A Family of Weighted Mean Based Optimal Fourth Order Methods for Solving System of Nonlinear Equations

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Abstract: Solving system of nonlinear equations is a problem of great importance in engineering sciences. In this work, we present a family of an optimal fourth order convergent root finding methods for solving nonlinear equations. We also extend our special cases to solve system of nonlinear equations. Numerical tests are performed for the presented methods in comparison with previous methods of this domain to signify the importance of the new methods.

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1. Introduction

Finding zeros of the nonlinear equations has remained as one of the most interesting problems of numerical analysis. One point methods like Newton-Raphson and Secant's method had remained as widely used methods in the past. With the passage of time the need to obtain more efficient and accurate methods was felt but by the use of one point methods it was not possible. As the informational efficiency of single step method cannot made greater than one even if additional derivative evaluations are involved [14]. Traub defined the informational efficiency (IE) of an iterative method (IM) as $IE(IM) = \frac{\zeta}{f_e}$, where ζ is the order of convergence and f_e is the number of functional evaluation per iteration. Thus, in order to increase the order and efficiency of iterative methods there was an emergent need of multipoint methods. Also, there are many problems in engineering sciences in which system of nonlinear equations are obtained that cannot be solved by direct methods. This area had also been addressed in parallel. Newton's method is commonly used as the first step to construct composite multipoint methods while the next steps are constructed in various ways such as by approximating the integral in Newton's theorem:

$$f(x) = f(x_i) + \int_{x_i}^{x} f'(t)dt. \tag{1}$$

Frontini [4], used rectangular rule to approximate integral in (1) to get the following cubically convergent scheme:

$$y_i = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'\left(\frac{x_i + y_i}{2}\right)} \tag{2}$$

Frontini and [5], also Sormani extended their scheme (2) for multivariate case as follows:

$$\mathbf{X}_{i+1} = \mathbf{X}_i - \left[\sum_{j=1}^{m} \mathbf{A}_j \mathbf{F}'(\mathbf{X}_i - \tau_j(\mathbf{F}'(\mathbf{X}_i))^{-1}\mathbf{F}(\mathbf{X}_i))\right]^{-1} \mathbf{F}(\mathbf{X}_i)$$

Weerakoon and Fernando [15], used trapezoidal rule to approximate the integral in (1) and obtained the following two step third order scheme in which first step is Newton's Method:

$$z_{i+1} = z_i - \frac{f(z_i)}{f'(z_i)},$$

$$x_{i+1} = x_i - \frac{2f(x_i)}{f'(x_i) + f'(z_{i+1})}. \tag{3}$$

(3)

The above method (3) is also called trapezoidal or arithmetic mean Newton's method. It is also notable that schemes (2) and (3) having cubic order of convergence are not optimal as conjectured by Kung and Traub [14]. If harmonic mean is used to approximate the integral in (1), a two point non-optimal cubically convergent method of Özban [12] is obtained as follows:

$$x_{i+1} = x_i - \frac{f(x_i)(f'(x_i) + f'(z_{i+1}))}{2f'(x_i)f'(z_{i+1})}, \tag{4}$$

where,

$$z_{i+1} = z_i - \frac{f(z_i)}{f'(z_i)}.$$

In the similar way, Homeier [6], also developed a cubically convergent scheme using inverse interpolation as follows:

$$x_{i+1} = x_i - \frac{f(x_i)}{2}\left(\frac{1}{f'(x_i)} + \frac{1}{f'(y_i)}\right). \quad (5)$$

Among two step methods there are some iterative schemes which uses two derivative evaluations and one functional evaluation with optimal order of convergence four known as Jarratt type schemes [7] and are given by: (6)

$$x_{i+1} = x_i - \left[1 - \frac{2}{3}\frac{f'(y_i) - f'(x_i)}{3f'(y_i) - f'(x_i)}\right]\frac{f(x_i)}{f'(x_i)}, \quad (6)$$

where,

$$y_i = x_i - \frac{2}{3}\frac{f(x_i)}{f'(x_i)}.$$

and the extension of the scheme (6) for multivariate case is given as [9]:

$$\mathbf{Y}^{(i)} = \mathbf{X}^{(i)} - \frac{2}{3}\mathbf{F}(\mathbf{X}^{(i)})^{-1}\mathbf{F}(\mathbf{X}^{(i)}),$$

$$\mathbf{X}^{(i+1)} = \mathbf{X}^{(i)} - \left[\mathbf{I} - \frac{2}{3}(3\mathbf{F}'(\mathbf{Y}^{(i)}) - \mathbf{F}'(\mathbf{X}^{(i)}))^{-1} \times (\mathbf{F}'(\mathbf{Y}^{(i)}) - \mathbf{F}'(\mathbf{X}^{(i)}))\right] \times \mathrm{F}'(\mathrm{X}^{(i)})^{-1}\mathrm{F}(\mathrm{X}^{(i)}). \quad (7)$$

In [10], Khattri developed the fourth order scheme based on Jarrat's first step which requires two evaluations of the derivative and one function evaluation:

$$y_i = x_i - \frac{2}{3}\frac{f(x_i)}{f'(x_i)},$$

$$x_{i+1} = x_i - \left[1 + \sum_{j=1}^{4}\beta_j\left(\frac{f'(y_i)}{f'(x_i)}\right)^j\right]\frac{f(x_i)}{f'(x_i)}, \beta_j \in R. \quad (8)$$

Chun [3], added his contribution by making another fourth order scheme given by:

$$y_i = x_i - \frac{2}{3}\frac{f(x_i)}{f'(x_i)},$$

$$x_{i+1} = x_i - \left[1 + \frac{3}{4}\frac{f'(x_i) - f'(y_i)}{f'(x_i)} + \frac{9}{8}\left(\frac{f'(x_i) - f'(y_i)}{f'(x_i)}\right)^2\right]\frac{f(x_i)}{f'(x_i)} \quad (9),$$

