A model of flows of distribution in the network

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Abstract. A network is a multichannel structure in which two points are linked by a number of lines consisting of sectors and routes. A mathematical model of network flows distribution based on the Erlang law was developed. We introduced our classification of nodes as well as the criteria of efficiency of flows distribution. The analytical dependence of such indices as the mean value of delay-in-queue and the weight of node, i.e. the mean number of requests on queue, on the Erlang time distribution parameters was deduced. The relevance of the developed model of network operation was empirically proved in the transportation network of an urban locality.

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Introduction

The notion of network is basic in the logistics of complex communication systems. A network is a multichannel structure in which two points are linked by a number of lines consisting of sectors and routes. Whether it is a transportation network or those of cable or cellar communications, the key problem is to cover the maximum area and provide an access from any point to others.

A transportation network gives us a vivid example of network operation. Mathematical models applied for analysis of transportation networks are various due to the problems solved, the data, the mathematical tools, and the degree of specification in traffic description [as in 1-4].

The first macroscopic model was suggested by M. Lighthill and G. Whitham in the middle of the 20th century [5]. The first microscopic models (e.g.'follow-the-leader' theory) which explicitly derived an equation of motion for each individual vehicle were also developed at that time (A. Reshel, L. Pipes, D. Gazis and others) [6].

F. A. Haight was the first to establish the mathematical investigation of traffic flow as a separate branch of applied mathematics [7]. At present there is voluminous literature on vehicle traffic flows. But the problem of efficient operation of transportation network is still relevant due to the accelerating growth of traffic volume.

A very important task is to develop a model of network operation which will allow adequate forecasting of the efficient distribution of network flows using the minimum number of initial data. Developing a microscopic model of transportation dynamics in the nodes and estimation of their influence on the network flows distribution is also relevant.

1. Mathematical model of network flow

We present a network in the traditional form of graph. A network, thus, is a graph each arc of which is assigned to a certain number.

A flow in the graph is a set of homogeneous objects (requests) sent from one node to another. A flow is, thus, a certain function prescribed for the graph arcs [8]. In the developed model, the flow in the graph is given as a function of density of arrival distribution (arrival times of service requests).

Let the time distribution in each flow of requests (channels) be Erlang, which will allow us to describe high density flows.

The Erlang density distribution is:

$$f^{(k)}(t) = \lambda(\lambda t)^{k-1} e^{-\lambda t} \ /(k-1)! \quad (t>0) \qquad (1)$$

The function of the k-th order Erlang distribution is:

$$F^{(k)}(t) = 1 - \sum_{n=0}^{k-1} \left((\lambda t)^n e^{-\lambda t}\right)/n! = 1 - R(k-1, \lambda t) \quad (t>0) \qquad (2)$$

Let us specify the basic notions used in this paper.

We will refer to the network flows as 'non-conflict' if they are not crossed in the given sector of network, and as 'conflict' otherwise. Let the node-points (the points of sources or consumption of information and those of conflict flows crossing) be the vertices of graph. These are formed by multichannel line crossing.

We suggest the following classification of node-points (NP). Some flows (the main ones) are freely passing the NP. The customers of the rest of flows (the secondary ones) are expecting sufficient time intervals between arrivals of main flows in order to cross the NP. We will call such a node-point a 'type 1 node'.

Let us consider a node-point where the channel is alternately blocked for one of the non-conflict flow group for a fixed time to enable the crossing of the NP which we call a 'type 2 node'.

We use the following notation: an arc is part of a multichannel line between two nodes.

Let:

$\{l_j\}$ – set of graph arcs;

$\{z_j\}$ – set of nodes.

In order to determine the indices of efficiency of network flow distribution and to solve the optimization problems we use the following notations:

1) $\omega_M(z_n)$ – mean delay-in-queue in the node;

z_n – requests of the chosen direction;

2) $\vec{\mu}(z_n)$ – weight of node z_n (node-point) for the given direction flow, i.e. the mean total delay-in-queue of all requests in the given direction flow in the node z_n per time unit;

3) $\mu(z_n)$ – weight of node z_n (node-point), i.e. the mean total delay-in-queue of all requests in the node z_n per time unit.

