Acceptable method for ordinary differential equations integration

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Abstract. Differential equations course plays a great role in the fundamental training of a future teacher in the aspect of the formation of a student's scientific world view, definite level of mathematical culture, definite level of methodological culture, especially on such components as understanding of applied and practical direction of mathematics education, mastering the method of mathematical modeling, the ability to fulfill inter and correlative connections between subjects. To the components of humanitarian potential of the differential equation teaching at pedagogical schools of higher education course except the above, we relate professional-pedagogical directivity of the course, at that, in comparison with other mathematical disciplines, there are greater opportunities for full realization of professional-pedagogical directivity of education. This imposes special obligations on a teacher of differential equations course in realization of the binarity principle – the most adequate combination of mathematical and methodological lines.

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In order to improve cognitive activity and enhance creative abilities of students in vocational training through the use of intra communication we will show that the first order differential equation can be integrated with only one method.

1^0. In mathematical analysis considered the issue of finding the unknown function, that is primitive in its derivative. This action is regarded as a convenient initial period of the simplest solutions of differential equations. Here students form their knowledge on conceptual level, master differential equation and its basic elements and get e possibility to apply them to the other equations.

Let x be the independent variable (x) is a continuously differentiable function defined on the interval and the relationship between them is given by the equation

Let $x \in \langle a;b \rangle$ x – independent variable, y(x) - is a continuously differentiable function defined on the interval $\langle a;b \rangle$ and the relationship between them is given by the equation

$$y' = f(x) \quad (1)$$

where $\langle a;b \rangle$ - a finite or infinite open or closed interval. Equation (1) – is the first order differential equation. Every function $y = \varphi(x)$ is continuously differentiable on the interval $\langle a;b \rangle$ together with its derivative, turns this equation into an identity is called a solution of the equation

$$\varphi'(x) = f(x), \ \forall x \in \langle a;b \rangle . \quad (2)$$

Equation (1) can be written through the differentials:

$$-f(x)dx + dy = 0 \quad (3)$$

Substituting the solution in this equation we obtain the identity

$$dy = \varphi'(x)dx \Rightarrow [-f(x) + \varphi'(x)]dx = 0, \ \forall x \in \langle a;b \rangle,$$

$$\Leftrightarrow$$

$$d\int [-f(x) + \varphi'(x)]dx = 0, \ \forall x \in \langle a;b \rangle$$

Hence

$$-\int f(x)dx + \int \varphi'(x)dx = C,$$

Or considering $\varphi(x) = y \Rightarrow \varphi' dx = dy$, then we obtain the identity

$$y = \int f(x)dx + C, \quad \forall x \in \langle a;b \rangle \quad (4)$$

when C=0, $\varphi(x) = \int f(x)dx$ we find a unique solution φ(x), satisfying the identity (2). If C is an arbitrary constant (possessing any numeric value), then the expression (4) gives the formula of the general solution, ie defines the general solution of the equation. Hence, for a fixed value of C (for example, when C = 0), the function gives a particular solution. Left-hand side of equation (3) on the basis of (4) satisfies the identity [1]:

$$-f(x)dx + dy = d\left[-\int f(x)dx + y\right],$$

or

$$-f(x)dx + dy = dU(x,y) \quad (5)$$

$$U(x,y) = -\int f(x)dx + y .$$

In other words, the left-hand side of equation (3) is equal to the total differential continuously differentiable function $U(x,y)$. This equation is called the total differential. Since $dU(x,y)=0$, then to solve $\varphi(x)$ here is the equation

$$U(x,y)=C,\ x\in\langle a;b\rangle,\quad y\in\langle\varphi(a),\varphi(b)\rangle=:\langle c;d\rangle$$

This expression is called the general solution of equation (3). The general integral is the implicit form of the general equation. Hence there is the general solution of the form (4). We now consider the question of finding the function $U(x,y)$. To do this, we write the equation (5)

$$-f(x)dx+dy=dU(x,y)=\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy,\quad \forall(x,y)\in D:D=\langle a;b\rangle\times\langle c;d\rangle.$$

so [2]

$$\frac{\partial U}{\partial x}=-f(x),\quad \frac{\partial U}{\partial y}=1 \quad (6)$$