Soleymani [13], replaced the parameter approach by the concept of weight function approach to construct a family of new optimal Jarratt type two step methods:

$$y_k = x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)},$$

$$x_{k+1} = x_k - \frac{2f(x_k)}{f'(x_k) + f'(y_k)}[G(\mu) \times H(\nu)]. \quad (10)$$

Babajee [1], extended one example of Soleymani's findings [13] for the multivariate case as follows:

$$\mathbf{X}^{(i+1)} = \mathbf{X}^{(i)} - (\mathrm{I} - (1/4)(\tau(\mathbf{X}^{(i)}) - \mathrm{I}) + (3/4)(\tau(\mathbf{X}^{(i)}) - \mathrm{I})^2)\mathbf{A}_1(\mathbf{X}^{(i)})^{-1}\mathrm{F}(\mathbf{X}^{(i)}), \quad (11)$$

where,

$$\mathbf{A}_1(\mathbf{X}^{(i)}) = (1/2)(\mathbf{F}'(\mathbf{X}^{(i)}) + \mathbf{F}'(\mathbf{Y}^{(i)})),$$

$$\tau(\mathbf{X}^{(i)}) = \mathbf{F}'(\mathbf{X}^{(i)})^{-1}\mathbf{F}'(\mathbf{X}^{(i)} - (2/3)\mathbf{U}(\mathbf{X}^{(i)}))$$

and $\mathbf{U}(\mathbf{X}^{(i)}) = \mathbf{F}'(\mathbf{X}^{(i)})^{-1}\mathbf{F}(\mathbf{X}^{(i)})$ and $\mathbf{I}$ may be the $n \times n$ identity matrix. It is noted that all the above mentioned methods can be extended for multivariate case and extensions for some of them are given in [1, 5, 9]. Recently, Soleymani [13], developed two new classes of fourth order optimal Jarratt type methods. Motivated by the research going on this direction, we, in this paper developed a fourth order weighted mean based optimal Jarratt type scheme by using combination of weight functions in the second step. We have also extended our method for system of nonlinear equations. Rest of the paper is organized as follows. The development of fourth order optimal method with its convergence analysis is given in Section 2. Its extension for the multivariate case with corresponding convergence analysis is presented in Section 3. Section 4 consists of numerical comparison for univariate functions and Section 5 is comprised of computational cost and numerical comparison for the multivariate case. Section 6 consists of concluding remarks.

2. New Family of Weighted Mean Based Methods and Its Convergence Analysis

Now, we propose an optimal weighted mean based family of fourth order methods. We are considering two steps scheme in which first step is similar to Jarratt's method and second step involves function evaluation at mid point with a combination of weight functions given as:

$$y_i = x_i - \beta \frac{f(x_i)}{f'(x_i)},$$

$$x_{i+1} = x_i - \left[S(\phi_i) \times T(\psi_i)\right] \frac{f(x_i)}{f'(z_i)}, \quad (12)$$

where, $z_i = \frac{x_i + y_i}{2}$, $\phi_i = \frac{f(x_i)}{f'(x_i)}$, $\psi_i = \frac{f'(z_i)}{f'(x_i)}$

and $T(\phi_i)$ and $S(\psi_i)$ represents real valued weight functions chosen such that new scheme (12) achieves optimal fourth order convergence as proved in the Theorem1 stated below.

Theorem 1 *Let $f : I \subseteq \Re \to \Re$ be a sufficiently differentiable function in the neighborhood of the root* □ *in the open interval I. Then, for $\beta = \frac{4}{3}$, the new without memory scheme (12) has optimal fourth order convergence such that*

$$S(1) = 1, \quad S'(1) = \frac{1}{4}, \quad S''(1) = \frac{3}{4}, \quad \left|S'''(1)\right| \le \infty,$$

$$T(0) = 1, \quad T'(0) = T''(0) = 0, \quad \left|T'''(0)\right| \le \infty, \quad (13)$$

and satisfies the following error equation:

$$e_{i+1} = \left(\frac{7}{3}c_2^3 - c_2 c_3 + \frac{1}{9}c_4 + \frac{32}{81}S'''(1)c_2^3 - \frac{1}{6}T'''(0)\right) e_i^4 + O(e_i^5). \quad (14)$$

Proof

Let $e_i = x_i - \omega$ be the error at ith computing step. By using Taylor's expansion of $f(x_i)$ about the root ω, we get:

$$f(x_i) = f'(\omega)(e_i + c_2 e_i^2 + c_3 e_i^3 + c_4 e_i^4) + O(e_i^5), \quad (15)$$

where, $c_i = \frac{f^{(i)}(\omega)}{i!\, f'(\omega)}, i \ge 2$ and first derivative in first step of our scheme can be computed as:

$$f'(x_i) = f'(\omega)(1 + 2c_2 e_i + 3c_3 e_i^2 + 4c_4 e_i^3) + O(e_i^4). \quad (16)$$

Using (15) and (16) in first step of (12), the Taylor's expansions for y_i and z_i and $f'(z_i)$ are given as:

$$y_i = \omega + (1-\beta)e_i + \beta c_2 e_i^2 - \beta(-2c_3 + 2c_2^2)e_i^3 - \beta(c_2(-3c_3 + 4c_2^2) - 3c_4 + 4c_2 c_3 + 2(3c_3 - 4c_2^2)c_2)e_i^4, \quad (17)$$

$$z_i = \omega + (1 - \frac{1}{2}\beta)e_i + \frac{1}{2}\beta c_2 e_i^2 - \frac{1}{2}\beta(-2c_3 + 2c_2^2)e_i^3 - \frac{1}{2}\beta (c_2(-3c_3 + 4c_2^2) - 3c_4 + 4c_2 c_3 + 2(3c_3 - 4c_2^2)c_2)e_i^4 + O(e_i^5).$$

$$f'(z_i) = 1 + 2c_2(1 - \frac{1}{2}\beta)e_i + (\beta c_2^2 + 3c_3(1 - \frac{1}{2}\beta)^2 e_i^2 + (3c_3(1 - \frac{1}{2}\beta)\beta c_2 - c_2\beta(-2c_3 + 2c_2^2) + 4c_4(1 - \frac{1}{2}\beta)^3)) e_i^3 + (3c_3(-(1 - \frac{1}{2}\beta)\beta(-2c_3 + 2c_2^2) + \frac{1}{4}\beta^2 c_2^2) + 6c_4(1 - \frac{1}{2}\beta)^2 \beta c_2) - c_2\beta(c_2(-3c_3 + 4c_2^2) - 3c_4 + 4c_2c_3 + 2(3c_3 - 4c_2^2)c_2) + 5c_5(1 - \frac{1}{2}\beta)^4)e_i^4 + O(e_i^5). \quad (18)$$