1.1. Model of the type 1 node operation

The operation of a type 1 node can be regarded as a mass service system. The service time is the waiting time for a customer of a secondary flow to cross the necessary main flows; the queue size is a number of customers waiting for the possibility to continue to move; the discipline (priority) is according to the order of arrival at the NP.

Let the service time be distributed by the exponential law with the μ parameter.

The customer flow is distributed by the generalised Erlang law with the (k+l)-folded $\lambda_0, \lambda_1, \ldots, \lambda_\kappa$ parameters. No more than n customers can be serviced at a time. To find out if there is a probability for m ($n > m$) customers to be serviced, we apply the pseudostates method [9].

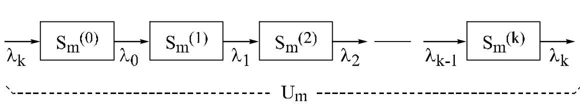

Figure 1. Pseudostates of the Erlang (k+l)-folded distribution

Let us take n service channels with an unlimited queue size. Make a differential equation system to determine probabilities of s arrivals in the queue:

$p_{n+s}(t)$ – probability of s arrivals in the queue at time t.

Let:

$p_{n+i}(t+\Delta t)$ – probability of s arrivals in the queue at time $(t+\Delta t)$, We denote $p_l^{(0)} \equiv p_l, l \in \mathrm{N} \cup \{0\}$.

$$p_{n+i}(t+\Delta t) \approx p_{n+i}(t)\big(1-(\lambda_0+n\mu)\Delta t\big)++p^{(k)}{}_{n+i-1}(t)\cdot\lambda_k\cdot\Delta t+n\mu p_{n+i+1}(t)\cdot\Delta t.$$

Dividing by Δt:

$$\frac{p_{n+i}(t+\Delta t)-p_{n+i}(t)}{\Delta t} \approx -p_{n+i}(t)(\lambda_0+n\mu)+p^{(k)}{}_{n+i-1}(t)\cdot\lambda_k+n\mu\, p_{n+i+1}(t). \quad (3)$$

With $\Delta t \to 0$ we have the following differential equations system:

$$p'_{n+i}(t) = -p_{n+i}(t)\cdot(\lambda_0+n\mu)+p^{(k)}_{n+i-1}(t)\cdot\lambda_k+n\mu\, p_{n+i+1}$$
$$(i=1;\ 2;\ 3;\ \ldots). \quad (4)$$

Similarly, for the instant of time with no queue when m ($m \le n$) service channels are busy:

$$p_m(t+\Delta t) \approx p_m(t)\cdot(1-(\lambda_0+m\mu)\Delta t)+(m+1)\mu\, p_{m+1}(t)\cdot\Delta t+p^{(k)}{}_{m-1}(t)\lambda_k\Delta t. \quad (5)$$

Hereof:

$$(p_m(t))'_t = -(\lambda_0+m\mu)p_m(t)+(m+1)\mu\cdot p_{m+1}+\lambda_k p^{(k)}_{m-1}(t), \quad (m=1,2,\ldots,n). \quad (6)$$

For the instant of time when the entire system is free, we have by analogy the following equation:

$$(p_0(t))'_t = -\lambda_0 p_0(t)+\mu\, p_1(t). \quad (7)$$

Assume that probabilities $p_m(t)$, $p_{n+1}(t)$, …, $p_{n+i}(t)$, … of system states $s_m^0, s_{n+1}^0, \ldots, s_{n+i}^0, \ldots$ are already known. Find the probabilities of transit states for subset U_m (using the graph in Fig.1):

$$p_m^{(j)}(t+\Delta t) \approx p_m^{(j)}(t)(1-\lambda_j\Delta t)+p_m^{(j-1)}(t)\cdot\lambda_{j-1}\Delta t. \quad (8)$$

Dividing by Δt find the limit for $\Delta t \to 0$:

$$\left(p_m^{(j)}(t)\right)'_t = -\lambda_j p_m^{(j)}(t)+\lambda_{j-1}p_m^{(j-1)}(t) \qquad (j=1,2,3,\ldots,k). \quad (9)$$

So, to find unknown probabilities we obtain the following system of differential equations:

The initial conditions are as follows:

$$p_0^{(0)}(0)=1, \quad p_m^{(j)}(0)=0, \quad (m=1,2,3,\ldots;\quad j=1,2,\ldots,k) \quad (10)$$

Denote $r_m(t)=P(U_m)$, i.e. $r_m(t)$ – probability of system state U_m.

