Estimation of $P(Y < X)$ Using Lower Record Data from the Exponentiated Weibull Distribution: Classical and MCMC Approaches

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Abstract: In this paper we consider the problem of estimating the stress-strength reliability $R = P[Y < X]$ when the available data is in the form of lower record values. The two-parameter exponentiated Weibull (EW) is considered, where both X and Y are independent and (EW) random variables with one different shape parameter, but having a common second shape parameter. The maximum likelihood estimator (MLE) and percentile bootstrap confidence intervals of R are provided. Also, we will apply Markov chian Monte Carlo (MCMC) techniques to study the Bayesian estimation of R and by Lindley's approximation technique as well. Assuming known common shape parameter, the MLE of R is obtained. The exact distributions of the MLEs of the unknown parameters are used to construct the exact confidence interval of R. Analysis of a simulated data set has also been presented for illustrative purposes. Monte Carlo simulations are performed to compare the different proposed methods.
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1. Introduction

Whenever the comparison of the distributions of two random variables X and Y is of interest, inferences about $R = P[Y < X]$ provide a way of summarizing this comparison in terms of a single parameter. The parameter R is frequently called the reliability or stress--strength parameter. The importance of R arises in industrial contexts, since the reliability of a component can be described in terms of the stress experienced by the component, described by Y, and the strength of the component available to overcome the stress, represented by X. If the stress exceeds the strength, the component fails, and otherwise it resists. In such a setting, reliability is thus defined as the probability of not failing, which is therefore $R = P[Y < X]$ and is thus desired to be close to one. However, R arises in many other areas of application aside from industry. In biometrical studies, the random variable Y may represent the remaining lifetime of a patient treated with a certain drug, while X represents the remaining lifetime when treated by another drug. Our interest lies in the fact of whether R is greater than or less than 0.5. Many authors assumed that X and Y belong to a certain family of probability distributions with unknown parameters and then considered the estimation problem of the reliability R, for example, the exponential distribution [Nadarajah (2003), Baklizi and El-Masri (2004) and Krishnamoorthy *et al*. (2007)], Logistic and Laplace distributions [Nadarajah (2004a, 2004b)], bivariate exponential [Nadarajah and Kotz (2006)], Weibull [Kundu and Gupta (2006), Kundu and Raqab (2009) and Amiri *et al*. (2013)], Burr distributions [Mokhlis (2005) and Shawky and Al-Gashgari (2007)], gamma distribution [Nadarajah (2005)] and Pareto distribution [Shawky and Al-Gashgari (2013)]. Also, see Kotz *et al*. (2003) and the references therein.

Weibull family is commonly used for modeling systems with monotone aging property. However, in reliability analysis, lifetimes can exhibit high initial failure rate (FR) and eventual high FR due to aging and wear-out, indicating a bathtub shape. Bathtub shaped FR functions are faced frequently in many practical situations. One of the bathtub shaped models is the exponentiated Weibull family introduced by Mudholkar and Srivastava (1993). It contains distributions with bathtub shaped and unimodal FR besides a broader class of monotone FR. Some recent results about this family can be found in the works of Mudholkar *et al*. (1995), Jiang and Murthy (1999), Nassar and Eissa (2003), Xie *et al*. (2004), Lai *et al*. (2004), Shen *et al*. (2009). The exponentiated Weibull family is a generalization of the commonly known Weibull distribution. It is quite adequate for modeling monotone as well as non-monotone failure rates which are quite common in reliability and biological studies.

The exponentiated Weibull distribution is a bathtub failure rate distribution with three parameters (σ, α and θ). α and θ are referred to as the shape parameters. If θ is equal to 1, the distribution is said to be Weibull distribution. The scale parameter σ will not determine the shapes of the FR curve.

The probability density function (pdf) and cumulative distribution function (cdf) of the Exponentiated Weibull distribution EW(α,θ), with two shape parameters, α and θ are given, respectively, by

$$f(x) = \alpha\theta x^{\alpha-1}e^{-x^{\alpha}}(1-e^{-x^{\alpha}})^{\theta-1},$$
$$x>0, \alpha>0,\ \theta>0, \qquad (1)$$
$$F(x) = (1-e^{-x^{\alpha}})^{\theta}. \qquad (2)$$

Record data arise in a wide variety of practical situations. Examples include industrial stress testing, meteorological analysis, hydrology, seismology, sporting and athletic events. For comprehensive accounts of the theory and applications of record values, we refer the readers to [1-4]. There are many situations in which only records are observed. Ultimate examples of such situations can be found from the website for Guinness World Records; see http://www.guin-nessworldrecords.com/. As an example is the situation of testing the breaking strength of wooden beams as described in Glick (1978). Hence, it is important that one has accurate estimation procedures based only on records.

Let X_1, X_2, ... be an infinite sequence of independent identically continuous random variables (r.v.'s). An observation X_j will be called an lower record value if $X_j<X_i$ for every i <j. We will assume that X_j occurs at time j, then the record time sequence is defined as, T_0=1 and $T_n=\min\{j : X_j<X_{T_{n-1}}\}$.The lower record value sequence R_0, R_1, ..., R_n is defined as $R_n = X_{T_n}$, $n = 0,1,2,...$.

Inference problems based on records with many lifetime models were considered by several authors, see Awad and Raqab (2000), Ahmadi and Balakrishnan (2005), Baklizi (2008a, b), Soliman *et al.* (2006), Wang and Shi (2013) and the references therein. Our interest in this paper is in estimating the stress-strength reliability $R=P(Y<X)$, where X and Y follow the EW distributions with common one shape parameter. We discuss the problem in the situation where the stress measurements and the strength measurements are both in terms of records. First, we will obtain the MLE of R in the general case (the common shape parameter is unknown). The MLE of the three unknown parameters can be obtained by solving one nonlinear equation. We provide a simple fixed point type algorithm to find the MLE. We will also propose the percentile bootstrap confidence intervals of R. A Bayes point estimator of R, and the corresponding credible interval using the MCMC sampling technique have been proposed. Second, assuming that the common shape parameter is known, the MLE of R is obtained. Using exact distributions of the MLEs of the two unknown parameters, we construct the exact confidence interval of R. In this case, Bayes estimators have been obtained using Lindley's approximations. The different proposed methods have been compared via Monte Carlo simulation.