In the similar manner, for the second step of (12), we have:

$$\phi_i = 1 + (-2c_2 + 2c_2(1 - \frac{1}{2}\beta))e_i + (-3c_3 + 4c_2^2 + \beta c_2^2 + 3c_3 (1 - \frac{1}{2}\beta)^2 - 4c_2^2(1 - \frac{1}{2}\beta))e_i^2 + (2c_2(1 - \frac{1}{2}\beta)(-3c_3 + 4c_2^2) + 3c_3(1 - \frac{1}{2}\beta)\beta c_2 - c_2\beta(-2c_3 + 2c_2^2) + 4c_4(1 - \frac{1}{2}\beta)^3 - 2(\beta c_2^2 + 3c_3(1 - \frac{1}{2}\beta)^2)c_2 - 4c_4 + 6c_2c_3 + 2(3c_3 - 4c_2^2)c_2)e_i^3) + O(e_i)^4. \quad (19)$$

Again using Taylor's expansion we accomplish:

$$S(\phi_i) = S(1) - 2S'(1)\beta c_2 e_i + (6S'(1)\beta c_2^2 - 6S'(1)\beta c_3 + 3S'(1)\beta^2 c_3 + 2S'(1)\beta^2 c_2^2)e_i^2 + \cdots + O(e_i^5). \quad (20)$$

By using (16) and (18) ,we get:

$$\psi_i = e_i - c_2 e_i^2 + (-2c_3 + 2c_2^2)e_i^3 + (c_2(-3c_3 + 4c_2^2) - 3c_4 + 4c_2c_3 + 2(3c_3 - 4c_2^2)c_2)e_i^4 + O(e_i^5). \quad (21)$$

Expanding Taylor's series of $T(\psi_i)$ about zero, we get:

$$T(\psi_i) = T(0) + T'(0)e_i + (-T'(0)c_2 + \frac{1}{2}T''(0))e_i^2 + \cdots + O(e_i^5) \quad (22)$$

Substituting $\beta = \frac{4}{3}$, (15), (18), (20) and (22) in (12) and using (13) we have:

$$x_{i+1} = \omega + (\frac{7}{3}c_2^3 - c_2c_3 + \frac{1}{9}c_4 + \frac{32}{81}S'''(1)c_2^3 - \frac{1}{6}T'''(0))e_i^4 + O(e_i^5), \quad (23)$$

gives the following error equation showing new scheme has optimal convergence order upto 4:

$$e_{i+1} = \left(\frac{7}{3}c_2^3 - c_2c_3 + \frac{1}{9}c_4 + \frac{32}{81}S'''(1)c_2^3 - \frac{1}{6}T'''(0)\right)e_i^4 + O(e_i^5). \quad (24)$$

3. Extension of New Family for Multivariate Case And Its Convergence Analysis

Now, we extend our new weighted mean based family of optimal methods (12) for solving systems of nonlinear equations. We present here a special case of our family (12) by defining following weight functions satisfying the conditions of Theorem 1.

$$S(\phi_i) = 1 + \frac{1}{4}\left(\frac{f'(z_i)}{f'(x_i)} - 1\right) + \frac{3}{8}\left(\frac{f'(z_i)}{f'(x_i)} - 1\right)^2, \quad (25)$$

$$T(\psi_i) = 1.$$

Thus, we obtain a new weighted mean based fourth order method as:

$$y_i = x_i - \frac{4}{3}\frac{f(x_i)}{f'(x_i)},$$

$$x_{i+1} = x_i - \left[\left(1 + \frac{1}{4}\left(\frac{f'(z_i)}{f'(x_i)} - 1\right) + \frac{3}{8}\left(\frac{f'(z_i)}{f'(x_i)} - 1\right)^2\right)\right] \times \frac{f(x_i)}{f'(z_i)}, \quad (26)$$

Where $z_i = \frac{x_i + y_i}{2}$.

Let $A_0 \subset D$, where A_0 is an open convex neighborhood of $\mathbf{X}^{(0)}$ and $\mathbf{F}: D \subseteq \Re^n \to \Re^n$ be sufficiently Fréchet differentiable in A_0, $\mathbf{X}^{(i)}$ be approximate root of the exact root $\mathbf{W}$ and $\mathbf{E}^{(i)} = \mathbf{X}^{(i)} - \mathbf{W}$. The i th derivative of $\mathbf{F}$ at $\mathbf{h} \in \Re^n$, $\mathbf{h} \geq 1$, is an $i-$ linear function $\mathbf{F}^{(i)}(\mathbf{h}): \Re^{(n)} \times \Re^{(n)} \times \ldots \times \Re^{(n)} \to \Re^{(n)}$ such that $\mathbf{F}^{(i)}(X^{(i)})\mathbf{h} \in \Re^n$. Now, the iterative method (26) is extended for solving system of nonlinear equations as follows:

$$\mathbf{X}^{(i+1)} = \mathbf{X}^{(i)} - [(\mathbf{I} + \frac{1}{4}(\phi(\mathbf{X}^{(i)}) - \mathbf{I}) + \frac{3}{8}(\phi(\mathbf{X}^{(i)}) - \mathbf{I})^2))] \times \mathbf{F}'(\mathbf{Z}^{(i)})^{-1}\mathbf{F}(\mathbf{X}^{(i)}) \quad (27)$$

where,

$$\mathbf{Y}^{(i)}(\mathbf{X}) = \mathbf{X}^{(i)} - (\frac{4}{3})[\mathbf{F}'(\mathbf{X}^{(i)})]^{-1}\mathbf{F}(\mathbf{X}^{(i)}), \ \phi(\mathbf{X}^{(i)}) = [\mathbf{F}'(\mathbf{X}^{(i)})]^{-1}\mathbf{F}'(\mathbf{Z}^{(i)}),$$

$\mathbf{I}$ is the $\mathbf{n} \times \mathbf{n}$ identity matrix and $\mathbf{F}(\mathbf{X}^{(i)}) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_2))a$.