According to the laws of chance:

$$r_m(t)=p_m(t)+\sum_{j=1}^{k}p_m^{(j)}. \quad (11)$$

Then, the mean delay-in-queue at time t:

$$M(l(t))=\sum_{i=1}^{\infty} i\cdot r_{n+i}(t); \quad (12)$$

$$\begin{cases} (p_m(t))'_t = -(\lambda_0 + m\mu)p_m(t) + (m+1)\mu \cdot p_{m+1} + \lambda_k p_{m-1}^{(k)}, (m = 1, 2, ..., n) \\ (p_{n+i}(t))'_t = -(\lambda_0 + n\mu)p_{n+i}(t) + \lambda_k p_{n+i-1}^{(k)}(t) + n\mu p_{n+i-1}(t), (i = 1,2,3,...) \\ (p_m^{(j)}(t))'_t = -\lambda_j p_m^{(j)}(t) + \lambda_{j-1} p_m^{(j-1)}(t), (j = 1,2,..., k) \end{cases}$$

the mean expected number of busy service channels:

$$M(N(t)) = \sum_{i=0}^{n-1} i \cdot r_i(t) + n\sum_{l=0}^{\infty} r_{n+l}(t); \quad (13)$$

the mean expected number of requests in the system (both serviced and waiting) is:

$$M(X(t)) = \sum_{m=0}^{\infty} i \cdot r_m(t). \quad (14)$$

For a long running network it is expedient to consider **a stationary process**. Then (at $t \to \infty$) probabilities p_m are constant, hence their derivatives equal 0, and the systems of differential equations will acquire a simpler form by virtue of:

$$\left(p_m^{(j)}(t)\right)'_t = \left(p_m(t)\right)'_t = \left(p_{n+i}(t)\right) = 0,$$

$$(j = 1,2,...,k;\ m = 1, 2, ..., n;\ i = 1,2,3,...)$$

The system solution is:

$$p_m^{(j)} = \frac{\lambda_0}{\lambda_j} p_m \qquad (j=1,...,k;\ m = 0, 1, 2, ...) \quad (15)$$

$$p_m = \frac{\left(\frac{\lambda_0}{\mu}\right)^m}{m!} p_0, \ (m = 0, 1, ..., n-1), \quad (16)$$

$$p_{n+i} = \frac{\left(\frac{\lambda_0}{\mu}\right)^{n+i}}{n! \cdot n^i} p_0, \ (i = 0, 1, 2, ...). \quad (17)$$

Let:

$$\alpha = \frac{\lambda_0}{\mu},\ b = \sum_{i=0}^{k} \frac{\lambda_0}{\lambda_i},\ r_m = P\{S \subset U_m\}.$$

Then:

$$r_m = p_m + \sum_{i=1}^{k} p_m^{(i)} = p_m \sum_{i=0}^{k} \frac{\lambda_0}{\lambda_i} = p_m \cdot b.$$

Find p_0 from: $\sum_{j=0}^{\infty} r_j = 1$.

$$1 = \sum_{j=0}^{\infty} r_j = \sum_{j=0}^{\infty} p_j b = \sum_{j=0}^{n} \frac{\alpha^j}{j!} p_0 b + \sum_{j=1}^{\infty} \frac{\alpha^{n+j}}{n! \cdot n^j} p_0 b = bp_0 \left(\sum_{j=0}^{n} \frac{\alpha^j}{j!} + \frac{\alpha^n}{n!} \sum_{j=1}^{\infty} \left(\frac{\alpha}{n}\right)^j \right). \quad (18)$$

A numerical series is convergent if the following condition is fulfilled:

$$\frac{\alpha}{n} < 1 \quad (19)$$

Denote by p_0 the probability that:

$$p_0 = \frac{1}{b\left(\sum_{j=0}^{n} \frac{\alpha^j}{j!} + \frac{\alpha^n}{n!} \frac{\alpha / n}{1 - \frac{\alpha}{n}} \right)}. \quad (20)$$