In the next Section, we will derive the MLE of R. The different confidence intervals of R are proposed in Section 3. In Section 4, we will discuss Bayes estimates of R when the common shape parameter α is unknown and we will construct the credible intervals using MCMC technique. The MLE of R and exact confidence intervals of R when the common shape parameter α is known are proposed in Section 5. Illustrative examples will be given in Section 6, and a comparison of the results are made as well. In Section 7, we will provide some simulation results in order to give an assessment of the performance of the different estimation methods and finally we will draw conclusions in Section 8.

2. Maximum likelihood estimator of *R*

Suppose that X is the strength of a component which is subject to stress Y. The system fails if and only if at any time the applied stress is greater than the strength. Let X be a random variable following an EW distribution with parameters α and θ (denoted by EW(α, θ)), and Y is another independent EW random variable with parameters α and β (denoted by EW(α, β)), then

$$R = (Y<X)$$
$$= \int_0^{\infty} P(Y<X|X=x)\,P(X=x)dx$$
$$= \frac{\theta}{\beta+\theta}. \qquad (3)$$

Let $\underline{x} = (x_{L(1)}, x_{L(2)}, \dots, x_{L(n)})$be the first lower record values of size n from EW(α, θ) and $\underline{y} = (y_{L(1)}, y_{L(2)}, \dots, y_{L(m)})$ be an independent set of the first lower record values of size m from EW(α, β). The likelihood functions for both observed records x and y are given, respectively, (see Arnold et al. 1998) by

$$L_1(\alpha,\theta\,\big|\underline{x}) = f(x_{L(n)})\prod_{i=1}^{n-1}\frac{f(x_{L(i)})}{F(x_{L(i)})}, \qquad (4)$$

and

$$L_2(\alpha,\beta\,\Big|\underline{y}) = g(y_{L(m)})\prod_{j=1}^{m-1}\frac{g(y_{L(j)})}{G(y_{L(j)})}, \qquad (5)$$

where f and F are respectively, the (pdf) and (cdf) of X, g and G are the (pdf) and (cdf) of Y , respectively. Substituting f, F, g and G in the likelihood functions we obtain

$$L_1(\alpha,\theta|\underline{x}) = \alpha^n\theta^n\Psi_1(\underline{x},\alpha)e^{\theta\log v_n}, \qquad (6)$$
$$L_2(\alpha,\beta\,|\underline{y}) = \alpha^m\beta^m\Psi_2(\underline{y},\alpha)\,e^{\beta\log w_m}, \qquad (7)$$

where

$$\Psi_1(\underline{x},\alpha) = \prod_{i=1}^{n} \frac{x_{L(i)}^{\alpha-1} e^{-x_{L(i)}^{\alpha}}}{v_i},$$

$$\Psi_2(\underline{y},\alpha) = \prod_{j=1}^{m} \frac{y_{L(j)}^{\alpha-1} e^{-y_{L(j)}^{\alpha}}}{w_j}, \quad (8)$$

$v_i = 1 - e^{-x_{L(i)}^{\alpha}}$ and $w_j = 1 - e^{-y_{L(j)}^{\alpha}}$, $i=1, \ldots, n$, $j=1, \ldots, m$.

Therefore, the joint Log-likelihood function of the observed records $\underline{x}$ and $\underline{y}$ is

$$\ell(\alpha,\theta,\beta|data) = (n+m)\log\alpha + n\log\theta + m\log\beta + \theta\log v_n + \beta\log w_m + (\alpha-1)\left(\sum_{i=1}^{n}\log x_{L(i)} + \sum_{j=1}^{m}\log y_{L(j)}\right) - \left(\sum_{i=1}^{n}\log v_i + \sum_{j=1}^{m}\log w_j\right) - \left(\sum_{i=1}^{n} x_{L(i)}^{\alpha} + \sum_{j=1}^{m} y_{L(j)}^{\alpha}\right). \quad (9)$$

The MLEs of α,θ and β denoted, respectively, by $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\beta}$ are obtained by solving the following equations

$$\frac{\partial\ell}{\partial\alpha} = \frac{n+m}{\alpha} + \left(\sum_{i=1}^{n}\log x_{L(i)} + \sum_{j=1}^{m}\log y_{L(j)}\right) - \left(\sum_{i=1}^{n} x_{L(i)}^{\alpha}\log x_{L(i)} + \sum_{j=1}^{m} y_{L(j)}^{\alpha}\log y_{L(j)}\right) - \eta_1(\underline{x},\alpha) - \eta_2(\underline{y},\alpha) + \theta\xi_1(x_{L(n)},\alpha) + \beta\xi_2(y_{L(m)},\alpha) = 0, \quad (10)$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \log\left(1 - e^{-x_{L(n)}^{\alpha}}\right) = 0, \quad (11)$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \log\left(1 - e^{-y_{L(m)}^{\alpha}}\right) = 0, \quad (12)$$

where

$$\eta_1(\underline{x},\alpha) = \sum_{i=1}^{n} \frac{x_{L(i)}^{\alpha} e^{-x_{L(i)}^{\alpha}} \log x_{L(i)}}{v_i},$$

$$\eta_2(\underline{y},\alpha) = \sum_{j=1}^{m} \frac{y_{L(j)}^{\alpha} e^{-y_{L(j)}^{\alpha}} \log y_{L(j)}}{w_j}, \quad (13)$$

$$\xi_1(x_{L(n)},\alpha) = \frac{x_{L(n)}^{\alpha} e^{-x_{L(n)}^{\alpha}} \log x_{L(n)}}{v_n},$$

and

$$\xi_2(y_{L(m)},\alpha) = \frac{y_{L(m)}^{\alpha} e^{-y_{L(m)}^{\alpha}} \log y_{L(m)}}{w_m}. \quad (14)$$