Integrating the first equality on x (taking into account that $U(x,y)$ depends on y)

$$U(x,y)=-\int f(x)dx+C(y)$$

Substituting it into the second equation we get $C'(y)=1$. Consequently, $C(y)=y$. Hence we obtain the general solution in the form (4)

$$U(x,y)=-\int f(x)dx+y=C\Leftrightarrow y=\int f(x)dx+C.$$

If the first of equation (6) is differentiable with respect to y, and the second - for x, we have

$$-\frac{\partial f}{\partial y}=\frac{\partial(1)}{\partial x}=0\Rightarrow\frac{\partial^2 U}{\partial x\partial y}=\frac{\partial^2 U}{\partial y\partial x}.$$

This is a sufficient condition to equation (3) to be a differential equation in exact differential.

The way of the proof for the equation (1) can be used for the equation $y'=f(y)$. For this keeping equation content we swap independent of x and y.

2^0. Now consider the general form of the equation (1).

$$M(x)dx+N(y)dy=0 \quad (7)$$

where $M(x)\in C\langle a;b\rangle,\ N(y)\in C\langle c;d\rangle.$ This equation is called a differential equation with separated variables, since each term has only a function of one variable. Equation (3) is of this kind: $N(y)=1,\forall y\in\langle c;d\rangle$.

Let search the function $U(x,y)$, satisfying the equation (5)

$$M(x)dx+N(y)dy=dU(x,y)=\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy,\ \forall(x,y)\in D:D=\langle a;b\rangle\times\langle c;d\rangle$$

(8)

Hence [2]

$$\frac{\partial U}{\partial x}=M(x),\quad \frac{\partial U}{\partial y}=N(y) \quad (9)$$

and

$$\frac{\partial^2 U}{\partial x\partial y}=\frac{\partial^2 U}{\partial y\partial x}=0,$$

that is,

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=0$$

From equations (9) we obtain

$$U(x,y)=\int M(x)dx+C(y)\Rightarrow C'(y)=N(y)\Rightarrow C(y)=\int N(y)dy.$$

Function $U(x,y)$ satisfying the equation (9) or (8) can be written as

$$U(x,y)=\int M(x)dx+\int N(y)dy=C,$$

That is, (7) is a differential equation in the dipole differentials. Therefore, its general integral has the form

$$\int M(x)dx+\int N(y)dy=C. \quad (10)$$

Consider the equation of the form

$$M_1(x)\cdot M_2(y)dx+N_1(x)\cdot N_2(y)dy=0 \quad (11)$$

where $M_i,N_i,\ i=\overline{1,2}$ - continuous functions. Equation (11) is called the equation with separable variables, since equation (11) reduces to equation (7). To do this considering $M_2(y)\cdot N_1(x)\neq 0$, it is sufficiently to divide both sides of this equation by this product. Thus taking (10) into account the general integral has the form [3]:

$$\int\frac{M_1(x)}{N_1(x)}dx+\int\frac{N_2(y)}{M_2(y)}dy=C \quad (12)$$

We must have an eye to the case $M_2(y)\cdot N_1(x)=0$. The resulting solutions $M_2(y)=0,\ N_1(x)=0$ of this equation satisfy the equation (11). It is necessary to check whether they are included in the formula (12). If included in the solution (12) then they are partial solutions, and if not, they will be special solutions.

Function $\mu(x,y)$ after multiplication by the equation will be a differential equation in total differentials and is called integrating factor, i.e.

$$\mu(x,y) := \frac{1}{N_1(x)M_2(y)}$$

Integrating factor of the equation (11).

3^0. Consider the general form of first order differential equations written using differentials.

$$M(x,y)dx + N(x,y)dy = 0, \quad (13)$$

where M, N - second measurement functions defined in the area of $D \subset R^2$ function [4]. If $M(x,y)$, $N(x,y)$ are homogeneous functions of the same degree, that is, if for any variable s-equalities

$$M(sx,sy) = s^m M(x,y), \ \forall(x,y) \in D$$

$$N(sx,sy) = s^m N(x,y), \ \forall(x,y) \in D$$

Then the equation (13) called homogeneous equation. If the equation is homogeneous, then if $s = \frac{1}{x}$ we obtain the following equality

$$M(x,y) = x^m M(1,\frac{y}{x}), \quad N(x,y) = x^m N(1,\frac{y}{x})$$

.