The following termonologies are defined in [11]:

$$[\mathbf{F}'(\mathbf{X}^{(i)})\mathbf{hk}]^T = \mathbf{k}^T\mathbf{S}(\mathbf{X}^{(i)})\mathbf{h} \quad (28)$$

and second derivative is

$$[\mathbf{F}''(\mathbf{X}^{(i)})\mathbf{hk}]^T = (\mathbf{k}^T\mathbf{S}_1(\mathbf{X}^{(i)})\mathbf{h}, \mathbf{k}^T\mathbf{S}_2(\mathbf{X}^{(i)})\mathbf{h}, \ldots, \mathbf{k}^T\mathbf{S}_m(\mathbf{X}^{(i)})\mathbf{h}),$$

where $S_1(\mathbf{x}), \ldots, S_m(\mathbf{x})$ are the Hessian matrices of $f_1, \ldots, f_m$ at $\mathbf{X}$ Thus, Taylor's series for $n-$ dimension case can be written as:

$$\mathbf{F}(\mathbf{X}^{(i)}) = \mathbf{F}'(W)\left[\mathbf{E}^{(i)} + \mathbf{C}_2(\mathbf{E}^{(i)})^2 + \mathbf{C}_3(\mathbf{E}^{(i)})^3 + \mathbf{C}_4(\mathbf{E}^{(i)})^4\right] + O((\mathbf{E}^{(i)})^5).$$

where,

$\mathbf{C_i} = \frac{1}{\mathbf{i}!}[\mathbf{F}'(\mathbf{X}_0)]^{-1}\mathbf{F}^{(\mathbf{i})}(\mathbf{X}_0)$, $\mathbf{F}'(\mathbf{X}_0)$ is continuous and non-singular and $\mathbf{X}^{(0)}$ is closer to $\mathbf{W}$. By using above Taylor's expansion, we can prove the following result.

Theorem 2 Let $\mathbf{A}_0 \subseteq \mathbf{D}$ be an open convex set containing the root $\mathbf{W}$ of $\mathbf{F(X) = 0}$ and $\mathbf{F}: \mathbf{D} \subseteq \Re^\mathbf{n} \to \Re^\mathbf{n}$, be four-times Fréchet differentiable in A_0 which has continuous and non-singular Jacobian matrix in D. Then, the new method (27) has order of convergence four.

Proof

Let $\mathbf{W}$ be the solution of nonlinear system $\mathbf{F(X) = 0}$, and $\mathbf{X}^{(0)}$ be an initial guess close to $\mathbf{W}$ then by Taylor expansion of $\mathbf{F}(\mathbf{X}^{(\mathbf{i})})$ about $\mathbf{W}$, we have:

$$F(X^{(i)}) = F'(W)\left[E^{(i)} + C_2(E^{(i)})^2 + C_3(E^{(i)})^3 + C_4(E^{(i)})^4\right] + O((E^{(i)})^5), \quad (29)$$

and

$$F'(X^{(i)}) = F'(W)\left[I + 2C_2(E^{(i)}) + 3C_3(E^{(i)})^2 + 4C_4(E^{(i)})^3\right] + O((E^{(i)})^4), \quad (30)$$

where $\mathbf{C}_i = \frac{1}{i!}[\mathbf{F}'(\mathbf{W})]^{-1}\mathbf{F}^{(i)}(\mathbf{W}), i \geq 2$ and

$\mathbf{E}^{(i)} = \mathbf{X}^{(i)} - \mathbf{W}.$

We calculate,

$$[\mathbf{F}'(\mathbf{X}^{(i)})]^{-1} = \begin{bmatrix} \mathbf{I} - 2\mathbf{C}_2(\mathbf{E}^{(i)}) + (4\mathbf{C}_2^2 - 3\mathbf{C}_3)(\mathbf{E}^{(i)})^2 \\ +(-8\mathbf{C}_2^3 + 12\mathbf{C}_2\mathbf{C}_3 - 4\mathbf{C}_4)(\mathbf{E}^{(i)})^3 \end{bmatrix} + \mathbf{O}((\mathbf{E}^{(i)})^4) \quad (31)$$

and

$$\frac{\mathbf{F}(\mathbf{X}^{(i)})}{\mathbf{F}'(\mathbf{X}^{(i)})} = \begin{bmatrix} \mathbf{E}^{(i)} - \mathbf{C}_2(\mathbf{E}^{(i)})^2 + 2(\mathbf{C}_2^2 - \mathbf{C}_3)(\mathbf{E}^{(i)})^3 \\ +(-4\mathbf{C}_2^3 + 7\mathbf{C}_2\mathbf{C}_3 - 3\mathbf{C}_4)(\mathbf{E}^{(i)})^4 \end{bmatrix} + \mathbf{O}((\mathbf{E}^{(i)})^5) \quad (32)$$

and using (32) we have,

$$\mathrm{Y}^{(i)} = -\frac{1}{3}(\mathrm{E}^{(i)}) + \frac{4}{3}\mathrm{C}_2(\mathrm{E}^{(i)})^2 - \frac{8}{3}(\mathrm{C}_2^2 - \mathrm{C}_3)(\mathrm{E}^{(i)})^3$$

$$+(-8\mathrm{C}_2\mathrm{C}_3 + \frac{16}{3}\mathrm{C}_2^3 + 4\mathrm{C}_4)(\mathrm{E}^{(i)})^4 + \mathrm{O}((\mathrm{E}^{(i)})^5). \quad (33)$$