From Equations (11) and (12), we thus have

$$\hat{\theta} = \frac{n}{\log(v_n)}, \quad (15)$$

$$\hat{\beta} = \frac{m}{\log(w_m)}, \quad (16)$$

and $\hat{\alpha}$ can be obtained as the solution of the following nonlinear equation

$$f(\alpha) = \frac{n+m}{\alpha} + \left(\sum_{i=1}^{n}\log x_{L(i)} + \sum_{j=1}^{m}\log y_{L(j)}\right) - \left(\sum_{i=1}^{n} x_{L(i)}^{\alpha}\log x_{L(i)} + \sum_{j=1}^{m} y_{L(j)}^{\alpha}\log y_{L(j)}\right) + \left(\frac{n\,\xi_1(x_{L(n)},\alpha)}{\log v_n}\right) + \left(\frac{m\,\xi_2(y_{L(m)},\alpha)}{\log w_m}\right) - \eta_1(\underline{x},\alpha) - \eta_2(\underline{y},\alpha). \quad (17)$$

Therefore, $\hat{\alpha}$ is obtained by solving a nonlinear equation of the form

$$h(\alpha) = \alpha, \quad (18)$$

where

$$h(\alpha) = (n+m)\Bigg[-\left(\sum_{i=1}^{n}\log x_{L(i)} + \sum_{j=1}^{m}\log y_{L(j)}\right) + \left(\sum_{i=1}^{n} x_{L(i)}^{\alpha}\log x_{L(i)} + \sum_{j=1}^{m} y_{L(j)}^{\alpha}\log y_{L(j)}\right) - \left(\frac{n\,\xi_1(x_{L(n)},\alpha)}{\log v_n}\right) - \left(\frac{m\,\xi_2(y_{L(m)},\alpha)}{\log v_m}\right) + \eta_1(\underline{x},\alpha) + \eta_2(\underline{y},\alpha)\Bigg]^{-1}.$$

Since $\hat{\alpha}$ is a fixed point solution of nonlinear equation (17), which is obtained by using a simple iterative scheme as follows:

$$h(\alpha_j) = \alpha_{j+1}, \quad (19)$$

where α_j is the j^{th} iterate of $\hat{\alpha}$. The iteration procedure should be stopped when $|\hat{\alpha}_{j+1} - \hat{\alpha}_j|$ is sufficiently small. Once we obtain $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\beta}$, the MLE of R becomes

$$\hat{R} = \frac{\hat{\theta}}{\hat{\beta}+\hat{\theta}}. \quad (20)$$

3. Bootstrap confidence intervals

In this section, we will construct the confidence intervals based on the following methods: (i) percentile bootstrap method (Boot-p) using the idea of Efron (1982), and (ii) bootstrap-t method (Boot-t) using the idea of Hall (1988). The algorithms for estimating the confidence intervals of R using both methods are illustrated below.

3.1 Percentile bootstrap method

Step 1: From the original two samples of lower record $\{x_{L(1)}, x_{L(2)}, \ldots, x_{L(n)}\}$ and $\{y_{L(1)}, y_{L(2)}, \ldots, y_{L(m)}\}$ compute ML estimates $\hat{\alpha}$, $\hat{\theta}$, $\hat{\beta}$ and $\hat{R}$.

Step 2: Using $\hat{\alpha}$ and $\hat{\theta}$ to generate a bootstrap lower record sample $\{x_{L(1)}^{*}, x_{L(2)}^{*}, \ldots, x_{L(n)}^{*}\}$ and similarly using $\hat{\alpha}$ and $\hat{\beta}$ to generate a bootstrap lower record sample $\{y_{L(1)}^{*}, y_{L(2)}^{*}, \ldots, y_{L(m)}^{*}\}$. Based on these data, we compute the bootstrap estimators say, $\hat{\alpha}^*$, $\hat{\theta}^*$, $\hat{\beta}^*$ and $\hat{R}^*$.

Step 3: Repeat step 2, N boot times.

Step 4: Let $G(x) = P(\hat{R}^* \leq x)$ be the cumulative distribution of $\hat{R}^*$. Define $\hat{R}_{Boot} = G^{-1}(x)$for a given x. The approximate 100(1-γ)% confidence interval of R is given by

$$(\hat{R}_{Boot-P}\left(\frac{\gamma}{2}\right), \hat{R}_{Boot-P}\left(1-\frac{\gamma}{2}\right). \qquad (21)$$

3.2 Bootstrap-t method

Step 1: From the original two samples of lower record $\{x_{L(1)}, x_{L(2)}, \ldots, x_{L(n)}\}$ and $\{y_{L(1)}, y_{L(2)}, \ldots, y_{L(m)}\}$ compute ML estimates $\hat{\alpha}$, $\hat{\theta}$, $\hat{\beta}$ and $\hat{R}$.

Step 2: Using $\hat{\alpha}$ and $\hat{\theta}$ to generate a bootstrap lower record sample $\{x^*_{L(1)}, x^*_{L(2)}, \ldots, x^*_{L(n)}\}$ and similarly using $\hat{\alpha}$ and $\hat{\beta}$ to generate a bootstrap lower record sample $\{y^*_{L(1)}, y^*_{L(2)}, \ldots, y^*_{L(m)}\}\}$. Based on these data, we compute the bootstrap estimators say, $\hat{\alpha}^*$, $\hat{\theta}^*$and $\hat{\beta}^*$, then compute the bootstrap estimate of R using (20), say $\hat{R}^*$ which follows the statistic $T^* = \frac{\sqrt{n}(\hat{R}^*-\hat{R})}{\sqrt{Var(\hat{R}^*)}}$,where $Var(\hat{R}^*)$ is obtained by the delta method (see Greene (2000).

