(G, e)\) and by Lemma 3.5,
$V^{-1} \in SO(G,e)$ such that $V \ast V \subset U$. Therefore, $V^2 \subset U$.

(ii) Since $(G, \ast, \tau)$ is an s-topological group. Therefore, for every $U \in \beta_e$, there exists $V \in SO(G,e)$ such that $i(V) = V^{-1} \subset U$ because the identity function $i: G \rightarrow G$ is semi continuous by Theorem 7 [2].

(iii) Since $(G, \ast, \tau)$ is an s-topological group, therefore by Lemma 3.9, $\xi_x: G \rightarrow G$ and $\tau_x: G \rightarrow G$ are $S$-homeomorphisms. Thus for each $U \in \beta_e$, containing $x$, there exists $V \in SO(G,e)$ such that $\tau_x(V) = V \ast x \subset U$. The following example shows that Theorem 3.11 (i) is not true for $S$-topological groups.

**Example 3.12.** $G = \{0,1,2\}$ is a group under addition modulo 3, and $\tau = \{\varphi, G, [0], [0,1]\}$ is a topology on $G$. $SO(G) = \{\varphi, G, [0], [0,1]\}$. Then $(G, +, \tau)$ is an S-topological group. For any $U \in \beta_e$, containing $x, e$, there exists $V \in SO(G,e)$ such that $V \ast x \subset U$, satisfying $V^2 \subset U$.

**Lemma 3.13.** If $(G, \ast, \tau)$ is an irresolute topological group, then the inverse mapping $i: G \rightarrow G$ defined by $i(x) = x^{-1}$ for all $x \in G$ is a semi homeomorphism.

**Theorem 3.14.** Let $(G, \ast, \tau)$ be an irresolute topological group and $\mu_e$ be the collection of all semi open neighbourhoods of the identity element $e$ of $G$. Then:

(i) For every $U \in \mu_e$, there is an element $V \in \mu_e$ such that $V^2 \subset U$.

(ii) For every $U \in \mu_e$, there is an element $V \in \mu_e$ such that $V \ast x \subset U$. For each $x \in U$.

(iii) For every $U \in \mu_e$, there is an element $V \in \mu_e$ such that $V \ast x \subset U$. For each $x \in U$.

(iv) For every $U \in \mu_e$, and $x \in G$, there is an element $V \in \mu_e$ such that $x \ast V \ast x^{-1} \subset U$.

**Proof (i).** Let $U \in \mu_e$. This implies that there exists $e \in U \subset G$ and $U_{\mu_e} = U$. Since $(G, \ast, \tau)$ is an irresolute topological group, there exist $V, V^{-1} \in \mu_e$ such that $V \ast V \subset U$. Therefore, $V^2 \subset U$.

(ii). Since $(G, \ast, \tau)$ is an irresolute topological group. Therefore, for every $U \in \mu_e$, there exists $V \in SO(G,e)$ such that $i(V) = V^{-1} \subset U$ because the identity function $i: G \rightarrow G$ is irresolute by Lemma 3.13.

(iii). Since $(G, \ast, \tau)$ is an irresolute topological group and, therefore, by Lemma 3.10, $\xi_x: G \rightarrow G$ and $\tau_x : G \rightarrow G$ are semi homeomorphisms. Thus for each $U \in \mu_e$, containing $x$, there exists $V \in SO(G,e)$ such that $\tau_x(V) = V \ast x \subset U$.

(iv). It follows from the fact that $\xi_x$ and $\tau_x^{-1}$ are semi homeomorphisms of $G$ and $\xi_x(e) = x$, so that there exists $V \in \mu_e$ such that $\xi_x(V) = x \ast V$, and $x \ast V$ is a semi open neighbourhood of $x$, and $r_{x^{-1}}(x \ast V) = x \ast V \ast x^{-1} \subset U$.

**Theorem 3.15.** If $(G, \ast, \tau)$ is an irresolute topological group, then it is an s-topological group.

**Proof.** Let $x, y \in G$ and let $W \subset G$ be an open neighbourhood of $x \ast y^{-1}$. Then $W \subset G$ is a semi open neighbourhood of $x \ast y^{-1}$. Since $(G, \ast, \tau)$ is an irresolute-topological group, there are semi open $U \subset G$ containing $x$ and $V \subset G$ containing $y$ such that $U \ast V^{-1} \subset W$. This implies that $(G, \ast, \tau)$ is an s-topological group.