Theorem 1. Let the service time for the flow requests have exponential distribution with parameter μ and the flow of requests have an Erlang distribution with (k+l)-folded parameters $\lambda_0, \lambda_1, ..., \lambda_K$. Then for a stationary process with an unlimited queue size:

$$p_m = \frac{\left(\frac{\lambda_0}{\mu}\right)^m}{m! \cdot b\left(\sum_{j=0}^{n} \frac{\alpha^j}{j!} + \frac{\alpha^n}{n!} \frac{\alpha / n}{1 - \frac{\alpha}{n}} \right)} \ (m = 0, 1, ..., n-1);$$

$$p_{n+i} = \frac{\alpha^{n+i}}{n! \cdot n^i \cdot b\left(\sum_{j=0}^{n} \frac{\alpha^j}{j!} + \frac{\alpha^n}{n!} \frac{\alpha / n}{1 - \frac{\alpha}{n}} \right)} \ (i = 0, 1, 2, ...);$$

Subject to Theorem 1, the probability of no more than s requests–in-queue is:

$$P_{n+s} = \sum_{m=0}^{n} r_m + \sum_{j=1}^{s} r_{n+j} = bp_0 \left(\sum_{m=0}^{n} \frac{\alpha^m}{m!} + \frac{\alpha^n}{n!} \sum_{j=1}^{s} \left(\frac{\alpha}{n}\right)^j \right). \quad (21)$$

The mathematical expectation of the number of requests-in-queue is found by the formula:

$$M(l) = \sum_{j=1}^{\infty} jr_{m+j} = bp_0 \frac{\alpha^n}{n!} \sum_{j=0}^{\infty} \left(\frac{\alpha}{n}\right)^j j = bp_0 \frac{\alpha^n}{n!} \frac{\alpha}{n} \sum_{j=0}^{\infty} \left(\left(\frac{\alpha}{n}\right)^j \right)' = bp_0 \frac{\alpha^{n+1}}{n! \cdot n \cdot \left(1 - \alpha/n\right)^2}. \quad (22)$$

The mathematical expectation of the number of busy servers:

$$M(N) = \sum_{j=0}^{n-1} jr_j + n\sum_{j=0}^{\infty} r_{n+j} = bp_0 \left(\sum_{j=0}^{n-1} \frac{\alpha^j}{j!} + \frac{n\alpha^n}{n!} \frac{1}{1 - \frac{\alpha}{n}} \right). \quad (23)$$

The mean number of customers being served and waiting in queue:

$$M(X) = \sum_{j=0}^{\infty} jr_j = bp_0 \left(\sum_{j=0}^{n} j\frac{\alpha^j}{j!} + \frac{\alpha^n}{n!} \sum_{j=1}^{\infty} (n+j) \left(\frac{\alpha}{n}\right)^j \right).$$

$$M(X) = bp_0 \left(\sum_{j=1}^{n} \frac{\alpha^j}{(j-1)!} + \frac{\alpha^{n+1}}{n!} \frac{1}{1 - \frac{\alpha}{n}} + \frac{\alpha^{n+1}}{n!\, n \cdot \left(1 - \alpha/n\right)^2} \right) \quad (24)$$

1.2. Calculation of the mean service time value in the type 1 node

Let a secondary flow customer need to cross L main flows in the type 1 node to go on moving. Assume that time intervals in the crossed flows are distributed by the Erlang law with the (k_1, ..., k_L)-folded $\lambda_0, \lambda_1, ..., \lambda_k$ parameters correspondingly. Take into account that a customer arrives at the NP at a random moment, irrespective of other flows customers.

Denote by T_i^1 time intervals between requests i and $(i+1)$ in the nearest main flow:

R_i^j – time intervals between requests i and $(i+1)$ in flow j, $j \in (1,2,3...,l)$;

$Y_0 = \min\limits_{j=2...L} \{R_0^1, R_0^j\}$; $Y_i = \min\limits_{j=2..L} \{T_i^1, R_i^j\}$, i=2, 3, 4, ...

Random variable Y_i has equal distribution when i = 2, 3, 4, …. We will use it in derivation of formulas below.

Assume that there is a possibility to cross an NP if Y_i is greater than or equal to some variable T_0.

The distribution function of random variable Y_i (the minimal of some random variables):

$$\Phi(y) = 1 - \prod_{i-1}^{L} \left(1 - F_i(y)\right) = P(Y_i < y), \quad (25)$$

where $F_i(y)$ = distribution function of random variable i.