Step 3: Repeat step 2, N boot times.

Step 4: For theT^* values obtained in step 2, determine the upper and lower bounds of the 100(1-γ)% confidence interval of R as follows: let $H(x) = P(T^* \leq x)$be the cumulative distribution function of T^*. For a given x, define

$$\hat{R}_{Boot-t} = \hat{R} + n^{-1/2}\sqrt{Var(\hat{R})}H^{-1}(x).$$

Similarly, $Var(\hat{R})$ can be computed using the same technique used in computing the $Var(\hat{R}^*)$. The approximate 100(1-γ)% confidence interval of R is given by

$$(\hat{R}_{Boot-t}\left(\frac{\gamma}{2}\right), \hat{R}_{Boot-t}\left(1-\frac{\gamma}{2}\right)). \qquad (22)$$

4. Bayes estimation of R Using MCMC

Recently, there is a vast amount of statistical literature on the MCMC methodology have appeared. The advantage of MCMC is that it gives not only a point estimate of the parameter, but also gives an interval estimation based on the final simulated empirical distribution. MCMC is essentially an iterative sampling algorithm, drawing values from the posterior distributions of the parameter in the model concerned. We consider the MCMC method to generate samples from the posterior distributions and then compute the Bayes estimates of R under lower record values from the EWD. A wide variety of MCMC schemes are available, and it can be difficult to choose among them. An important sub-class of MCMC methods is Gibbs sampling and more general Metropolis-Hastings (M-H) algorithm (Metropolis *et al*. (1953), Hastings (1970)). For more details about MCMC and the related methodologies, one can refer to Gentle (1998), Chen *et al*. (2000) and Robert and Casella (2004).

This section describes Bayesian MCMC methods that have been used to estimate R based on lower record values from the EWD. The Bayesian approach is introduced and its computational implementation with MCMC algorithms is described. Gibbs sampling procedure and Metropolis--Hastings (M-H) Method are used to generate samples from the posterior density function and in order to compute the Bayes point estimates, we then construct the corresponding credible intervals based on the generated posterior samples, as well.

It is assumed that(α, θ, β)have independent gamma priors with the pdf's

$$\pi_1(\alpha|a_1, b_1) = \begin{cases} \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha} & if \ \ \alpha>0 \\ 0 & if \ \ \alpha\leq 0, \end{cases} \qquad (23)$$

$$\pi_2(\theta|a_2, b_2) = \begin{cases} \frac{b_2^{a_2}}{\Gamma(a_2)}\theta^{a_2-1}e^{-b_2\theta} & if \ \ \theta>0 \\ 0 & if \ \ \theta\leq 0, \end{cases} \qquad (24)$$

$$\pi_3(\beta|a_3, b_3) = \begin{cases} \frac{b_3^{a_3}}{\Gamma(a_3)}\beta^{a_3-1}e^{-b_3\beta} & if \ \ \beta>0 \\ 0 & if \ \ \beta\leq 0, \end{cases} \qquad (25)$$

where a_1, b_1, a_2, b_2, a_3 and b_3 are chosen to reflect prior knowledge about α, θ and β. Note that when $a_i = b_i = 0, i = 1, 2$ and 3, it is corresponding to the case of non-informative priors. We assume that the parameters are mutually independent. The deduced posterior distribution is proportional to the product of the prior and the likelihood function given by

$$\pi^*(\alpha, \theta, \beta|data) \propto \alpha^{(n+m+a_1-1)}\theta^{(n+a_2-1)}\beta^{(m+a_3-1)} \times \Psi_1(\underline{x}, \alpha)\Psi_2(\underline{y}, \alpha)\, e^{-b_1\alpha} \times e^{-\theta[b_2 - T_1(x_{L(n)},\alpha)]}e^{-\beta[b_3 - T_2(y_{L(m)},\alpha)]}, \qquad (26)$$

where $\Psi_1(x,\alpha)$, $\Psi_2(y,\alpha)$ are given in (8) and

$$T_1(x_{L(n)}, \alpha) = log v_n \text{ and } T_2(y_{L(m)}, \alpha) = log\omega_m, \qquad (27)$$

$$\pi_1^*(\alpha|\theta, \beta, data) \propto \alpha^{(n+m+a_1-1)}\Psi_1(\underline{x}, \alpha)\Psi_2(\underline{y}, \alpha) \times e^{-b_1\alpha}e^{-\theta[b_2 - T_1(x_{L(n)},\alpha)]}e^{-\beta[b_3 - T_2(y_{L(m)},\alpha)]}. \qquad (28)$$

Similarly, the full posterior conditional distribution for θ and β are given by

$$\pi_2^*(\theta|\alpha, data) \sim Gamma(n + a_2, b_2 - T_1(x_{L(n)}, \alpha)), \qquad (29)$$

$$\pi_3^*(\beta|\alpha, data) \sim Gamma(m + a_3, b_3 - T_2(y_{L(m)}, \alpha)), \qquad (30)$$

where$T_1(x_{L(n)}, \alpha)$ and $T_2(y_{L(m)}, \alpha)$are given in (27).

It follows from Equations (29) and (30) that the samples of θ and β can be easily generated using any gamma generating routine. However, in our case, the conditional posterior distribution of α in Equation (28) cannot be reduced analytically to well known distributions and therefore it is not possible to simplify it directly by standard methods, but its graph indicates that it is similar to the normal distribution. So, to generate random numbers from this distribution, we use the Metropolis-Hastings (M-H) method with normal proposed distribution.

Therefore, the algorithm of Gibbs sampling is as follows

Step 1: Start with an ($\alpha^{(0)}=\hat{\alpha}$) and set $t=1$.