**Theorem 3.16.** If $(G, \ast, \tau)$ is an Ir-irresolute topological group, then it is an S-topological group.

**Proof.** Since irresolute mapping is a semi continuous mapping, therefore an Ir-irresolute topological group $(G, \ast, \tau)$ is an S-topological group.

**Remark 3.17:**

Irreducible topological group $\Rightarrow$ s-topological group $\Rightarrow$ s-topological group $\Rightarrow$ S-topological group.

The following example shows that S-topological group may not be an s-topological group.

**Example 3.18.** Let $G = \{-1,1\}$ be a group under multiplication and $\tau = \{\varphi, G, [1], G\}$ a topology on $G$. In [15], it is shown that $(G, \ast, \tau)$ is an S-topological group but not a topological group. Here $(G, \ast, \tau)$ is not an s-topological group because the only semi open neighbourhood of $-1$ is $G$ and if we choose $[1]$ for $x = y = -1$, as an open neighbourhood of $x \ast y^{-1}$. Now, the only possibility for the selection of semi open neighbourhoods in $G$ containing $-1,1$ is $G, G$ but, then $G \ast G \notin \{1\}$. This shows that $(G, \ast, \tau)$ is not an s-topological group.

The following example shows that $(G, \ast, \tau)$ is an S-topological group but non of the followings: s-topological group, irresolute topological group and topological group.

**Example 3.19.** Suppose $G = \{0,1,2\}$ is a group under addition modulo 3 and topology on $G$ is $\tau = \{\varphi, G, [0], [0,1]\}$. Then $(G, +_3, \tau)$ is an S-topological group but not an s-topological group, therefore not an irresolute topological group. $(G, +_3, \tau)$ is not a topological group. The following example shows that Irr-topological group may neither be irresolute topological group nor topological group.

**Example 3.20.** $G = \{1, \omega, \omega^2\}$ is a group under multiplication, where $\omega$ is a complex cube root of unity and topology on $G$ is $\tau = \{\varphi, G, [0], [0,1]\}$. Then $(G, \ast, \tau)$ is an Ir-irresolute topological group but not an irresolute topological group, and $(G, \ast, \tau)$ is not a topological group.
Theorem 3.21. Let $V$ be a semi open neighbourhood of $e$ in $s$-topological group $(G, \ast, \tau)$. Then $V \subset sCl(V) \subset V^2$.

Proof. Note that $s \in sCl(V)$ if and only if every semi open neighbourhood of $s$ meets $V$. Since $s \ast V^{-1}$ is a semi open neighbourhood of $s$, it must meet $V$. Thus there is $t \in V$ of the form $s \ast v^{-1}$, where $v \in V$. But then $s = t \ast v \in V \ast V = V^2$ and $sCl(V) \subset V^2$. Thus $V \subset sCl(V) \subset V^2$.

Theorem 3.22. Let $(G, \ast, \tau)$ be an $s$-topological group. Then for every subset $A$ of $G$ and every open neighbourhood $U$ of the neutral element $e$, $sCl(A) \subset A \ast U$.

Proof. Since $(G, \ast, \tau)$ is an $s$-topological group. Then for every open neighbourhood $U$ of $e$ by Theorem 3.11, there exists $V \in SO(G, e)$ such that $V^{-1} \subset U$. Let $x \in sCl(A)$ and $x \ast V$ is a semi open neighbourhood of $x$. Therefore, there is $a \in A \ast x \ast V$, that is $a \in x \ast V$. This implies that $a = x \ast b$ for some $b \in V$. Then $x = a \ast b^{-1} \in a \ast V^{-1} \subset A \ast U$. Hence $sCl(A) \subset A \ast U$.

Corollary 3.23. Let $(G, \ast, \tau)$ be an irresolute topological group. Then for every subset $A$ of $G$ and every open neighbourhood $U$ of the neutral element $e$, $sCl(A) \subset A \ast U$.

Theorem 3.24. Let $(G, \ast, \tau)$ be an $s$-topological group and $\beta_e$ a base of the space $(G, \tau)$ at the neutral element $e$. Then for every subset $A$ of $G$, $sCl(A) = \cap \{A \ast U; U \in \beta_e\}$.