Find the mean waiting time for an opportunity to continue motion:

$$Z = \sum_{i=0}^{Y} T_i, \quad (26)$$

where T_i = time between two arrivals (random variable).

$$m_z = M(Z) = \sum_{m=1}^{\infty} M(Z/Y=m)\cdot P_m = \sum_{m=1}^{\infty}\left(\frac{k_1+1}{2\lambda_1} + (m-1)\frac{k_1}{\lambda_1}\right)P_m =$$
$$= \frac{k_1+1}{2\lambda_1}\sum_{m=1}^{\infty} P_m + \frac{k_1}{\lambda_1}\sum_{m=1}^{\infty}(m-1)P_m, \quad (27)$$

where Y = number of customers having passed before an opportunity to continue motion (random variable).

Probability P_m that m customers have passed can be expressed by the following formula:

$$P_0 = P(Y_0 > T_0)$$
$$P_m = P(Y_0 < T_0)\cdot P(Y_{m-1} < T_0)^{m-1} P(Y_m > T_0)), \quad m > 0 \quad (28)$$

Subject to the foregoing formulas, proceed with calculation of the waiting time:

$$m_z = \frac{k_1+1}{2\lambda_1}\sum_{m=0}^{\infty} P(Y_0 < T_0)\cdot P(Y_m < T_0)^m P(Y_{m+1} > T_0) +$$
$$+ \frac{k_1}{\lambda_1}\sum_{m=0}^{\infty} mP(Y_0 < T_0)\cdot P(Y_m < T_0)^m P(Y_{m+1} > T_0) =$$
$$= \frac{k_1+1}{2\lambda_1} P(Y_0 < T_0)\cdot P(Y_{m+1} > T_0)\frac{1}{1-P(Y_m < T_0)} +$$
$$+ \frac{k_1}{\lambda_1} P(Y_0 < T_0)\cdot P(Y_m < T_0)P(Y_{m+1} > T_0)\left(\frac{1}{1-P(Y_m < T_0)}\right)_P^{/} =$$
$$= P(Y_0 < T_0)P(Y_{m+1} > T_0)\left(\frac{(k_1+1)}{2\lambda_1(1-P(Y_m < T_0))} + \frac{k_1\cdot P(Y_m < T_0)}{\lambda_1(1-P(Y_m < T_0))^2}\right) =$$
$$= P(Y_0 < T_0)P(Y_{m+1} > T_0)\cdot\frac{k_1+1+(k_1-1)P(Y_m < T_0)}{2\lambda_1(1-P(Y_m < T_0))^2}.$$

Theorem 2. If the number of servers $n = 1$, the mean waiting time (in seconds) for an opportunity to cross L flows of requests distributed by Erlang with the (k_1, ..., k_L)-folded parameters $\lambda_1, ..., \lambda_L$ correspondingly is:

$$(m_z) = \left(1 - \prod_{i=1}^{L}\left[\frac{1}{k_i}\sum_{n=0}^{ki-1} R(n, T_0\lambda_i)\right]\right)\times$$
$$\times\frac{k_1+1+(k_1-1)\left(1 - R(k_1-1, T_0\lambda_1)\cdot\prod_{i=2}^{L}\left[\frac{1}{k_i}\sum_{n=0}^{ki-1} R(n, T_0\lambda_i)\right]\right)}{2\lambda_1\left(R(k_1-1, T_0\lambda_1)\cdot\prod_{i=2}^{L}\left[\frac{1}{k_i}\sum_{n=0}^{ki-1} R(n, T_0\lambda_i)\right]\right)}.$$

The mean delay (in seconds) at the NP of one customer of a secondary direction with the Erlang distribution with parameters λ and k:

$$W_H = m_Z \cdot M(l) = \frac{\alpha\cdot m_z}{1-\alpha}, \quad (29)$$

where $R(k-1, \lambda t) = \sum_{n=0}^{k-1}\left((\lambda t)^n e^{-\lambda t}\right)/n!$,

$$\alpha = \frac{\lambda}{k\cdot\mu}, \quad \mu = 1/(m_z),$$

T_0 = acceptable time interval (sec.) before continuing motion.