Thus, we get

$$\mathbf{Z}^{(i)} = \mathbf{W} + (\frac{1}{3})\mathbf{E}^{(i)} + \frac{2}{3}\mathbf{C}_2(\mathbf{E}^{(i)})^2 + (\frac{4}{3}\mathbf{C}_3 - \frac{4}{3}\mathbf{C}_2^2)(\mathbf{E}^{(i)})^3 +$$

$$(\frac{8}{3}\mathrm{C}_2^3 - \frac{14}{3}\mathrm{C}_2\mathrm{C}_3 + 2\mathrm{C}_4)(\mathrm{E}^{(i)})^4 + \mathrm{O}(\mathrm{E}^{(i)})^5 \quad (34)$$

The Jacobian matrix $\mathrm{F}'(\mathrm{Z}^{(i)})$ has the following Taylor's expansion:

$$\mathrm{F}'(\mathrm{Z}^{(i)}) = 1 + \frac{2}{3}\mathbf{C}_2(\mathrm{E}^{(i)}) + (\frac{4}{3}\mathbf{C}_2^2 + \frac{1}{3}\mathbf{C}_3)(\mathrm{E}^{(i)})^2 +$$

$$(4\mathbf{C}_2\mathbf{C}_3 - (\frac{8}{3}\mathbf{C}_2^3) + \frac{4}{27}\mathrm{C}_4)(\mathrm{E}^{(i)})^3 + \mathrm{O}(\mathrm{E}^{(i)})^4 \quad (35)$$

We obtain,

$$\frac{\mathrm{F}'(\mathrm{Z}^{(i)})}{\mathrm{F}'(\mathrm{X}^{(i)})} = 1 - \frac{4}{3}\mathbf{C}_2\mathrm{E}^{(i)} + (-\frac{3}{8}\mathbf{C}_3 + 4\mathbf{C}_2^2)(\mathrm{E}^{(i)})^2 +$$

$$(\frac{2}{3}\mathbf{C}_2(4\mathbf{C}_2^2 - 3\mathbf{C}_3) + \frac{22}{3}\mathbf{C}_2\mathbf{C}_3 + 2\mathbf{C}_2(\frac{4}{3\,2}\mathbf{C}_3 - \frac{4}{3}\mathbf{C}_2^2)$$

$$-\frac{104}{27}\mathrm{C}_4 - 2(\frac{4}{3}\mathbf{C}_2^2 + \frac{1}{3}\mathbf{C}_3)\mathbf{C}_2 + 2(-4\mathrm{C}_2^2 + 3\mathrm{C}_3)\mathrm{C}_2)$$

$$(\mathrm{E}^{(i)})^3 + \mathrm{O}(\mathrm{E}^{(i)})^4. \quad (36)$$

Using (32), (35)and (36) in second step of (27), we attain with following error term

$$\mathbf{E}^{(i+1)} = (\frac{7}{3}\mathbf{C}_2^3 - \mathbf{C}_2\mathbf{C}_3 + \frac{1}{9}\mathbf{C}_4)(\mathbf{E}^{(i)})^4 + O(\mathbf{E}^{(i)})^5, \quad (37)$$

which shows that the proposed method (27) has fourth order convergence.

4. Numerical Comparisons for Univariate Functions

In this section, we compare our new optimal fourth order family of methods for solving non-linear equations (12) (SMNF1) with Newton's method (NM), Khattri's method (KM) [10], Chun's method (CM) [3] and Solemani's method (SM) [13]. We use Maple 7 for all the computations. The test functions are given in Table 1 taken from literature. The stopping criterion is $|f(x_m)| + |x_m - \omega| < 10^{-800}$ with a precision of 1000 decimal digits for approximating the roots. Tables 2-6 shows the number of iterations, n, the absolute values of the function $|f(x_m)|$ and the absolute values of the approximated root and the exact root, $|x_m - \omega|$ for each iterative step.

Table 1: Test Functions and Their Roots

Numerical Example	Exact Zero
$f_1(x) = \sin 3x + x\cos x$	ω = 0.0000000000000000
$f_2(x) = x + \cos x^2 - \frac{1}{2}$	ω = -0.4747149936699329
$f_3(x) = (x+2)e^x - 1$	ω = -0.4428544010023885
$f_4(x) = \sin^{-1}(x^2 - 1) - \frac{1}{2}x + 1$	ω = 0.5948109683983691
$f_5(x) = x^2e^x - \sin x$	ω=0.0000000000000000