Hence, applying the substitution $y = xz$ it can lead to an equation with a variable in a form

$$x^m[M(1,z) + zN(1,z)]dx + x^{m+1}N(1,z)dz = 0$$

.

Integrating factor for this equation is the function

$$\mu(x,y) := \frac{1}{x^m[M(1,z) + zN(1,z)]}$$

.

A type of equation

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \quad (14)$$

is called reducible to homogeneous equation [5]. The variables x, y introduce new variables u, v by replacing

$$a_1x + b_1y + c_1 =: v; \ a_2x + b_2y + c_2 =: u$$

.

Hence

$$dv = a_1dx + b_1dy; \ \ du = a_2dx + b_2dy$$

and instead of the equation (14) we have the equation

$$\frac{dv}{du} = \frac{a_1 + b_1y'}{a_2 + b_2y'} = \frac{a_1 + b_1 f\left(\frac{v}{u}\right)}{a_2 + b_2 f\left(\frac{v}{u}\right)} =: g\left(\frac{v}{u}\right),$$

This is a homogeneous equation. It is by the way of substitute $v = uz, \ \ z = z(u)$ has the look (11):

$$z + uz' = g(z), \ (z - g(z))du + udz = 0$$

Integrating factor is the function

$$\mu(x,y) := \frac{1}{u[z - g(z)]}.$$

4^0. Consider the first order differential equation of the general form

$$M(x,y)dx + N(x,y)dy = 0 \quad (15)$$

where M, N - function of the second measurement in the area $D \subset R^2$, continuously differentiable. In the particular case, as was shown above, in D there is a function defined in this area, continuously differentiable function $U(x,y)$ satisfying

$$dU(x,y) = M(x,y)dx + N(x,y)dy \quad (16)$$

If there is a function $U(x,y)$, equation (15) would be an equation in clandestine differentials. Necessary and sufficient condition for the equation to be a differential equation in exact differentials is the equality [1]

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \ \ \forall(x,y) \in D. \quad (17)$$

If (15) is a total differential equation, then by equation (15) and (16) it can be written as a general integral

$$U(x,y) = C$$

Consider the case when the equation (15) is not an exact differential equation, that is $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, it is possible to find an integrating factor. If there is such a factor, then consider the formula to find the integrating factor. Let $\mu(x,y)$ be such a factor,that is $\mu \in C^1(D)$ then there exists $U(x,y)$ satisfying the identity

$$\mu M dx + \mu N dy = dU, \ \ \forall(x,y) \in D.$$

Hence on the basis of (17) we obtain

$$\frac{\partial(\mu M)}{\partial y}=\frac{\partial(\mu N)}{\partial x},\ \forall(x,y)\in D.$$

If you open the derivatives, the equation for the integrating factor has the form [6]

$$\frac{\partial\mu}{\partial y}M+\mu\frac{\partial M}{\partial y}=\frac{\partial\mu}{\partial x}N+\mu\frac{\partial N}{\partial x}. \quad (18)$$

We consider two cases, when it is possible to find a relatively easy solution of equation (18). Suppose $\mu=\mu(x)$, that is, the integrating factor is a function of x alone. Hence

$$\frac{\partial\mu}{\partial y}=0,\ \frac{\partial M}{\partial x}=\frac{\partial\mu}{\partial x}$$

and then equation (18) can be reduced to

$$\frac{\partial\mu}{\partial x}N=\mu\left(\frac{\partial\mu}{\partial y}-\frac{\partial N}{\partial x}\right).$$

or a ratio of

$$\frac{\frac{\partial\mu}{\partial y}-\frac{\partial N}{\partial x}}{N}=:a(x) \quad (19)$$

Is a function of only one x, then for the equation (15) integrating factor is the function

$$\mu(x)=e^{\int a(x)dx}.$$

Similarly, if the ratio

$$\frac{\frac{\partial\mu}{\partial y}-\frac{\partial N}{\partial x}}{-M}=:b(y)$$

Is a function of only one x, then for the equation (15) integrating factor is the function

$$\mu(y)=e^{\int b(y)dy}.$$

With the help of an integrating factor we find the solution of linear equations of the first order.