Step2: Generate $\theta^{(t)}$ from

$$Gamma\left(n+a_2, b_2-T_1\left(x_{L(n)},\alpha^{(t-1)}\right)\right).$$

Step 3: Generate $\beta^{(t)}$ from Gamma$\left(m+a_3, b_3-T_2\left(y_{L(m)},\alpha^{(t-1)}\right)\right)$.

Step 4: Using M-H (see, Metropolis et al. (1953)), to generate $\alpha^{(t)}$ from (28) with $N(\alpha^{(t-1)},\sigma^2)$ proposed distribution, where σ² is variances-covariances matrix.

Step 5: Compute $\alpha^{(t)}$, $\theta^{(t)}$ and $\beta^{(t)}$, then deduce $R^{(t)}=\frac{\theta^{(t)}}{\beta^{(t)}+\theta^{(t)}}$.

Step 6: Set $t = t+1$.

Step 7: Repeat steps 2-5 N times.

Step 8: We obtain the Bayes MCMC point estimate of R as

$$E(R|data)=\frac{1}{N-M}\sum_{i=M+1}^{N}R^{(i)}, \quad (31)$$

where M is the burn-in period (that is, a number of iterations before the stationary distribution is achieved), and posterior variance of R becomes

$$\hat{V}(R|data)=\frac{1}{N-M}\sum_{i=M+1}^{N}\left(R^{(i)}-\hat{E}(R|data)\right)^2. \quad (32)$$

Step 9: To compute the credible intervals of R, as it is well known, we take the quantiles of the sample as the endpoints of the interval. Order $R^{(M+1)}, R^{(M+2)},\ldots,R^{(N)}$ as $R_{(1)}, R_{(2)}, \ldots, R_{(N-M)}$. Then the 100(1-γ)% symmetric credible interval is

$$\left(R_{\left(\frac{\gamma}{2}(N-M)\right)}, R_{\left(\left(1-\frac{\gamma}{2}\right)(N-M)\right)}\right). \quad (33)$$

Next, we consider the special case when the common shape parameter α is known.

5. Estimation of R if α is known

In this section, we consider the estimation of R and the corresponding highest posterior density (HPD) intervals when α is known. Therefore, assume that $x_{L(1)}, x_{L(2)},\ldots,x_{L(n)}$ are the first lower record values observed from $EWD(\alpha,\theta)$ and $y_{L(1)}, y_{L(2)},\ldots,y_{L(m)}$ are the first lower record values observed from $EWD(\alpha,\beta)$, based on the samples we want to estimate R. Recent works on interval estimation of R are discussed in Shoukri *et al.* (2005), Baklizi (2008a; 2008b) and Rezaeia *et al.* (2010). First, we will consider the MLE of R and its distributional properties.

5.1 MLE of R

Based on section 2, it is clear that the MLE of R, say $\hat{R}$, will be

$$\hat{R}=\frac{\hat{\theta}}{\hat{\beta}+\hat{\theta}}. \quad (34)$$

where

$$\hat{\theta}=\frac{n}{\log\left(1-e^{-x_{L(n)}^{\alpha}}\right)}, \text{ and}$$

$$\hat{\beta}=\frac{m}{\log\left(1-e^{-y_{L(m)}^{\alpha}}\right)}. \quad (35)$$

Therefore

$$\hat{R}=\frac{n\log\left(1-e^{-y_{L(m)}^{\alpha}}\right)}{m\log(\nu_n)+n\log(\omega_m)}. \quad (36)$$

To study the confidence interval of R, we need to study the distribution of $\hat{R}$ as well as the distributions of $\hat{\theta}$and $\hat{\beta}$. Consider first$\hat{\theta}=\frac{n}{\log\left(1-e^{-r_{L(n)}^{\alpha}}\right)}$, Arnold *et al.* (1998) obtained the probability density function (pdf) of $R_n$as follows

$$f_{R_n}(r_n)=f(r_n)\frac{[-\log(F(r_n))]^{n-1}}{(n-1)!}, \quad (37)$$

under the $EW(\alpha,\theta)$ distribution

$$f_{R_n}(r_n)=\frac{\alpha\theta^n r_n^{\alpha-1}}{\Gamma(n)}e^{-r_n^{\alpha}}.$$

$$(1-e^{-r_n^{\alpha}})^{\theta-1}\{-\log(1-e^{-r_n^{\alpha}})\}^{n-1},$$

$$r_n>0. \quad (38)$$

Consequently, the (pdf) of $Z_1=\hat{\theta}=\frac{n}{\log\left(1-e^{-r_{L(n)}^{\alpha}}\right)}$ is given by

$$f_{Z_1}(z_1)=\frac{(n\theta)^n}{\Gamma(n)z_1^{n+1}}e^{-\frac{n\theta}{z_1}},\ z_1>0. \quad (39)$$

This is the inverted gamma distribution. Similarly, the (pdf) of $Z_2=\hat{\beta}=\frac{m}{\log(\omega_n)}$ is given by

$$f_{Z_2}(z_2)=\frac{(m\beta)^m}{\Gamma(m)z_2^{m+1}}e^{-\frac{m\beta}{z_2}},\ z_2>0. \quad (40)$$