Proof. In view of Theorem 3.22, we only have to verify that if $x \in sCl(A)$, then there exists $U \in \beta_e$ such that $x \in A \ast U$. Since $x \in A$, then by definition there exists a semi open neighbourhood $W$ of $e$ such that $x \in W \ast A = \varphi$. Take $U$ in $\beta_e$ satisfying the condition $U^{-1} \subset W$. Then $x \in U^{-1} \cap A = \varphi$, that is $\{x\} \cap A = U = \varphi$. This implies that $x \notin A \ast U$.

Theorem 3.25. If $(G, \ast, \tau)$ is an $s$-topological group, then $(G, \tau)$ is $s$-regular and semi-$T_2$ space.

Proof. Suppose that $F \subset G$ is closed and $s \notin F$. Multiply by $s^{-1}$ allows us to assume that $s = e$. Since $F$ is closed, $W = G - F$ is an open neighbourhood of $e$. By Theorem 3.11, there exists $V \in SO(G, e)$ such that $V^2 \subset W$. Thus by Theorem 3.22, $sCl(V) \subset W$. Then $U = G - sCl(V)$ is a semi open neighbourhood containing $F$ which is disjoint from $V$. This proves that $(G, \ast, \tau)$ is $s$-regular. That is, $e \in V \in SO(G, e)$ and $e \neq y \in F \subset U \in SO(G)$ such that $V \cap U = \varphi$. Hence $G$ is semi-$T_2$ space.

Theorem 3.26. Let $(G, \ast, \tau)$ be an irresolute topological group, $F$ a semi closed subset of $G$, and $K$ is an $s$-compact subset of $G$ such that $F \cap K = \varphi$. Then there is semi open neighbourhood $V$ of $e$ such that $F \cap V \ast K = \varphi$ (and a semi open neighbourhood $V'$ of $e$ such that $F \cap K \ast V' = \varphi$).

Proof. Let $x \in K$, so $x \in G - F$, and $G - F = F^c$ is a semi open neighbourhood of $x$. Therefore, $F^c \ast x^{-1}$ is a semi open neighbourhood of $e$. By Theorem 3.14, there is a semi open neighbourhood $W_x$ of $e$ such that $W_x \ast W_{x^c} \subset F^c \ast x^{-1}$. Now $K \cap U_{x \in k} W_x \ast x$. Since $K$ is $s$-compact so $K \subset \cup_{i=1}^{n} W_{x_i} \ast x_i = \cup_{i=1}^{n} W_i \ast x_i$. Take $V = \cap_{i=1}^{n} W_i$ for any $x \in K, x \in W_i \ast x_i$ for some $i$. By construction of $V$, we have $V \subset W_i$ for each $i$. This implies $V \ast x \in W_i \ast W_i \ast x_i \subset F^c$. In other words, $F \cap V \ast x = \varphi$. Since this is true for any $x \in K$, we now have $F \cap V \ast K = \varphi$.

Theorem 3.27. If $K$ is $s$-compact, then $y \ast K^{-1}$ is $s$-compact in a $s$-topological group $(G, \ast, \tau)$.

Proof. Let $(U_a; a \in I)$ be a cover of $y \ast K^{-1}$, where $U_a \in SO(G)$). Then $y \ast K^{-1} \subset \cup_{a \in I} U_a$. This implies that $K^{-1} \subset \cap_{a \in I} U_{a^{-1}}$. This implies that $K \subset \cup_{a \in I} y \ast U_{a^{-1}}$. Therefore, there is a finite set $I_0$ such that $K \subset \cup_{a \in I_0} y \ast U_{a^{-1}}$. This implies that $y \ast K^{-1} \subset \cup_{a \in I_0} U_a$. This implies that $y \ast K^{-1} \subset \cup_{a \in I_0} U_a$. That is $y \ast K^{-1}$ has a finite subcover in $G$. Hence $y \ast K^{-1}$ is $s$-compact.

Theorem 3.28. Let $(G, \ast, \tau)$ be an s-topological group, $K$ a compact subset of $G$, and $F$ a semi closed subset of $G$. Then $F \ast K$ and $K \ast F$ are semi closed subsets of $G$.