Specify the criteria of efficiency of flows distribution for the type 1 node:

1) $\omega_M(z_n) = \dfrac{\sum_{i\in M}\left(\frac{\lambda i}{ki}\cdot W_{Hi}\right)}{\sum_{i\in M}\frac{\lambda i}{ki}}$, where M = set of chosen directions;

2) $\vec{\mu}(z_n) = \sum_{i\in M}\dfrac{\frac{\lambda i}{ki}W_{Hi}}{3600}$, where M = set of chosen directions;

3) $\mu(z_n) = \sum_{i\in\Omega} \frac{\frac{\lambda i}{ki} W_{Hi}}{3600}$, where Ω = set of all directions.

1.3. Model of the type 2 node operation

Consider a type 2 node-point.

Find the mean value of variable N which is the number of arrivals at the node for interval of time *t* (*0; t*).

Consider a restoration function $H(t) = M(N_t)$ – mathematical expectation of arrivals per time *t*. The Laplace transform for this function will hold [10]:

$$H^*(s) = \frac{f^*(s)}{s(1-f^*(s))}. \quad (30)$$

Here $f^*(s)$ is the image of time distribution function. For Erlang:

$$f^*(s) = \frac{\lambda^k}{(\lambda+s)^k}. \quad (31)$$

Then the image of restoration function is:

$$H^*(s) = \frac{\lambda^k}{s((\lambda+s)^k - \lambda^k)}. \quad (32)$$

Therefore, $H^*(s)$ can be reduced to common fractions containing the following terms [as in 10]:

1) from pole $s = 0$;
2) from nonzero poles in the points which are the roots of the equation $f^*(s) = 1$.

$$H^*(s) = \frac{1}{\mu}\frac{1}{s^2} + \frac{\sigma^2 - \mu^2}{2\mu^2}\frac{1}{s} + R^*(s),$$

where $\mu = \frac{\lambda}{k}$; $\sigma^2 = \frac{\lambda^2}{k^2}$; $R^*(s)$ is a rational function with the poles in the points which are the roots of the equation $f^*(s) = 1$.

The nonzero roots are:

$$s_p = \lambda \cdot \left(e^{2\pi pi/k} - 1\right), \quad p = 1,2,\dots,k-1$$ (where *i* is an imaginary unit).

Any simple nonzero root s_p in expansion $H^*(s)$ fits a fraction:

$$\frac{-1}{s_p \cdot \left(f^*(s_p)\right)' \cdot \left(s - s_p\right)} = \frac{\lambda + s_p}{k \cdot s_p \cdot (s - s_p)},$$

that is

$$H^*(s) = \frac{\lambda}{ks^2} + \frac{1}{s}\frac{1-k}{2k} + \sum_{p=1}^{k-1} \frac{\lambda + s_p}{k \cdot s_p \cdot (s - s_p)}. \quad (33)$$

Hereof, find the original, i.e. the restoration function $H(t)$ is the number of arrivals at the given point of road in interval of time *t* (*0; t*).

Each fraction $\frac{1}{s - s_p}$ fits original $e^{s_p t}$, since $\operatorname{Re} s < 0$, then $R(t)$ decreases with time *t*.

Find the restoration function for individual values of parameters *k*.

Parameter $k = 2$. In the case when Erlang has $k = 2$, the equation holds:

$$\lambda^2 = (\lambda + s)^2. \quad (34)$$

Then

$$H^*(s) = \frac{\lambda}{2s^2} - \frac{1}{4s} + \frac{1}{4(s+2\lambda)}. \quad (35)$$

Find the original by image:

$$H(t) = \frac{\lambda t}{2} - \frac{1}{4} + \frac{1}{4}e^{-2\lambda t}. \quad (36)$$

Parameter $k = 3$. With the third-order Erlang the following equation must hold:

$$\lambda^3 = (s + \lambda)^3. \quad (37)$$

The simple nonzero complex roots for the given equation are:

$$s_{1,2} = -\frac{3}{2}\lambda \pm i\frac{\sqrt{3}}{2}\lambda. \quad (38)$$

By analogy:

$$H(t) = \frac{\lambda t}{3} - \frac{1}{3} + \frac{1}{3\sqrt{3}}e^{-3\lambda t/2}\left[\sin\left(\frac{\lambda t\sqrt{3}}{2}\right) + \sqrt{3}\cos\left(\frac{\lambda t\sqrt{3}}{2}\right)\right]. \quad (39)$$