5^0. Equation of the form

$$y'+p(x)y=q(x) \quad (20)$$

Is alled inhomogeneous linear differential equation of the first order, where $p(x),\ q(x)$ are continuous functions in the interval $\langle a;b\rangle$ [7]. Equation (20) can be written using differentials

$$[p(x)y-q(x)]dx+dy=0,$$

where

$M(x,y)=p(x)y-q(x),\ N(x,y)=1$ and condition (19) is done. Hence

$$\mu(x)=e^{\int p(x)dx}$$

is an integrating factor, multiplying the last equation we obtain the equation by a factor of total differential [8]

$$[p(x)y-q(x)]e^{\int p(x)dx}dx+e^{\int p(x)dx}dy=0$$

Therefore, form the equality

$$\frac{\partial U}{\partial x}=[p(x)y-q(x)]e^{\int p(x)dx},\quad \frac{\partial U}{\partial y}=e^{\int p(x)dx}$$

we get

$$U(x,y)=\int[p(x)y-q(x)]e^{\int p(x)dx}dx+C(y)$$

$$\frac{\partial U}{\partial y}=\int p(x)e^{\int p(x)dx}dx+C'(y)=e^{\int p(x)dx}\Rightarrow$$

$$C'(y)=0\Rightarrow C(y)=C.$$

If C=0. then

$$U(x,y)=y\int e^{\int p(x)dx}dx-\int q(x)e^{\int p(x)dx}dx$$

$$y\int e^{\int p(x)dx}dx-\int q(x)e^{\int p(x)dx}dx=C.$$

Hence we obtain the formula of the general solution of equation (20) in the form [2]

$$y=e^{-\int p(x)dx}\left(C+\int q(x)e^{\int p(x)dx}dx\right).$$

In the course of solving problems we must be able to freely choose the textbooks and methods, in fact, consider simple and convenient cases as a mathematical model, base on their classification and level.

Integration of linear homogeneous equation with constant coefficients.

Consider the linear homogeneous equation with constant coefficients.

$$a_0y''+a_1y'+a_2y=f(x)$$

where $a_i\ (i=0,1,2)$ real numbers, $f(x)$ known continuous function defined in the interval $\langle a,b\rangle$. This equation can be written in short form:

$$L(D)y=f(x) \quad (21)$$

The corresponding homogeneous equation is

$$L(D)y=0. \quad (22)$$

You can always find a solution to equation (22). Using the method of variation of parameters you can solve an equation (21) [9].

Consider the case when a particular solution of equation (21) can be found by the algebraic way. It is known that the general solution of equation (21) is the sum of the particular solution found and the general solution of equation (22). Consider the case where the function on the right side of equation (21) consists of the polynom product and an exponential function, i.e., has the form:

$$L(D)y = P_2(x)e^{\alpha x} \quad (23)$$

For definiteness, $P_2(x)$ - is a polynomial of second degree.

A particular solution of this equation does not depend only on the right side of the form, but the form of the left side, namely, whether or not the number is a αroot or not of the characteristic equation

$$M(\lambda) = a_0\lambda^2 + a_1\lambda + a_2 = 0 \quad (24)$$

To find a particular solution we consider three cases [1].

1) Number α is not a root of the characteristic equation, i.e, $M(\alpha) \neq 0$. Then a particular solution of equation (23) will be in the form

$$y = ze^{\alpha x} \; . \quad (25)$$

Substituting this function into equation (23) we obtain the following equation

$$M(\alpha)z + \frac{1}{1!}M'(\alpha)z' + \frac{1}{2!}M''(\alpha)z'' = P_2(x) \quad (26)$$

Here

$$M^{(i)}(\lambda) = \left.\frac{d^i M(\lambda)}{d\lambda^i}\right|_{\lambda=\alpha}, \; i = 0,1,2.$$