Proof. If $F \ast K = G$ we are done, so let $y \in G - F \ast K$. This means $F \cap y \ast K^{-1} = \varphi$. Since $K$ is s-compact so by Theorem 3.27, $y \ast K^{-1}$ is s-compact. By Theorem 3.26, there is a semi open neighbourhood $V$ of $e$ such that $F \cap V \ast y \ast K^{-1} = \varphi$. That is, $F \ast K \cap V \ast y = \varphi$. Since $V \ast y$ is semi open neighbourhood of $y$ contained in $G - F \ast K$, we have $F \ast K$ is semi closed and similar arguments for the proof of $K \ast F$.

Theorem 3.29. A non-empty subgroup $H$ of an s-topological group $G$ is semi open if and only if its semi interior is non empty.

Proof. Assume that $x \in sInt(H)$. Then by definition there is a semi open set $V$ such that $x \in V \subset H$. For every $y \in H$, we have $y \ast V \subset y \ast H = H$. Since $V$ is semi open so is $y \ast V$, we conclude that $H = \cup \{y \ast V; y \in H\}$ is a semi open set as the union of semi open sets is semi open. Conversely is very easy approach.

Theorem 3.30. If $U \in SO(G, \tau)$, then the set $L = \cup_{n=1}^{\infty} U^n$ is a semi open set in an s-topological group $(G, \ast, \tau)$.

Proof. Since $U$ is semi open in an $s$-topological group $(G, \ast, \tau)$. Then by Lemma 3.5, $U \ast U = U^2 \in SO(G, \tau)$, $U^2 \ast U = U^3 \in SO(G, \tau)$ and similarly $U^4, U^5, \ldots$ all are semi open sets in $G$. Thus the
set $L = \bigcup_{n=1}^{\infty} U^n$ being the union of semi open sets is a semi open set.

**Lemma 3.31.** If $(G,\ast,\tau)$ is an s-topological group, then the inverse map $i: G \to G$ defined by $i(x) = x^{-1}$ for all $x \in G$ is an S-homeomorphism.

**Theorem 3.32.** Let $A$ be a subset of an s-topological group $(G,\ast,\tau)$. Then $(sInt(A))^{-1} = sInt(A^{-1})$.

**Proof.** By Lemma 3.31, the inverse mapping $i: G \to G$ is an S-homeomorphism, then $sInt(i(A)) = sInt(A^{-1}) = i(sInt(A)) = (sInt(A))^{-1}$.

**Theorem 3.33.** Let $(G,\ast,\tau)$ be an irresolute topological group. Then for any symmetric subset $A$ of $G$, the closure of $A$ is also symmetric in $G$.

**Proof.** By Lemma 3.13, the inverse mapping $i: G \to G$ defined by $i(x) = x^{-1}$, for each $x \in G$ is a semi homeomorphism. Hence $i(Cl(A)) = (Cl(A))^{-1} = Cl(A^{-1}) = Cl(A)$ because $A^{-1} = A$. That is, $(Cl(A))^{-1} = Cl(A)$.

**Definition 3.34.** Suppose $U$ is a semi open neighbourhood of the neutral element $e$ of an s-topological group $(G,\ast,\tau)$. A subset $A$ of $G$ is called $U$-semi disjoint if $f \notin a \ast U$, for any disjoint $a, b \in A$.

**Definition 3.35.** A collection $\tau_\ast$ of subsets of a topological space $(G, \tau)$ is semi discrete, provided each $x \in G$ has a semi open neighbourhood that intersects at most one member of $\tau_\ast$.

**Theorem 3.36.** Let $U$ and $V$ be semi open neighbourhoods of the neutral element $e$ in an s-topological group $(G,\ast,\tau)$ such that $V^a \subset U$ and $V^{-1} = V$. If a subset $A$ of $G$ is $U$-semi disjoint, then the family of semi open sets $\{a \ast V: a \in A\}$ is semi discrete in $G$.

**Proof.** It suffices to verify that, for every $x \in G$, a semi open neighbourhood $x \ast V$ of $x$ intersects at most one element of the family $\{a \ast V: a \in A\}$. Suppose to the contrary that, for some $x \in G$, there exists distinct elements $a, b \in A$ such that $x \ast V \cap a \ast V \neq \emptyset$, and $x \ast V \cap b \ast V \neq \emptyset$. Then $x^{-1} \ast a \in V^x$ and $b^{-1} \ast x \in V^b$, where $b^{-1} \ast a = (b^{-1} \ast x)(x^{-1} \ast a) \in V^a \subset U$. This implies that $a \in b \ast U$, which contradicts the assumption that $A$ is $U$-semi disjoint.

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