To find the (pdf) of $\hat{R}=\frac{\hat{\theta}}{\hat{\beta}+\hat{\theta}}=\frac{Z_1}{Z_1+Z_2}=\frac{1}{1+Z_2/Z_1}$, consider the quotient Z_2/Z_1. Note that, by the properties of the inverted gamma distribution and its relation with the gamma distribution we have$\frac{n\,\theta}{Z_1}\sim\mathrm{Gamma}(n,1)$and $\frac{m\,\beta}{Z_2}\sim\mathrm{Gamma}(m,1)$. Hence $\frac{2n\,\theta}{Z_1}\sim\chi^2_{2n}$ and $\frac{2m\,\beta}{Z_2}\sim\chi^2_{2m}$. Note that, by the independence of two random quantities, we have $\frac{2n\,\theta/2nZ_1}{2m\beta/2mZ_2}\sim F_{2n,2m}$, hence$\frac{Z_2}{Z_1}\sim\frac{\beta}{\theta}F_{2n,2m}$, a scaled F distribution. It follows that the distribution of $\hat{R}$ is that$\frac{1}{1+\frac{\beta}{\theta}F_{2n,2m}}$. Hence $\frac{1}{\hat{R}}=1+\left(\frac{1}{R}-1\right)F(2n,2m)$, then

$$R\sim\frac{F(2n,2m)}{F(2n,2m)+(\frac{1}{\hat{R}}-1)}.$$

The 100(1-γ)% confidence interval of R is given by

$$\left[\frac{F_{\gamma/2}(2n,2m)}{F_{\gamma/2}(2n,2m)+(\frac{1}{\hat{R}}-1)},\frac{F_{1-\gamma/2}(2n,2m)}{F_{1-\gamma/2}(2n,2m)+(\frac{1}{\hat{R}}-1)}\right]. \quad (41)$$

5.2 Bayes estimation of R

In this subsection, we will obtain the Bayes estimate of R under the assumption that the shape parameters θ and β are random variables. It is assumed that θ and β have independent gamma priors given in (24) and (25), respectively, with the parameters (a_2, b_2) and (a_3, b_3). The posterior pdf's of θ and β are given by (29) and (30) respectively. Since the priors θ and β are independent, then, using a standard transformation techniques and after some manipulations, the posterior (pdf) of $R=\frac{\theta}{\beta+\theta}$ will be

$$f_R(r)=C\,r^{n+a_2-1}(1-r)^{n+a_3-1}/\left[r\left(b_2-T_1\left(x_{L(n)},\alpha\right)+(1-r)(b_3-T_2\left(y_{L(m)},\alpha\right)\right]^{n+m+a_2+a_3}\right], \quad 0<r<1, \quad (42)$$

and 0 otherwise,
where

$$C=\frac{\Gamma(n+m+a_2+a_3)}{\Gamma(n+a_2)\Gamma(m+a_3)}\times\{b_2-T_1\left(x_{L(n)},\alpha\right)\}^{n+a_2}\times\{b_3-T_2\left(y_{L(m)},\alpha\right)\}^{m+a_3}. \quad (43)$$

There is no explicit expression for the posterior mean or median of (42). On the other hand, the posterior mode can be easily obtained as follows

$$\frac{d}{dr}f_R(r)=\{r^{A_1-1}(1-r)^{A_2-1}[2r^2(B_1-B_2)+r(2B_2-2B_1-A_2B_1+A_1B_2)+A_1B_2]\}/[B_2(1-r)\ +\ B_1r]^{A_1+A_2+3},$$

where
$B_1=b_2-T_1\left(x_{L(n)},\alpha\right)$, $B_2=b_3-T_2\left(y_{L(m)},\alpha\right)$, $A_1=n+a_2-1$ and $A_2=m+a_3-1$. Note that, for $r\in(0,1)$, $\left(\frac{d}{dr}\right)f_R(r)\ =0$ has only two roots. Using the fact that $\lim_{r\to 0^+}\left(\frac{d}{dr}\right)f_R(r)>0$ and $\lim_{r\to 1^-}\left(\frac{d}{dr}\right)f_R(r)<0$, it easily follows that the density function $f_R(r)$ has a unique mode. The posterior mode can be obtained as the unique root, which lies between 0 and 1, of the following quadratic equation:

$$2r^2(B_2-B_1)+r(2B_1-2B_2+A_2B_1-A_1B_2)-A_1B_2=0 \quad (44)$$

Now, consider the following loss function:

$$L(a,b)=\begin{cases}1 & if\ |a-b|>\varepsilon,\\ 0 & if\ |a-b|\le\varepsilon.\end{cases} \quad (45)$$

It is known that the Bayes estimate with respect to the above loss function (45) is the mid point of the `modal interval' of length 2ε of the posterior distribution. Therefore, the posterior mode is an approximate Bayes estimator of R with respect to the above loss function when the constant ε is small enough.

As mentioned above, the Bayes estimate of R under squared error loss cannot be computed analytically. Alternatively, using the approximate method of Lindley (1980), it can be easily seen that the approximate Bayes estimate of R, say $\tilde{R}_{Bayes}$ relative to squared error loss function is

$$\tilde{R}_{Bayes}=\breve{R}\left[1+\frac{\breve{\theta}\breve{R}^2[\breve{\theta}(n+a_2-1)-\breve{\beta}(m+a_3-1)]}{\breve{\beta}^2(n+a_2-1)(m+a_3-1)}\right], \quad (46)$$

where

$$\tilde{\theta}=\frac{n+a_2-1}{b_2-T_1\left(x_{L(n)},\alpha\right)},\quad \tilde{\beta}=\frac{m+a_3-1}{b_3-T_2\left(y_{L(m)},\alpha\right)}\ \text{and}\ \tilde{R}=\frac{\tilde{\theta}}{\tilde{\beta}+\tilde{\theta}}. \quad (47)$$

For comparison purposes, we also compute a highest posterior density (HPD) interval of R . Based on the discussion of Soliman and Al-Aboud (2008), and due to the unimodality of the posterior distribution (42), the 100(1-γ)% HPD interval $[\omega_L,\ \omega_U]$ for R is given by solving simultaneously the following nonlinear equations

$$\int_{\omega_L}^{\omega_U}f_R(r|data)=1-\gamma \quad \text{and} \quad f_R(\omega_L|data)=f_R(\omega_U|data). \quad (48)$$

We can employ Newton-Raphson iteration to solve the equations in (48) and hence the HPD interval is obtained.