Table 2: Comparison of Various Iterative Methods for $f_1(x)$

$f_1(x)$ $x_0 = 0.4$	**NM**	**KM**	**CM**	**SM**	**SMNF1**
n	9	D	D	D	7
$\|x_1-\omega\|$	$3.02\ 10^{-1}$	$2.56\ 10^{0}$	$7.05\ 10^{-1}$	$6.07\ 10^{-1}$	$3.80\ 10^{-1}$
$\|f_1(x_1)\|$	$1.075\ 10^{0}$	$1.16\ 10^{0}$	$1.39\ 10^{0}$	$1.46\ 10^{0}$	$1.26\ 10^{0}$
$\|x_2-\omega\|$	$9.4\ 10^{-2}$	$3.0\ 10^{0}$	$2.10\ 10^{0}$	$35854.07\ 10^{0}$	$1.43\ 10^{-3}$
$\|f_1(x_2)\|$	$3.72\ 10^{-1}$	$2.64\ 10^{0}$	$1.03\ 10^{0}$	$21526.38\ 10^{0}$	$5.69\ 10^{-3}$
$\|x_3-\omega\|$	$2.13\ 10^{-3}$	$2.11\ 10^{0}$	$95.83\ 10^{0}$	35853.13	$4.12\ 10^{-12}$
$\|f_1(x_3)\|$	$8.54\ 10^{-3}$	$1.03\ 10^{0}$	$2.23\ 10^{0}$	$10417.75\ 10^{0}$	$1.64\ 10^{-11}$
$\|x_4-\omega\|$	$2.43\ 10^{-8}$	$883.95\ 10^{0}$	$95.80\ 10^{0}$	$35853.41\ 10^{0}$	$2.88\ 10^{-46}$
$\|f_1(x_4)\|$	$9.74\ 10^{-8}$	$344.38\ 10^{0}$	$8.81\ 10^{-8}$	$290.79\ 10^{0}$	$1.15\ 10^{-45}$
$\|x_5-\omega\|$	$3.61\ 10^{-23}$	$884.35\ 10^{0}$	$95.80\ 10^{0}$	$35853.42\ 10^{0}$	$6.91\ 10^{-183}$
$\|f_1(x_5)\|$	$1.44\ 10^{-22}$	$1.11\ 10^{0}$	$3.90\ 10^{-157}$	$1.55\ 10^{-4}$	$2.76\ 10^{-182}$
$\|x_6-\omega\|$	$1.17\ 10^{-67}$	$884.35\ 10^{0}$	$95.80\ 10^{0}$	$35853.42\ 10^{0}$	$2.28\ 10^{-729}$
$\|f_1(x_6)\|$	$4.71\ 10^{-67}$	$5.21\ 10^{-13}$	$4.56\ 10^{-635}$	$1.25\ 10^{-29}$	$9.15\ 10^{-729}$

Table 3: Comparison of Various Iterative Methods for $f_2(x)$

$f_2(x)$ $x_0 = -0.3$	**NM**	**KM**	**CM**	**SM**	**SMNF1**
n	10	5	5	6	5
$\|x_1-\omega\|$	$1.12\ 10^{-2}$	$3.64\ 10^{-3}$	$8.67\ 10^{-5}$	$1.13\ 10^{-3}$	$1.43\ 10^{-4}$
$\|f_2(x_1)\|$	$1.36\ 10^{-2}$	$4.41\ 10^{-3}$	$1.05\ 10^{-4}$	$1.37\ 10^{-3}$	$1.28\ 10^{-3}$
$\|x_2-\omega\|$	$6.99\ 10^{-5}$	$2.06\ 10^{-14}$	$2.53\ 10^{-17}$	$2.0\ 10^{-12}$	$1.26\ 10^{-12}$
$\|f_2(x_2)\|$	$8.47\ 10^{-5}$	$2.5\ 10^{-14}$	$3.07\ 10^{-17}$	$2.43\ 10^{-12}$	$1.52\ 10^{-12}$
$\|x_3-\omega\|$	$2.67\ 10^{-9}$	$2.12\ 10^{-55}$	$1.84\ 10^{-67}$	$1.98\ 10^{-47}$	$2.55\ 10^{-48}$
$\|f_2(x_3)\|$	$3.24\ 10^{-9}$	$2.58\ 10^{-55}$	$2.23\ 10^{-67}$	$2.40\ 10^{-47}$	$3.09\ 10^{-48}$
$\|x_4-\omega\|$	$3.91\ 10^{-18}$	$2.41\ 10^{-219}$	$5.20\ 10^{-268}$	$1.91\ 10^{-187}$	$4.30\ 10^{-191}$
$\|f_2(x_4)\|$	$4.74\ 10^{-18}$	$2.92\ 10^{-219}$	$6.31\ 10^{-268}$	$2.32\ 10^{-187}$	$5.21\ 10^{-191}$
$\|x_5-\omega\|$	$8.36\ 10^{-36}$	$3.97\ 10^{-875}$	0	$1.65\ 10^{-747}$	0
$\|f_2(x_5)\|$	$1.01\ 10^{-35}$	$4.82\ 10^{-875}$	0	$2.00\ 10^{-747}$	0
$\|x_6-\omega\|$	$3.82\ 10^{-71}$	0	0	0	0
$\|f_2(x_6)\|$	$4.63\ 10^{-71}$	0	0	0	0

Table 4: Comparison of Various Iterative Methods for $f_3(x)$

$f_3(x)$ $x_0 = -0.47$	**NM**	**KM**	**CM**	**SM**	**SMNF1**
n	9	5	5	5	5
$\|x_1-\omega\|$	$5.20\ 10^{-4}$	$1.78\ 10^{-6}$	$8.70\ 10^{-7}$	$2.47\ 10^{-8}$	$2.39\ 10^{-7}$
$\|f_3(x_1)\|$	$8.54\ 10^{-4}$	$2.93\ 10^{-6}$	1.4310^{-6}	$4.06\ 10^{-8}$	$3.93\ 10^{-7}$
$\|x_2-\omega\|$	$1.88\ 10^{-7}$	$3.03\ 10^{-23}$	$8.54\ 10^{-25}$	$1.39\ 10^{-32}$	$1.35\ 10^{-27}$
$\|f_3(x_2)\|$	$3.08\ 10^{-7}$	$4.99\ 10^{-23}$	$1.40\ 10^{-24}$	$2.29\ 10^{-32}$	$2.22\ 10^{-27}$
$\|x_3-\omega\|$	$2.46\ 10^{-14}$	$2.54\ 10^{-90}$	$7.92\ 10^{-97}$	$1.41\ 10^{-129}$	$1.39\ 10^{-108}$
$\|f_3(x_3)\|$	$4.04\ 10^{-14}$	$4.17\ 10^{-90}$	$1.30\ 10^{-96}$	$2.31\ 10^{-129}$	$2.28\ 10^{-108}$
$\|x_4-\omega\|$	$4.21\ 10^{-28}$	$1.24\ 10^{-358}$	$5.85\ 10^{-385}$	$1.47\ 10^{-517}$	$1.55\ 10^{-432}$
$\|f_3(x_4)\|$	$6.91\ 10^{-28}$	$2.04\ 10^{-358}$	$9.61\ 10^{-385}$	$2.42\ 10^{-517}$	$2.54\ 10^{-432}$
$\|x_5-\omega\|$	$1.23\ 10^{-55}$	0	0	0	0
$\|f_3(x_5)\|$	$2.02\ 10^{-55}$	0	0	0	0
$\|x_6-\omega\|$	$1.05\ 10^{-110}$	0	0	0	0
$\|f_3(x_6)\|$	$1.73\ 10^{-110}$	0	0	0	0