We introduce the notation

$$p_i = \frac{M^{(i)}(\alpha)}{i!}, \; i = 0,1,2$$

whence $p_0 = M(\alpha) \neq 0, \; p_2 = a_0$. Using the notation (26), the equation can be written as

$$p_0 z + p_1 z' + p_2 z'' = P_2(x) \quad (27)$$

Particular solution of equation (27) will be sought in the form of a polynomial of second degree $z = N_2(x)$. Substituting it into equation (27) and having differentiated the resulting equation we received the following system twice:

$$\begin{cases} p_0 N_2(x) + p_1 N_2'(x) + p_2 N_2''(x) = P_2(x) \\ p_0 N_2'(x) + p_1 N_2''(x) = P_2'(x) \\ p_0 N_2''(x) = P_2''(x) \end{cases} \quad (28)$$

This is triangular inhomogeneous linear algebraic system. So

$$N_2(x) = \frac{1}{P_0^3}\begin{vmatrix} P_2(x) & p_1 & p_2 \\ P_2'(x) & p_0 & p_1 \\ P_2''(x) & 0 & p_0 \end{vmatrix} \quad (29)$$

Z value found by substituting into (25) we find the particular solution of equation (23) as follows:

$$y_д = e^{\alpha x}N_2(x) \quad (30)$$

2) Let the number α is a simple root of the characteristic equation (24). Then $M(\alpha) = p_0 = 0, \; M'(\alpha) = p_1 \neq 0$ and equation (27) can be written as

$$p_1 z' + p_2 z'' = P_2(x). \quad (31)$$

If we denote by $z' = u$, we obtain

$$p_1 u + p_2 u' = P_2(x) \quad (32)$$

and hence

$$u = \frac{1}{P_1^2}\begin{vmatrix} P_2(x) & p_2 \\ P_2'(x) & p_1 \end{vmatrix} = N_2(x)$$

will be its solution. Then $z = \int N_2(x)dx = xQ_2(x)$ will be a solution of $y = xe^{\alpha x}Q_2(x)$, and a solution of equation (23) [10].

3) If the number of α is twofold root of characteristic equation (24), then equation (27) can be written as

$$p_2 z'' = P_2(x) \Rightarrow z'' = \frac{1}{p_2}P_2(x)$$

and $z = x^2 R_2(x)$ would be a solution of this equation. Then

$$y = x^2 e^{\alpha x}R_2(x)$$

gives the solution of equation (23).

The introduction of the offered thesis was realized by teaching mathematics (lectures giving, delivering practical lessons) at pedagogical institutes of higher education, participating at conferences and seminars according to the theme of the research. The elaborated syllabus is used by students and mathematics teachers in the educational process.

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References

1. Suleimenov, Zh.S., 2004. Methodical System of Teaching Differential Equations for the Students of Physics and Mathematics Departments of the University: diss… Doctor of Pedagogical Sciences. Almaty, 275 p.
2. Matveyev, N.M., 1988. Differential Equations. Moscow: Prosbeschenie, 254 p.
3. Samoylenko, A.M., S.A. Krivoshey and N.A. Perestok, 2006. Differential Equations. Practical course. Moscow: High School, 383 p.
4. Trench, W.F., 2002. Differential equations with boundary value problems. San Antonia, Texas, USA. 797 p.
5. Birkhoff, G., and G.-C. Rota, 1962. Ordinary Differential Equations, Blaisdell Publ. Co., Waltham, Mass., 409 p.
6. Tenenbaum, M. and H. Pollard, 1963. Ordinary Differential Equations: An elementary textbook for students of mathematics, engineering, and the sciences. Courier dover publications, 808 p.
7. Mohan, C. J., 2006. Ordinary differential equations: modern perspective. Oxford, U.K.: Alpha Science International Ltd., 261 p.
8. Larson, R., R.P. Hostetler, B.H. Edwards, 2006. Calculus: Early Transcendental Functions. Brooks Cole, 1139 p.
9. Pontryagin, L.S., 1970. Ordinary Differential Equations. Moscow: Nauka, 331 p.
10. Arnold, V.I., 1984. Ordinary Differential Equations. Moscow: Nauka, 271 p.

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