6. Illustrative example

In this section, we will simulate 6 lower record values from the exponentiated Weibull (EWD(2,3)) and 6 lower record values from the EWD(2,4).Therefore, R_{Exact}=0.4286. The data has been truncated after four decimal places as presented below. The x lower record values are

1.0186, 0.6453, 0.5815, 0.5814, 0.4749, 0.4480

and the corresponding y lower record values are

1.6081, 1.2491, 0.8616, 0.8109, 0.7017, 0.6161.

Now we consider two cases:

Case (1), when α is unknown:

Based on the above data, we plot the profile log-likelihood function of α in Fig. 1. It is an upside down function and it has a unique maximum. We obtain the MLE of α using the iterative procedure (18). We use the stopping criterion which states the iteration stops whenever two consecutive values are less than 10^{-6}, the iteration stops after 14 steps giving the MLE of $\hat{\alpha}$= 2.5021. Now using (15) and (16), we obtain the

MLEs of $\hat{\theta}$= 2.8907 and $\hat{\beta}$= 4.4213 hence, by (20), $\hat{R}$ = 0.3953. The 95% confidence, credible intervals and corresponding length are reported in Table (1) using exact confidence interval (41), parametric percentile bootstrap methods and MCMC technique.

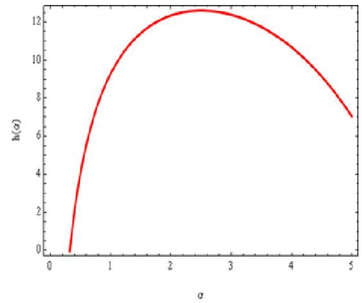

Fig.1: The profile likelihood of α for given data set presented above.

Table(1): Two-sided 95 % confidence and credible intervals of R when α=2, θ=3 and β=4 with prior 0.

Methods	$\hat{R}$	95% CI	Length	$\hat{R}$	95% CI	Length
α is unknown				α is known		
MLE	0.3953	(0.1663, 0.6818)	0.5155	0.4034	(0.1710, 0.6890)	0.5180
Boot-p	0.4829	(0.2250, 0.7499)	0.5249	0.4933	(0.2330, 0.7497)	0.5168
Boot-t	0.4896	(0.1624, 0.8635)	0.7011	0.4960	(0.1739, 0.8476)	0.6737
Bayes	0.3868	(0.1529, 0.6602)	0.5074	0.4005	(0.1620, 0.6709)	0.5089

Case (2), when α is known:

Let α=2. In this case, we obtain the MLEs of θ and β as, 3.5195 and 5.2060, respectively. Therefore, the MLE of *R* becomes $\hat{R}$= 0.4034. The corresponding 95% confidence, credible intervals and corresponding length are also reported in Table (1) using MLE (41), parametric percentile bootstrap methods and MCMC technique. The posterior probability density function (42) of *R* for the given data set, is plotted in Fig. 2.

The simulation number of R and Histogram of R generated by MCMC are plotted in Figs 3-4.

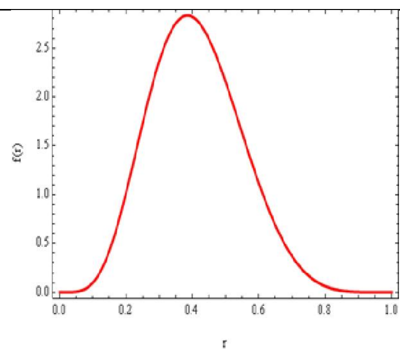

Fig. 2: Posterior probability density of R for given data set presented above.

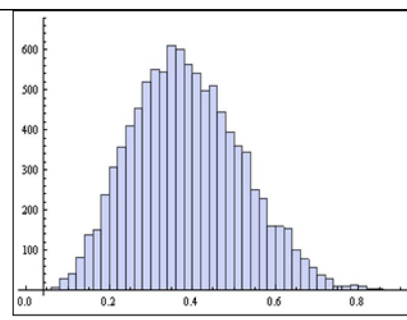

Fig. 3: Simulation number of R generated by MCMC method.

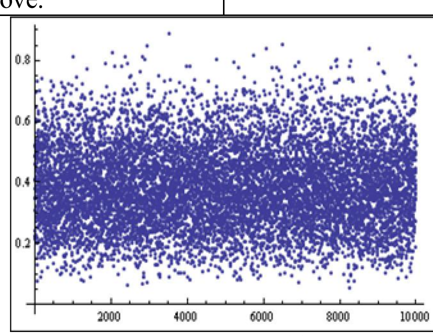

Fig.4: Histogram of R generated by MCMC method

7. Simulation results

In this section, we will report some numerical experiments performed to evaluate the behavior of the proposed methods for different samples, different parameter values, and different hyper parameters. We consider two cases separately to draw inference on R, namely when (i) common shape parameter α is unknown (ii) common shape parameter α is known. In both cases we consider the different sample sizes (n and m), and different hyperparameters (a_1, b_1, a_2, b_2, a_3, b_3). In first case we take α=2, θ=1.5 and β=3 and in the second case we take α=2, θ=3 and β=2.5. Without loss of generality we take α=2 in both cases considered. All the results are based on 1000 replications.

The first case (when α is unknown): From the sample, we estimated α using the iterative algorithm (18). We used the initial estimate to be 2 and the iterative process stops when the difference between the two consecutive iterates are less than 10^{-7}. Once we have reached an estimate for α, we estimate θ and β by using (15) and (16) respectively. Finally, using (20), we obtain the MLE of R. To find the Bayes MCMC estimates, we used the non informative gamma priors for the three parameters (we call it prior 0). Non-informative prior (a_1=b_1=a_2=b_2=a_3=b_3=0) provides prior distributions which are not proper. Informative priors, including prior 1, a_1=3, b_1=2, a_2 =2, b_2=1, a_3=2, b_3=1, and by using the previous fixed values of α, θ and β, are used as well. We have computed the Bayes estimates and 95% probability intervals based on 10,000 MCMC samples and have discarded the first 1000 values as `burn-in'. We give the average Bayes estimates, mean squared errors (MSEs), coverage percentages, and average probability interval lengths based on 1000 replications in Table 2.