Table 5: Comparison of Various Iterative Methods for $f_4(x)$

$f_4(x)$ $X_0=0.8$	**NM**	**KM**	**CM**	**SM**	**SMNF1**
n	10	7	5	6	6
$\|x_1-\omega\|$	$1.45\ 10^{-2}$	$1.04\ 10^{-2}$	$1.79\ 10^{-4}$	$1.41\ 10^{-3}$	$1.41\ 10^{-3}$
$\|f_4(x_1)\|$	$1.54\ 10^{-2}$	$1.10\ 10^{-2}$	$1.89\ 10^{-4}$	$1.49\ 10^{-3}$	$1.49\ 10^{-3}$
$\|x_2-\omega\|$	$5.67\ 10^{-5}$	$1.58\ 10^{-9}$	$4.86\ 10^{-17}$	$3.92\ 10^{-12}$	$3.92\ 10^{-12}$
$\|f_4(x_2)\|$	$6.01\ 10^{-5}$	$1.67\ 10^{-9}$	$5.14\ 10^{-17}$	$4.15\ 10^{-12}$	$4.15\ 10^{-12}$
$\|x_3-\omega\|$	$8.54\ 10^{-10}$	$8.28\ 10^{-37}$	$2.63\ 10^{-67}$	$2.37\ 10^{-46}$	$2.37\ 10^{-46}$
$\|f_4(x_3)\|$	$9.04\ 10^{-10}$	$8.77\ 10^{-37}$	$2.78\ 10^{-67}$	$2.51\ 10^{-46}$	$2.51\ 10^{-46}$
$\|x_4-\omega\|$	$1.94\ 10^{-19}$	$6.16\ 10^{-146}$	$2.25\ 10^{-268}$	$3.17\ 10^{-183}$	$3.17\ 10^{-183}$
$\|f_4(x_4)\|$	$2.06\ 10^{-19}$	$6.52\ 10^{-146}$	$2.39\ 10^{-268}$	$3.35\ 10^{-183}$	$3.35\ 10^{-183}$
$\|x_5-\omega\|$	$1.00\ 10^{-38}$	$0.41\ 10^{-500}$	0	$1.01\ 10^{-730}$	$1.01\ 10^{-730}$
$\|f_4(x_5)\|$	$1.06\ 10^{-38}$	$2.00\ 10^{-582}$	0	$1.07\ 10^{-730}$	$1.07\ 10^{-730}$
$\|x_6-\omega\|$	$2.67\ 10^{-77}$	$4.14\ 10^{-500}$	0	0	0
$\|f_4(x_6)\|$	$2.83\ 10^{-77}$	0	0	0	0

Table 6: Comparison of Various Iterative Methods for $f_5(x)$

$f_5(x)$ $x_0 = 0.22$	NM	KM	CM	SM	SMNF1
n	10	D	D	8	6
$\|x_1-\omega\|$	$2.09\ 10^{-1}$	$8.46\ 10^{0}$	$1.57\ 10^{0}$	$6.11\ 10^{-1}$	$2.49\ 10^{-1}$
$\|f_5(x_1)\|$	$2.44\ 10^{-1}$	$8.36\ 10^{-1}$	$1.51\ 10^{0}$	$7.77\ 10^{-1}$	$2.95\ 10^{-1}$
$\|x_2-\omega\|$	$1.96\ 10^{-2}$	$11.00\ 10^{0}$	$1.24\ 10^{11}$	$1.41\ 10^{-1}$	$4.70\ 10^{-3}$
$\|f_5(x_2)\|$	$1.99\ 10^{-2}$	$9.97\ 10^{-1}$	D	$1.57\ 10^{-1}$	$4.70\ 10^{-3}$
$\|x_3-\omega\|$	$3.53\ 10^{-4}$	$1.12\ 10^{9}$	D	$7.90\ 10^{-4}$	$2.14\ 10^{-9}$
$\|f_5(x_3)\|$	$3.53\ 10^{-4}$	$9.96\ 10^{-1}$	D	$7.91\ 10^{-4}$	$2.14\ 10^{-11}$
$\|x_4-\omega\|$	$1.25\ 10^{-7}$	$1.12\ 10^{9}$	D	$2.28\ 10^{-12}$	$9.6\ 10^{-35}$
$\|f_5(x_4)\|$	$1.25\ 10^{-7}$	$1.70\ 10^{-1}$	D	$2.28\ 10^{-12}$	$9.6\ 10^{-35}$
$\|x_5-\omega\|$	$1.56\ 10^{-14}$	$1.12\ 10^{9}$	D	$1.59\ 10^{-46}$	$3.89\ 0^{-136}$
$\|f_5(x_5)\|$	$1.56\ 10^{-14}$	$9.4\ 10^{-6}$	D	$1.59\ 10^{-46}$	$3.89\ 10^{-136}$
$\|x_6-\omega\|$	$2.44\ 10^{-28}$	$1.12\ 10^{9}$	D	$3.82\ 10^{-183}$	$9.72\ 10^{-266}$
$\|f_5(x_6)\|$	$2.44\ 10^{-28}$	$3.91\ 10^{-27}$	D	$3.82\ 10^{-183}$	$9.72\ 10^{-266}$