Parameter $k = 4$. The nonzero roots are:

$$s_1 = \lambda\left(e^{\frac{\pi}{2}i} - 1\right) = -\lambda + i\lambda;$$

$$s_2 = \lambda\left(e^{i\pi} - 1\right) = -2\lambda;$$

$$s_3 = \lambda\left(e^{\frac{3\pi}{2}i} - 1\right) = -\lambda - i\lambda.$$

$$R^*(s) = \frac{1}{4}\left(\frac{1}{2\cdot(s+2\lambda)} + \frac{i}{(-1+i)\cdot(s-\lambda(-1+i))} + \frac{i}{(1+i)\cdot(s-\lambda(-1-i))}\right)$$

Find the original by image:

$R(t)=\frac{1}{4}\left(\frac{1}{2}e^{-2\lambda t}+\frac{i}{-1+i}e^{\lambda(-1+i)t}+\frac{i}{1+i}e^{\lambda(-1-i)t}\right)=\frac{1}{8}e^{-2\lambda t}+\frac{1}{4}e^{-\lambda t}(\cos(\lambda t)+\sin(\lambda t))$

Then, the restoration function is:

$$H(t)=\frac{\lambda}{8}t-\frac{3}{8}+\frac{1}{8}e^{-2\lambda t}+\frac{1}{4}e^{-\lambda t}(\cos(\lambda t)+\sin(\lambda t)) \quad (40)$$

We define as a delay in the type 2 node the waiting time in the case when motion in the given direction is not allowed.

Calculate the expected total delay of the flow requests for T_i seconds (the time interval during which motion in the given direction is not allowed. Assume that at time $t=0$ the queue in the node is empty.

Divide interval $(0;T_i)$ into n parts by points $t_1, t_2, ..., t_n$; $\Delta t_i = t_i - t_{i-1}$. Then the total delays of all requests in the given flow for one operation cycle $T = T_1 + T_2$ approximately equals:

$W(T_i,\lambda)\approx\sum_{i=1}^{n}H_\lambda(t_i)\cdot\Delta t_i$. If we take the limit with $\max\{\Delta t_i\}\to 0$, we receive: $W(T_i,\lambda)=\int_0^{T_1}H_\lambda(t)dt$ (arrivals per second).

So, the total delays of all requests in the given flow per hour is :

$$W(T_i,\lambda)\cdot\frac{3600}{T}\cdot\frac{1}{3600}=\frac{W(T_i,\lambda)}{T} \text{ (arrivals per hour)} \quad (41)$$

Parameter $k=2$:

$$W(T_i,\lambda)=\int_0^{T_i}H_\lambda(t)dt=\frac{\lambda T^2{}_i}{4}-\frac{T_i}{4}-\frac{e^{-2\lambda T_i}}{8\lambda}+\frac{1}{8\lambda},\ i\in\{1;2\}, \quad (42)$$

Parameter $k=3$:

$$W(T_i,\lambda)=\int_0^{T_i}\left(\frac{\lambda t}{3}-\frac{1}{3}+\frac{1}{3\sqrt{3}}e^{-\frac{3\lambda t}{2}}\sin\frac{\lambda t\sqrt{3}}{2}+\frac{1}{3}e^{-\frac{3\lambda t}{2}}\cos\frac{\lambda t\sqrt{3}}{2}\right)dt$$

$$W(T_i,\lambda)=\frac{\lambda}{6}T_i-\frac{1}{3}T_i-\frac{1}{9\lambda}e^{-\frac{3}{2}\lambda T_i}\cdot\left(2\cos\frac{\lambda T_i\sqrt{3}}{2}-\sin\frac{\lambda T_i\sqrt{3}}{2}\right)+\frac{2}{9\lambda}. \quad (43)$$

Parameter $k=4$.

$$W(T_i,\lambda)=\int_0^{T_i}H(t)dt=\int_0^{T_i}\left(\frac{\lambda}{8}t-\frac{3}{8}+\frac{1}{8}e^{-2\lambda t}+\frac{1}{4}e^{-\lambda t}(\cos(\lambda t)+\sin(\lambda t))\right)dt$$

$$W(T_i,\lambda)=\frac{\lambda}{8}T_i^2-\frac{3}{8}T_i-\frac{1}{16\lambda}e^{-2\lambda T_i}-\frac{1}{4\lambda}e^{-\lambda T_i}\cos(\lambda T_i)+\frac{5}{16\lambda} \quad (44)$$

Now define the number of arrivals in the queue in each flow for the type 2 node.