Table (2): Simulation results and estimation of the parameters when α=2, θ=1.5, β=3 using prior 0.

(n, m)	R_{Exact}	MLE		Bayes using MCMC			
		Mean	MSE	Mean	MSE	Length	Coverge
(6, 6)	0.3333	0.3168	0.0286	0.3215	0.0271	0.5749	0.933
(7, 6)		0.3545	0.0243	0.3478	0.0240	0.5502	0.927
(8, 6)		0.3178	0.0221	0.3625	0.0211	0.5355	0.946
(7, 7)		0.3497	0.0217	0.3189	0.0198	0.4923	0.934
(8, 7)		0.3286	0.0196	0.3496	0.0186	0.4702	0.961
(8, 8)		0.3301	0.0191	0.3521	0.0157	0.4431	0.932
(9, 8)		0.3222	0.0173	0.3201	0.0136	0.4277	0.948
(9, 9)		0.3554	0.0152	0.3399	0.0130	0.4235	0.940
α=2, θ=1.5, β=3 using prior 1.							
(6, 6)	0.3333	0.3621	0.0252	0.3399	0.0229	0.5233	0.980
(7, 6)		0.3418	0.0250	0.3621	0.0195	0.5147	0.943
(8, 6)		0.3145	0.0231	0.3711	0.0193	0.5122	0.961
(7, 7)		0.3522	0.0197	0.3523	0.0186	0.5076	0.936
(8, 7)		0.3296	0.0187	0.3210	0.0182	0.5002	0.954
(8, 8)		0.3199	0.0181	0.3355	0.0166	0.4836	0.971
(9, 8)		0.2986	0.0159	0.3448	0.0141	0.4599	0.993
(9, 9)		0.3644	0.0136	0.3099	0.0110	0.4431	0.969

For the second case (when α is known), we obtain the estimates of R by using the ML method and Lindley's approximation approach. We have calculated the exact confidence intervals and HPD interval of R by using the same non informative prior (prior 0) and an informative prior, including (prior 1), we compute the average estimates of R, mean squared errors (MSEs), coverage percentages, and average probability interval lengths based on 1,000 replications. The results are given in Table 3.

Table (3): Simulation results and estimation of the parameters when α=2, θ=3, β=2.5 using prior 0.

(n,m)	R_{Exact}	MLE				Bayes using Lindely			
		Mean	MSE	Length	Coverge	Mean	MSE	Length	Coverge
(6,6)	0.5455	0.5113	0.0286	0.4996	0.938	0.3215	0.0271	0.5749	0.933
(7,6)		0.4975	0.0243	0.4870	0.936	0.3478	0.0240	0.5502	0.927
(8,6)		0.4922	0.0221	0.4737	0.923	0.3625	0.0211	0.5355	0.946
(7,7)		0.3497	0.0217	0.4709	0.933	0.3189	0.0198	0.4923	0.934
(8,7)		0.3286	0.0196	0.4597	0.934	0.3496	0.0186	0.4702	0.961
(8,8)		0.3301	0.0191	0.4467	0.951	0.3521	0.0157	0.4431	0.932
(9,8)		0.3222	0.0173	0.4347	0.941	0.3201	0.0136	0.4277	0.948
(9,9)		0.3554	0.0152	0.4214	0.942	0.3399	0.0130	0.4235	0.940
α=2, θ=3, β=2.5 using prior 1.									
(6, 6)	0.5455	0.3621	0.0252	0.4986	0.922	0.3399	0.0229	0.5233	0.980
(7, 6)		0.3418	0.0250	0.4867	0.934	0.3621	0.0195	0.5147	0.943
(8, 6)		0.3145	0.0231	0.4761	0.956	0.3711	0.0193	0.5122	0.961

(7, 7)		0.3522	0.0197	0.4735	0.940	0.3523	0.0186	0.5076	0.936
(8, 7)		0.3296	0.0187	0.4579	0.932	0.3210	0.0182	0.5002	0.954
(8, 8)		0.3199	0.0181	0.4481	0.945	0.3355	0.0166	0.4836	0.971
(9, 8)		0.2986	0.0159	0.4330	0.930	0.3448	0.0141	0.4599	0.993
(9, 9)		0.3644	0.0136	0.4271	0.945	0.3099	0.0110	0.4431	0.969

8. Conclusions

The purpose of this paper is to develop the classical and MCMC Bayesian analysis for R when both samples on X and Y are in the form of lower record values, observed from independent EW random variables with one different shape parameter. We considered the general case when all the parameters are unknown, and when the common shape parameter is known. In the first case the MCMC method provided an alternative method for parameters estimation of the EWD, and also, for obtaining both point and interval estimators of the stress-strength reliability model R. The results obtained hrerein show that MCMC is more flexible in comparing with the traditional methods such as MLE based on the set of lower record values. We hope that our investigation will be useful for researchers dealing with the kind of data considered in this paper. From the results, we observe the following:

(i) When the common shape parameter α is unknown, it is observed that the Bayes estimator using MCMC technique works quite well. We have used the MCMC sample to construct confidence intervals that works quite well. When the common shape parameter α is known we proposed a maximum likelihood estimators and Bayes estimators based on the approximate method of Lindley. The confidence interval based on the exact distribution of the MLE works very well. Also, we recommend using a highest posterior density (HPD) interval.

(ii) Tables 2-3 show that when m=n and m,n increase then MSEs, and average confidence interval lengths, credible interval lengths of the MLEs and Bayes estimators decrease and the coverage percentages reach the nominal level in most cases.

(iii) From Tables 2 and 3, it is clear that the Bayes estimators based on informative priors (prior 1) perform, are much better than the Bayes estimators based on non informative priors (prior 0) or MLEs in terms of biases, MSEs, and lengths of credible intervals.

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