*D stands for divergen

5. Computational Cost and Numerical Comparison for Multivariate Case

Table 7: Test Functions With Their Exact Roots and Initial Guesses

	Numerical Example	Exact Zero	Starting vector
Example 1	$f = x - y - 19$ $g = \dfrac{y^3}{6} - x^2 - 17$	$\mathbf{W} = \begin{bmatrix} 5.000000 \\ 6.000000 \end{bmatrix}$	$\mathbf{X}^{(0)} = \begin{bmatrix} 5.1 \\ 6.1 \end{bmatrix}$
Example 2	$f = \cos y - \sin x$ $g = z^x - \dfrac{1}{y}$ $h = e^x - z^2$	$\mathbf{W} = \begin{bmatrix} 0.909569 \\ 0.661227 \\ 1.575834 \\ 1.575834 \end{bmatrix}$	$\mathbf{X}^{(0)} = \begin{bmatrix} 1.0 \\ 0.5 \\ 1.5 \end{bmatrix}$
Example 3	$f = xy + z(x+y)$ $g = wy + z(w+y)$ $h = wx + z(w+x)$ $i = wx + wy + xy - 1$	$\mathbf{W} = \begin{bmatrix} 0.577350 \\ 0.5577350 \\ 0.577350 \\ -0.288675 \end{bmatrix}$	$\mathbf{X}^{(0)} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -0.2 \end{bmatrix}$

Table 8: Comparison of Various Iterative Methods for Solving System of Nonlinear Equations

Numerical Example		**NM**	**KM**	**BM**	**SMNF2**
Example 1	Iterations	7	5	4	4
	$\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\|_\infty$	4.98 10^{-115}	1.18 10^{-376}	2.59 10^{-103}	7.10 10^{-108}
	$\|\mathbf{F}(\mathbf{X}^{(k-1)})\|_\infty$	3.31 10^{-115}	2.13 10^{-375}	4.64 10^{-102}	1.27 10^{-106}
Example 2	Iterations	9	6	6	6
	$\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\|_\infty$	6.42 10^{-108}	2.76 10^{-210}	1.00 10^{-299}	1.18 10^{-356}
	$\|\mathbf{F}(\mathbf{X}^{(k-1)})\|_\infty$	6.46 10^{-108}	2.79 10^{-210}	1.02 10^{-299}	1.23 10^{-356}
Example 3	Iterations	8	5	5	5
	$\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\|_\infty$	4.40 10^{-145}	1.30 10^{-198}	3.49 10^{-238}	1.02 10^{-256}
	$\|\mathbf{F}(\mathbf{X}^{(k-1)})\|_\infty$	4.51 10^{-145}	1.30 10^{-198}	4.02 10^{-238}	1.02 10^{-256}

We also compare our extended method (27) (SMNF2) for solving systems of nonlinear equations with Newton's method (NM) [14], Khattri's method (KM) [10] and Babajee's method (BM) [1], to show that the new method is more efficient. We use Maple 7 for all the computations and testing of some known examples [1] given in Table 1. In case of solving system of equations the stopping criterion is $\|\mathbf{F}(\mathbf{X}^{(i-1)})\|_\infty + \|\mathbf{X}^{(i)}-\mathbf{X}^{(i-1)}\|_\infty < 10^{-100}$ correct to 500 decimal digits. Table 7 consists of test functions for solving systems of nonlinear equations along with their exact zeros. Table 8 shows the comparison of Newton's method (NM), fourth order Khattri's method (KM) [10], fourth order Babajee's method (BM) [1] and proposed method (27) (SMNF2) which includes number of iterations and error deviation. These methods are also compared in terms of arithmetic and computational cost in Table 9.

To see the effectiveness of newly developed methods for systems of nonlinear equations, we calculate the computational cost. Calculation of computational cost for solving systems of nonlinear equations is entirely different from calculating classical efficiency index for single nonlinear equations. For calculating the computational cost following facts are under consideration. The cost of computations for the function $\mathbf{F}(\mathbf{X})$ is n and computational cost for Jacobian $\mathbf{F}'(\mathbf{X})$ is n^2. Newly developed methods also contains LU factorization and the solution of linear system of the type, $\mathbf{F}'(\mathbf{X}^{(i)})^{-1}\mathbf{F}(\mathbf{X}^{(i)})$. (38)

The cost of LU factorization is $\frac{2}{3}n^3$ and computational cost for solving linear system is $2n^2$ in case of matrix vector multiplication. Therefore, combined cost of solving (38) is $\frac{2}{3}n^3+2n^2$ and this cost is $2n^3+2n^2$ in case of matrix-matrix multiplication is involved such as $\phi(\mathbf{X}^{(i)})=[\mathbf{F}'(\mathbf{X}^{(i)})]^{-1}\mathbf{F}'(\mathbf{Z}^{(i)})$. The cost of computations is also effected by number of scalar products, matrix products, matrix additions, matrix subtractions, vector additions, vector subtractions, vector multiplications along with decomposition of the first derivative into LU and the resolution of the triangular linear systems for the methods considered here. Table 9 demonstrates the total cost of computations and number of functional evaluations for execution of fourth order Khattri's method (KM) [10], fourth order Babajee's method (BM) [1], Chun's method (CM) [3] and proposed method (27) (SMNF2).

Table 9: Comparison of Computational Cost of Various Iterative Methods for Solving Systems

Methods	Convergence Order	Number of Functional Evaluations	Total Cost of Method
BM	4	$2n^2+n$	$\frac{13}{3}n^3+12n^2+4n$
KM	4	$2n^2+n$	$\frac{17}{3}n^3+7n^2+6n$
CM	4	$2n^2+n$	$\frac{11}{3}n^3+10n^2+4n$
SMNF2	4	$2n^2+n$	$\frac{13}{3}n^3+12n^2+4n$

6. Conclusions

From tables 2-8, it is concluded that our method (SMFN2) is comparable with the recent well known methods for solving system of nonlinear as well as for single non-linear equations in terms of computational cost.Moreover table 9 shows that our method is also comparable with the known methods for solving systems in terms of computational cost.

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