If the number of arrivals at the NP for one operation cycle is less than he number of arrivals leaving the NP for the interval when motion in the given direction is allowed, the queue does not grow and becomes empty for one cycle.

Let $nA1$ be the number of non-conflict flows in direction *A* of line *1*,

h – the mean time (in seconds) between two arrivals of the same flow;

λAi – the Erlang distribution parameter for the *i*-th flow of direction *A*.

If $\sum_{i=1}^{nA1}H_{\lambda A i}(T)-\frac{T_2}{h}nA1 < 0$, then the queue in this direction becomes empty foe one cycle and the total delays per hour are:

$$(T_\Sigma)_{P1}=\left(\sum_{i=1}^{nA1}W(T_1,\lambda Ai)\right)\frac{1}{(T_1+T_2)}, \quad (45)$$

T_1 = time when motion in direction *A* of line *1* is not allowed (sec.);

T_2 = time when motion of the flows of line 2 is not allowed (sec.);

$T = T_1+T_2$.

Similar are the considerations for other flow directions.

Specify the criteria of efficiency of flows distribution for the type 2 node:

1) $\omega_M(z_n)=\frac{\sum_{i\in M}W(T_a,\lambda_i)}{\sum_{i\in M}H(T_a,\lambda_i)}$, где *M* = set of chosen directions, $a\in\{1;2\}$

2) $\vec{\mu}(z_n)=\frac{\sum_{i\in M}W(T_a,\lambda_i)}{T}$, где *M* = set of chosen directions, $a\in\{1;2\}$;

3) $\mu(z_n)=\frac{\sum_i W(T_1,\lambda_i)+\sum_j W(T_2,\lambda_j)}{T}$.

2. Representation of urban transportation network

An urban transportation network can be represented as an oriented graph, where nodes correspond to intersections and arcs to road segments linking two neighboring intersections.

Simple intersections are assumed as type 1 nodes, intersections with regulated traffic as type 2 nodes. Then the above described mathematical model can apply to the operation of urban transportation network [11].

According to different investigations into the field of traffic flow and the experiments conducted by the authors, time intervals between vehicles in the flow fit to the Erlang distribution. Depending on the intensity of traffic flow parameter k varies from $k=1$ to $k=4$.

Since the Erlang distribution has two parameters, we have to find unknown parameters experimentally in order to obtain the dependence between X_i (the time interval between two successive arrivals at the given intersection for vehicles moving in the same direction) and number n_i of such intervals resulted from the experiment.

There are two unknown parameters k and λ in the Erlang distribution, so we need two equations to find these parameters by method of moments.

1) Let the initial theoretical moment of the first order be the first empirical moment:

$$M(X)=\overline{x}_B;$$

2) Let the corrected sample variance be the central theoretical moment of the second order:

$$D(X)=\hat{s}^2.$$

By solving the system we obtain the following estimations of the desired parameters:

$$\lambda^*=\frac{\overline{x}_B}{\hat{s}^2},\quad k^*=\frac{\overline{x}_B^2}{\hat{s}^2}. \qquad (46)$$

Gamma distribution is a more general case of Erlang with non-integer k. And the gamma distribution parameters were worked out by formulas (46). In order to obtain the Erlang distribution parameters, assign to k the next to $k*$ integer value, and to λ the following value:

$$\lambda=\frac{k}{\overline{\overline{x}}_B}.$$

Empirical moments are calculated by formulas:

$$\overline{x}_B=\frac{\sum_{i=1}^{N} n_i x_i}{N},\quad \hat{s}^B=\frac{\sum_{i-1}^{N} n_i\left(x_i-\overline{x}_B\right)^2}{N-1}.$$

3. Conclusions

The developed by the authors mathematical model of network operation is based on the Erlang distribution which allows us to describe flows of rather high density. The model takes into account the intensity of requests in each separate direction which makes the area of its application wider and enables modeling reorganized network structures without additional data collection. We introduced the algorithms of optimization of transportation network operation in the previous papers (e.g. [12, 13, 14]). The relevance of the developed model of network operation has been empirically proved in an urban transportation network.

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